Pricing Variance Swaps with Stochastic Volatility

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Abstract—**Following the pricing approach proposed by Zhu & Lian [19], we present an exact solution for pricing variance swaps with the realized variance in the payoff function being a logarithmic return of the underlying asset at some pre-specified discrete sampling points. Our newly-found pricing formula is based on the Heston's [8] two-factor stochastic volatility model. The discovery of this exact and closed-form solution has significantly improved the computational efficiency involved in computing the value of variance swaps with discrete sampling points.**

Keywords: variance swaps, Heston model, explicit formulae, stochastic volatility

1 Introduction

Volatility and variance swaps are essentially forward contracts on annualized realized volatility or variance that provide an easy way for investors to trade future realized volatility or variance against the current implied volatility or variance. There is no cost to enter these contracts as the initial value is typically set to zero at the inception of a contract. The long position of a variance swap pays a fixed delivery price at expiry and receives the floating amounts of annualized realized variance, whereas the short position is just the opposite. Most variance swaps are over-the-counter (OTC) contracts. However, there are some popularly traded exchange-listed volatility-based products. For example, Chicago Board Options Exchange (CBOE) launched 3-month variance futures on S&P 500 in May 2004, and 12-month variance futures in March 2006. It can be imagined that recent market turmoil due to the US subprime crisis would further enhance the trading of volatility-based financial derivatives, and thus greatly promote research in this area.

Although the history of trading variance swaps is relatively short, it has drawn considerable research interests, in terms of developing appropriate valuation approaches and trading strategies. In the literature, there have been two types of valuation approaches, numerical methods and analytical methods. Some typical papers in pricing variance swaps include [2, 3, 6, 9, 12, 19] etc.

The solution approach presented here is very similar to

that presented in Zhu & Lian [19], except that we now adopt the realized variance defined in Broadie & Jain [2] as the sum of squared log return of the underlying asset. Different from the solution approach proposed by Broadie & Jain [2], which is primarily based on integrating the underlying stochastic processes directly, we price discretely-sampled variance swaps based on Heston's two-factor stochastic volatility model by analytically solving the associated governing PDE in two stages which was firstly introduced by Little & Pant [12]. In this way, the nature of stochastic volatility is included in the model and most importantly, a closed-form exact solution can be worked out, even when the sampling times are discrete.

For the easiness of reference, we shall start with a description of our solution approach and our analytical formula for the variance swaps in Section 2. Our conclusions are stated in Section 3.

2 Our Solution Approach

In this section, we use Heston's [8] stochastic volatility model to describe the dynamics of the underlying asset. To evaluate the discretely-sampled realized variance swaps, we employ the dimension reduction technique proposed by Little & Pant [12] to analytically solve the associated governing PDE.

2.1 The Heston Model

It is a well-known fact by now that the Black-Scholes model [1] may fail to reflect certain features of the financial market reality due to some unrealistic assumptions, such as the constant volatility assumption. In an attempt to remedy the drawback of the constant volatility assumption in the Black-Scholes model, many models have been proposed to incorporate stochastic volatility [8, 13, 15]. Among all the stochastic volatility models in the literature, model proposed by Heston [8] has received the most attention since it can give a satisfactory description of the underlying asset dynamics [5, 14].

In the Heston [8] model, the underlying asset S_t is modeled by the following diffusion process with a stochastic instantaneous variance v_t, under the risk-neutral probability measure.

$$dS_t = rS_t dt + \sqrt{v_t} S_t d\widetilde{B}_t^S$$
$$dv_t = \kappa^*(\theta^* - v_t)dt + \sigma_V \sqrt{v_t} d\widetilde{B}_t^V \qquad (1)$$

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where θ^* is the long-term mean of the variance, κ^* is a mean-reverting speed parameter of the variance, σ_V is the so-called volatility of volatility. The two Wiener processes $d\widetilde{B}_t^S$ and $d\widetilde{B}_t^V$ describe the random noise in asset and variance respectively. They are assumed to be correlated with a constant correlation coefficient ρ, that is $(d\widetilde{B}_t^S, d\widetilde{B}_t^V) = \rho dt$. The stochastic volatility process is the familiar squared-root process. To ensure the variance is always positive, it is required that $2\kappa^*\theta^* \geq \sigma^2$. For the rest of this paper, our analysis will be based on the risk-neutral probability measure. The conditional expectation at time t is denoted by $E_t^Q = E^Q[\cdot \mid \mathscr{F}_t]$, where $\mathscr{F}_t$ is the filtration up to time t.

