A Note on the Dynamics of Target Leverage Ratios

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Abstract

In this communication we have proposed a simple model for the dynamics of the target leverage ratio of a firm and performed a theoretical investigation of its time path. The gap between a firm's current target leverage ratio and the long-term target ratio is found to diminish exponentially, and the decay rate is determined by the volatility of the leverage ratio and the long-term target ratio only. The time-varying target leverage ratios can be readily incorporated into the dynamics of the leverage ratios of individual firms, and the default probabilities of individual firms can be generated to assess the default risks of the firms.

Keywords: Target leverage ratios; Default risk; Fokker-Planck equation; Mean-reverting dynamics

1. Introduction

The trade-off theory of capital structure and its extensions, namely the dynamic trade-off models, say that firms choose how much debt and how much equity to use by balancing the costs and benefits. In other words, firms select optimal leverage ratios to balance the dead-weight costs of bankruptcy and the tax saving benefits of debt. In spite of many criticisms in the literature, the trade-off theory is well supported by both empirical and theoretical studies (Flannery and Rangan, 2006; Graham and Harvey, 2001; Robert, 2002; Hovakimian *et al.*, 2001; Korajczyka and Levy, 2003; Hennessy and Whited, 2005; Titman and Tsyplakov, 2007; Childs *et el.*, 2005), and thus remains one of the dominant theories of corporate capital structure as taught in the main corporate finance textbooks. Flannery and Rangan (2006) find substantive evidence that non-financial firms identified and pursued long-run target leverage ratios during the 1966-2001 period. Their empirical model accounts for the potentially dynamic nature of a firm's capital structure. Specificallly, firms that are under- or over-leveraged actively adjust their leverage ratios to offset the observed gap between their actual and target leverage ratios. Empirical evidence by Graham and Harvey (2001) also shows that 81% of firms consider a target leverage ratio or range when making their debt decisions.

Moreover, Flannery and Rangan (2006) find that firms adjust towards time-varying target leverage ratios, which depend on plausible firm features. Their base specification indicates that the typical firm's target debt ratio varies quite a lot. Consistent with this finding, Robert (2002) examines the dynamic properties of the capital structure of firms during the 1980-1998 period in a state-space framework and finds that firms gradually adjust their capital structure to a time-varying target, as opposed to a fixed level. In addition, mean reversion in leverage may be due to firm's credit considerations. Hovakimian *et al.* (2001) empirically find that the target ratio may change over time as the firm's profitability and stock price change. Korajczyka and Levy (2003) demonstrate that macroeconomic conditions affect firm's target leverages which could thus be cyclical and time varying.

The empirical findings about target leverage ratios call for the stationary-leverage model f or pricing corporate bonds, which has been studied by Collin-Dufresne and Goldstein (2001). The Collin-Dufresne and Goldstein model (hereafter referred to as the CG model) is based on the structural approach in Merton

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(1974) and considers a mean-reverting liability that is the default barrier. The leverage ratio is defined as a ratio of the liability to the asset value of a firm. The model parameters in the CG model are all constant. These assumptions make the leverage ratio approach a constant target liability-to-asset (leverage) ratio over time. Collin-Dufresne and Goldstein observe empirically that the long-term target ratio is close to the average leverage ratio of BBB-rated firms. They conclude that accounting for a bond issuer's ability to control its level of outstanding debt in the model has a significant impact on credit spread predictions. It helps reconcile some predictions of credit spreads with empirical observations. These include credit spreads that are larger for low-leverage firms and less sensitive to changes in firm value, and upward sloping term structures of credit spreads of speculative-grade bonds.

In view of the evidence for the existence of time-varying target ratios, Hui *et al.* (2006) extend the CG model to incorporate a time-dependent target leverage ratio. In their time-dependent stationary-leverage model (hereafter referred to as the TDSL model) a firm's liability is assumed to be governed by a mean-reverting stochastic process whilst the firm value follows a simple lognormal process. By incorporating time-dependent model parameters in the model, the target leverage ratio is thus time-dependent. The TDSL model can then reflect the movement of a firm's initial target leverage ratio towards a long-term target ratio over time. Using some simple scenarios about the movement, the TDSL model shows that the incorporation of time-varying traget leverage ratios into a structural credit risk model is capable of producing term structures of probabilities of default that are consistent with the default rates reported by Standard & Poor's, in particular for ratings of BBB and below. Hence, it