2.2 Variance Swaps

In this subsection, we introduce the variance swaps and provide various definitions of realized variance and the corresponding strike prices for variance swaps.

Variance swaps are forward contracts on the future realized variance of the returns of the specified underlying asset. The payoff at expiry for the long position of a variance swap is equal to the annualized realized variance over a pre-specified period minus a pre-set delivery price of the contract multiplied by a notional amount of the swap in dollars per annualized volatility point, whereas the short position is just the opposite. More specifically, the value of a variance swap at expiry can be written as $V_T = (RV - K_{var}) \times L$, where the RV is the annualized realized variance over the contract life $[0, T]$, K_{var} is the annualized delivery price for the variance swap, which is set to make the value of a variance swap equal to zero for both long and short positions at the time the contract is initially entered. To a certain extent, it reflects market's expectation of the realized variance in the future. L is the notional amount of the swap in dollars per annualized volatility point squared and T is the life time of the contract.

At the beginning of a contract, it is clearly specified the details of how the realized variance should be calculated. Important factors contributing to the calculation of the realized variance include underlying asset(s), the observation frequency of the price of the underlying asset(s), the annualization factor, the contract lifetime, the method of calculating the variance. Some typical formulae [9, 12] for the measure of realized variance are

$$V(0, N, T) = \frac{AF}{N} \sum_{i=1}^{N} \log^2(\frac{S_{t_i}}{S_{t_{i-1}}}) \times 100^2 \qquad (2)$$

where S_{t_i} is the closing price of the underlying asset at the i-th observation time t_i, and there are altogether N observations. AF is the annualized factor converting this expression to an annualized variance. We assume equally-spaced discrete observations in this paper so that the annualized factor is of a simple expression $AF = \frac{1}{\Delta t} = \frac{N}{T}$.

In the literature, these two definitions have been alternatingly used to measure the realized variance, even though in practice most of the contracts appear to be drawn up using the definition $V(0, N, T)$ for the realized variance.

In the risk-neutral world, the value V_t of a variance swap at time t is the expected present value of the future payoff. This should be zero at the beginning of the contract since there is no cost to enter into a swap. Therefore, the fair variance delivery price can be easily defined as $K_{var} = E_0^Q[V(0, N, T)]$, after setting the value of $V_t = 0$ initially. The variance swap valuation problem is therefore reduced to calculating the expectation value of the future realized variance in the risk-neutral world.

2.3 Pricing Approach

Our solution approach has been described in details in Zhu & Lian [19]. However, for the completeness of this paper and easiness of reference, we shall outline our approach again to show how it leads to an analytical solution for the fair delivery price of a variance swap with the realized variance being defined the sum of log-return of the underlying asset.

As illustrated in (2), the expected value of realized variance in the risk-neutral world is defined as:

$$E_0^Q[V(0, N, T)] = E_0^Q[\frac{1}{N\Delta t} \sum_{i=1}^{N} \log^2(\frac{S_{t_i}}{S_{t_{i-1}}})] \times 100^2$$
$$= \frac{100^2}{N\Delta t} \sum_{i=1}^{N} E_0^Q[\log^2(\frac{S_{t_i}}{S_{t_{i-1}}})]$$
$$\qquad (3)$$

So the problem of pricing variance swap is reduced to calculating the N expectations in the form of:

$$E_0^Q[\log^2(\frac{S_{t_i}}{S_{t_{i-1}}})] \qquad (4)$$

for some fixed equal time period Δt and N different tenors $t_i = i\Delta t$ $(i = 1, \cdots, N)$. In the rest of this section, we will focus our main attention on calculating the expectation of this expression. As shall be shown later, we need to consider two cases, $i = 1$ and $i > 1$ separately, due to the difference in the calculation procedures. The expectation in (4) is calculated by computing the expectation of $\log^2(\frac{S_{t_i}}{S_{t_{i-1}}})$ for each fixed t_i and t_{i-1}, which are given constants once a discretization is made along the time axis.

Firstly we consider the case $i > 1$. In this case the time $t_{i-1} > 0$ and thus $S_{t_{i-1}}$ is also an unknown at the current time $t = 0$. Therefore, the payoff function depends on two unknown variables $S_{t_{i-1}}$ and S_{t_i} which are the underlying prices in the future. This two-dimensional payoff function makes the problem extremely difficult to deal with. We will however show that the problem could be solved by

firstly introducing a new variable I_t and then decomposing the original problem into two one-dimensional problems which could be relatively easier to be solved analytically. This technique was first proposed by Little & Pant [12].

Introduce a new variable I_t

$$I_t = \int_0^t \delta(t_{i-1} - \tau) S_\tau d\tau = \begin{cases} S_{t_{i-1}}, & t_{i-1} \le t \le t_i \\ 0, & 0 \le t < t_{i-1} \end{cases} \quad (5)$$

where the $\delta(\cdot)$ is the Dirac delta function.

We now consider a contingent claim $U_i = U_i(S, v, I, t)$ whose payoff at expiry t_i is $\log^2(\frac{S_{t_i}}{I_{t_i}})$. Following the standard analysis of Asian options with stochastic volatility [7], we obtain the PDE for U_i (Subscripts have been omitted in the PDE without ambiguity).

$$\frac{\partial U_i}{\partial t} + \frac{1}{2} v S^2 \frac{\partial U_i^2}{\partial S^2} + \rho \sigma_V v S \frac{\partial U_i^2}{\partial S \partial v} + \frac{1}{2} \sigma_V^2 v \frac{\partial U_i^2}{\partial v^2} + rS \frac{\partial U_i}{\partial S}$$
$$+ [\kappa^*(\theta^* - v)] \frac{\partial U_i}{\partial v} - r U_i + \delta(t_{i-1} - t) \frac{\partial U_i}{\partial I} = 0$$
$$(6)$$

The terminal condition is

$$U_i(S_{t_i}, v_{t_i}, I_{t_i}, t_i) = \log^2(\frac{S_{t_i}}{I_{t_i}}) \quad (7)$$

The Feynman-Kac theorem [11] states that the solution of the PDE system satisfies:

$$E_0^Q[\log^2(\frac{S_{t_i}}{I_{t_i}})] = e^{r t_i} U_i(S_0, v_0, I_0, 0) \quad (8)$$

Thus it is sufficient to solve the PDE (6) with terminal condition (7) to obtain the expectation (4) we require. To solve this PDE system, we need to utilize the properties of variable I_t and the Dirac delta function in the equation.

The property of Dirac delta function indicates that any time away from t_{i-1} the PDE (6) could be reduced as

$$\frac{\partial U_i}{\partial t} + \frac{1}{2} v S^2 \frac{\partial U_i^2}{\partial S^2} + \rho \sigma_V v S \frac{\partial U_i^2}{\partial S \partial v} + \frac{1}{2} \sigma_V^2 v \frac{\partial U_i^2}{\partial v^2} + rS \frac{\partial U_i}{\partial S}$$
$$+ [\kappa^*(\theta^* - v)] \frac{\partial U_i}{\partial v} - r U_i = 0$$
$$(9)$$

This means that we have managed to get rid of variable I_t in the equation except at the time t_{i-1}. However, we cannot declare that we have succeeded in getting rid of one spatial dimension due to the presence of I_t in the terminal condition (7). To deal with the I_t in the terminal condition, we need to use the so-called jump condition.

As mentioned previously, $I_t = 0$, $t < t_{i-1}$ and $I_t = S_{t_{i-1}}$, $t \ge t_{i-1}$. The variable I_t therefore experiences a jump in value across time t_{i-1}. The no-arbitrary assumption leads to the condition that $U_i(S, v, I, t)$ should remain continuous across time t_{i-1}. This is a jump condition at time t_{i-1} because a "jump" has occurred in the independent variable I_t, rather than occurring on the dependent

variable $U_i(S, v, I, t)$ as in the definition of the jumps in traditional sense [17]. Mathematically, the jump condition is of the form

$$\lim_{t \uparrow t_{i-1}} U_i(S, v, I, t) = \lim_{t \downarrow t_{i-1}} U_i(S, v, I, t) \quad (10)$$

From this viewpoint, we can equivalently solve the PDE (9) with terminal condition (7) and jump condition (10) in order to get the expectation we are interested in. Furthermore, inspired by the property of variable I_t, we consider dividing the time domain $[0, t_i]$ into two parts $[0, t_{i-1}]$ and $[t_{i-1}, t_i]$ since during each of the two time sub-domains, I_t could be regarded as constant. Hence, it is an intelligent idea to solve the PDE system by two stages, the first stage in $[t_{i-1}, t_i]$ and the second stage in $[0, t_{i-1}]$. During each of the two stages the PDE systems have one dimension less than the original PDE system. The obtained solution of the first stage will provide the terminal condition for PDE system in second stage through the jump condition (10). We need to remark that this is one of the key features of this paper. Little & Pant [12] were the first to use this dimension reduction approach which provides many computational benefits in their instantaneous local volatility model. In this paper, the approach is applied to the stochastic volatility model and provides us with closed-form solution.

Now, the PDE system (6) could be equivalently expressed by two PDE systems as

$$\begin{cases} \frac{\partial U_i}{\partial t} + \frac{1}{2} v S^2 \frac{\partial U_i^2}{\partial S^2} + \rho \sigma_V v S \frac{\partial U_i^2}{\partial S \partial v} + \frac{1}{2} \sigma_V^2 v \frac{\partial U_i^2}{\partial v^2} \\ + rS \frac{\partial U_i}{\partial S} + [\kappa^*(\theta^* - v)] \frac{\partial U_i}{\partial v} - r U_i = 0 \\ U_i(S, v, I, t_i) = \log^2(\frac{S}{I}), t_{i-1} \le t \le t_i \end{cases}$$
$$(11)$$

and

$$\begin{cases} \frac{\partial U_i}{\partial t} + \frac{1}{2} v S^2 \frac{\partial U_i^2}{\partial S^2} + \rho \sigma_V v S \frac{\partial U_i^2}{\partial S \partial v} + \frac{1}{2} \sigma_V^2 v \frac{\partial U_i^2}{\partial v^2} \\ + rS \frac{\partial U_i}{\partial S} + [\kappa^*(\theta^* - v)] \frac{\partial U_i}{\partial v} - r U_i = 0 \\ \lim_{t \uparrow t_{i-1}} U_i(S, v, I, t) = \lim_{t \downarrow t_{i-1}} U_i(S, v, I, t), 0 \le t \le t_{i-1} \end{cases}$$
$$(12)$$

Note that I_t is a fixed number $I_t = S_{t_{i-1}}$ in the domain $t_{i-1} \le t \le t_i$ and $I_t = 0$ in $0 \le t < t_{i-1}$. We firstly analytically solve the PDE system (11) using generalized Fourier transform method.

Proposition 1 [1] *If the underlying asset follows the dynamic process (1) and a European-style derivative written on this underlying asset has a payoff function $U(S, v, T) = H(S)$ at expiry T, then the solution of the*

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[1]The proof of this proposition can be obtained from the authors on request.

associated governing PDE system of the derivative value

$$\begin{cases} \dfrac{\partial U}{\partial t} + \dfrac{1}{2}vS^2\dfrac{\partial U^2}{\partial S^2} + \rho\sigma_V vS\dfrac{\partial U^2}{\partial S\partial v} + \dfrac{1}{2}\sigma_V^2 v\dfrac{\partial U^2}{\partial v^2} \\ +rS\dfrac{\partial U}{\partial S} + [\kappa^*(\theta^* - v)]\dfrac{\partial U}{\partial v} - rU = 0 \\ U(S,v,T) = H(S) \end{cases} \quad (13)$$

can be expressed in closed form as:

$$U(x,v,t) = \mathcal{F}^{-1}[e^{C(\omega,T-t)+D(\omega,T-t)v}\mathcal{F}[H(e^x)]] \quad (14)$$

using generalized Fourier transform method, where $x = \ln S$, $j = \sqrt{-1}$ and ω is the Fourier transform variable, and

$$\begin{cases} C(\omega,\tau) = r(\omega j - 1)\tau + \dfrac{\kappa^*\theta^*}{\sigma_V^2}[(a+b)\tau - 2\ln(\dfrac{1-ge^{b\tau}}{1-g})] \\ D(\omega,\tau) = \dfrac{a+b}{\sigma_V^2}\dfrac{1-e^{b\tau}}{1-ge^{b\tau}} \\ a = \kappa^* - \rho\sigma_V\omega j, b = \sqrt{a^2 + \sigma_V^2(\omega^2 + \omega j)}, g = \dfrac{a+b}{a-b} \end{cases}$$
$$(15)$$

It should be noted that Formula (14) has been deliberately left in a rather general form. This is because the payoff function $H(S)$ hasn't been specified yet. In fact, Proposition 1 in this general form is applicable to most derivatives, as long as their payoffs depend only on the spot price S of underlying asset at expiry. The original result of Heston [8] is actually a special case covered by this proposition.

However, for some payoffs, the Fourier transform in Proposition 1 has to be interpreted as the Generalized Fourier transform, which is a useful tool for pricing derivatives. For most popularly used financial derivatives, such as vanilla call options with $H(S) = \max(S - K, 0)$, performing the generalized Fourier transform is straightforward. The main difficulties with this approach, however, are associated with the Fourier inverse transform needed to be performed, if one wishes to reduce the computational time substantially. For our specific case, $H(S) = \log^2(\frac{S}{I})$, the Fourier inverse transform could be explicitly worked out and hence the solution can be written in a much simple and elegant form.

Based on the generalized Fourier transform, we can perform the transformation as

$$\mathcal{F}[x^n] = 2\pi j^n \delta^{(n)}(\omega) \quad (16)$$

where $j = \sqrt{-1}$, n is any integer and $\delta^{(n)}(\omega)$ is the n-th order derivative of the generalized delta function satisfying

$$\int_{-\infty}^{\infty} \delta^{(n)}(\omega)\Phi(\omega)d\omega = (-1)^n\Phi^{(n)}(0) \quad (17)$$

In our specified case PDE (11), $H(S) = \log^2(\frac{S}{I})$. By setting $x = \ln S$ and noting I a constant, we perform the generalized Fourier transform to the payoff function $H(x)$ with regards to x.

$$\mathcal{F}[(x-\log I)^2] = 2\pi[-\delta^{(2)}(\omega)-2j\delta^{(1)}(\omega)\log(I)+\delta(\omega)\log^2 I] \quad (18)$$

Using the Proposition 1, the solution of PDE (11) is given by

$$\begin{aligned} U_i&(S,v,I,t) \\ &= -f^{(2)}(0) + 2jf^{(1)}(0)\log(I) + f(0)\log^2 I \end{aligned} \quad (19)$$

where $f(\omega) = e^{C(\omega,t_i-t)+D(\omega,t_i-t)v+x\omega j}$, with $x = \log S$ and $t_{i-1} \leq t \leq t_i$. The terms $f^{(2)}(0)$ and $f^{(1)}(0)$ can be easily computed, using symbolic calculation packages, such as Maple 10.

Now, we have succeeded in obtaining the solution for the PDE system (11), which is the first stage in calculating $E_0^Q[\log^2(\frac{S_{t_i}}{S_{t_{i-1}}})]$.

To finish off the calculation, we need to move to the second stage, i.e. solving the PDE system (12), after the imposition of the jump condition (10). As we shall show later, the simple form of solution (19) has paved an easy way of obtaining an analytical solution in the second stage.

By noting the fact that $\lim_{t\downarrow t_{i-1}}\log S_t = \log I$ due to the definition of I, we obtain

$$\lim_{t\downarrow t_{i-1}} U_i(S,v,I,t) = e^{-r\Delta t}g(v) \quad (20)$$

where $g(v)$ is the expression

$$g(v) = (D^{(1)})^2 v^2 + (2C^{(1)}D^{(1)} - D^{(2)})v + (C^{(1)})^2 - C^{(2)} \quad (21)$$

resulting from computing all the derivatives in (19) with $C^{(1)} = \frac{\partial C(\omega,\Delta t)}{\partial \omega}|_{\omega=0}$, $C^{(2)} = \frac{\partial^2 C(\omega,\Delta t)}{\partial \omega^2}|_{\omega=0}$. $D^{(1)}$ and $D^{(2)}$ are defined similarly. $C(\omega,\tau)$ and $D(\omega,\tau)$ are given in Eq. (15).

Eq. (20) is now the terminal condition for the PDE system (12) in the period $0 \leq t \leq t_{i-1}$, according to the jump condition (10).

It should be noticed that the terminal condition (20) for the PDE system (12) in the period $0 \leq t \leq t_{i-1}$ happens to contain one independent variable, v, only. One can thus take the advantage of this fact and solve the problem neatly with the following proposition.

Proposition 2 *If the underlying asset follows the dynamic process (1), the derivative written on some stochastic aggregated property of this underlying asset with payoff function depending on the v_T only, i.e.,*

$U(S, v, T) = G(v_T)$ *at expiry T will satisfy the PDE*

$$\begin{cases} \dfrac{\partial U}{\partial t} + \dfrac{1}{2}vS^2\dfrac{\partial U^2}{\partial S^2} + \rho\sigma_V vS\dfrac{\partial U^2}{\partial S\partial v} + \dfrac{1}{2}\sigma_V^2 v\dfrac{\partial U^2}{\partial v^2} \\ +rS\dfrac{\partial U}{\partial S} + [\kappa^*(\theta^* - v)]\dfrac{\partial U}{\partial v} - rU = 0 \\ U(S, v, T) = G(v) \end{cases} \quad (22)$$

The solution of this PDE can be obtained analytically in the form of

$$U(S, v, t) = \int_0^{+\infty} e^{-r(T-t)}G(v_T)p(v_T|v_t)dv_T. \quad (23)$$

where

$$p(v_T|v_t) = ce^{-W-V}\left(\frac{V}{W}\right)^{q/2}I_q(2\sqrt{WV}), q = \frac{2\kappa^*\theta^*}{\sigma_V^2} - 1,$$

$$W = cv_t e^{-\kappa^*(T-t)}, c = \frac{2\kappa^*}{\sigma_V^2(1 - e^{-\kappa^*(T-t)})} \quad (24)$$

and $I_q(\cdot)$ is the modified Bessel function of the first kind of order q.

The proof of Proposition 2 is trivial, as it is actually implied by the Feynman-Kac formula, which states that the solution of PDE (22) can be derived from the conditional expectation of the payoff function under the risk-neutral probability measure. Hence, the solution can be expressed in form as

$$U(S, v, t) = E_t^Q[e^{-r(T-t)}G(v_T)] \quad (25)$$

where the associated two processes S_t and v_t follow the stochastic processes in (1), respectively. The expectation is actually not related to the process S_t since the payoff function is independent of S. The process v_t is the well-known CIR squared-root process [4] which is associated with the noncentral chi-square distribution, $\chi^2(2V; 2q+2, 2W)$, with $2q+2$ degrees of freedom and parameter of non-centrality $2W$ proportional to the current variance, v_t. Once we realized that the needed transition probability density function $p(v_T|v_t)$ has been given in [4], as shown in Eq. (24), the proof naturally follows.

Using the Proposition 2, we can express the solution of PDE system (12) as

$$U_i(S, v, I, t) = \int_0^{\infty} e^{-r(t_{i-1}-t)}e^{-r\Delta t}g(v_{t_{i-1}})p(v_{t_{i-1}}|v_t)dv_{t_{i-1}} \quad (26)$$

where $0 \le t < t_{i-1}$, $g(v_{t_{i-1}})$ and $p(v_{t_{i-1}}|v_t)$ are given in Eq. (21) and Eq. (24) respectively. This means for each $i > 1$ the expectation (4) has been found by solving the PDE systems (11) and (12) in two stages,

$$E_0^Q[\log^2(\frac{S_{t_i}}{S_{t_{i-1}}})] = e^{rt_i}U_i(S_0, v_0, I_0, 0)$$

$$= \int_0^{\infty} g(v_{t_{i-1}})p(v_{t_{i-1}}|v_0)dv_{t_{i-1}} \quad (27)$$

As Zhang & Zhu [18] commented in their paper, the integration in the above equation usually cannot be explicitly carried out; we had initially decided to leave our final solution in this integral form too. However, after a careful examination of the properties of the integrand, we realized that the elegant form of $g(v)$, which is the solution of the first stage, could be further explored again. Utilizing the characteristic function of noncentral chi-squared distribution [10], we have successfully carried out the above integral analytically and obtain a fully closed-form solution as our final solution for the price of a variance swap with the realized variance defined by (2). This has made our solution in a remarkably simple form as

$$E_0^Q[\log^2(\frac{S_{t_i}}{S_{t_{i-1}}})] = g_i(v_0) \quad (28)$$

where

$$g_i(v_0) = \int_0^{\infty} g(v_{t_{i-1}})p(v_{t_{i-1}}|v_0)dv_{t_{i-1}}$$

$$= (D^{(1)})^2\left(\frac{\widetilde{q} + 2W_i + (\widetilde{q} + W_i)^2}{c_i^2}\right)$$

$$+ (2C^{(1)}D^{(1)} - D^{(2)})\left(\frac{\widetilde{q} + W_i}{c_i}\right) + (C^{(1)})^2 - C^{(2)} \quad (29)$$

$c_i = \frac{2\kappa^*}{\sigma_V^2(1 - e^{-\kappa^* t_{i-1}})}$, $W_i = c_i v_0 e^{-\kappa^* t_{i-1}}$ and $\widetilde{q} = \frac{2\kappa^*\theta^*}{\sigma_V^2}$.
To a certain extent, it is even simpler than that of the classic Black-Scholes formula, because the latter still involves the calculation of the cumulative distribution function, which is an integral of a smooth real-value function, whereas there is no need to calculate any integral at all in our final solution!

Utilizing (28), the summation in (24) can now be carried out all the way except for the very first period with $i = 1$.

We need to treat the case $i = 1$, separately, simply because in this case we have $t_{i-1} = 0$ and $S_{t_{i-1}} = S_0$, which is the current underlying asset price and is a known value, instead of an unknown value of $S_{t_{i-1}}$ for any other cases with $i > 1$. So the expectation that needs to be calculated in this special case is reduced to

$$E_0^Q[\log^2(\frac{S_{t_i}}{S_{t_{i-1}}})] \quad (30)$$

which can be easily derived by invoking Proposition 1 directly,

$$E_0^Q[\log^2(\frac{S_{t_i}}{S_{t_{i-1}}})] = g(v_0) \quad (31)$$

Summarizing the calculation procedure discussed above, we finally obtain the fair strike price for the variance swap as:

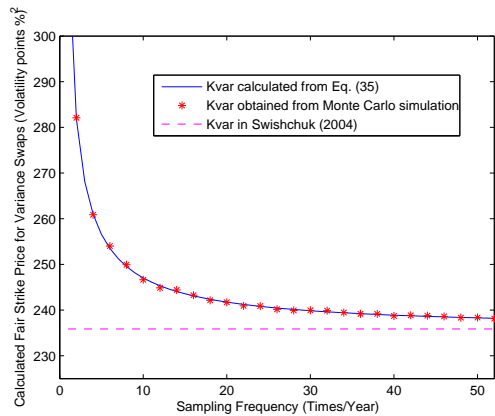$$K_{var} = E_0^Q[V(0, N, T)] = \frac{1}{T}[g(v_0) + \sum_{i=2}^{N}g_i(v_0)] \times 100^2 \quad (32)$$

Figure 1: A Comparison Of Fair Strike Values Based on the Discrete Model, Continuous Model and the Monte Carlo Simulations

N is a finite number denoting the total sampling times of the swap contract. The above equation gives a fair strike price for variance swaps in a simple and closed-form solution. This formula is obtained by solving the intricate associated PDE in two stages. In these two stages, PDEs are cracked separately and, unexpectedly, the two corresponding solutions of these two stages can be expressed in a simple and neat form. Hence, the final solution (32) for the variance swap is written in analytical and closed form.

2.4 Verification

To ensure the correctness of our solution, we have used Monte Carlo method to simulate the underlying process (1) and calculate realized variance according to definition (2). Shown in Fig. 1 are the results of the strike price of variance swaps obtained with the numerical implementation of Formula (32), and those from Monte Carlo simulations. One can clearly observe that the results from our exact solution perfectly match the results from the Monte Carlo simulations. The values of the relative difference is less than 0.1% already when the number of paths reaches 200,000 in Monte Carlo simulation. Such a relative difference is further reduced when the number of paths is increased; demonstrating the convergence of the Monte Carlo simulation towards our exact solution. We have also compared our solution with that of continuous sampling time obtained by Swishchuk [16]; the difference between the two becomes more significant when the the sampling frequency becomes small.

3 Conclusion

In this paper, we have applied the approach proposed by Zhu & Lian [19] to price variance swaps with the realized variance defined as the sum of the log return of the underlying asset, and obtained a closed-form exact solution based on the Heston stochastic volatility model.

Our newly-found pricing formula can be used to significantly improve the computational efficiency involved in computing the value of variance swaps with discrete sampling points.

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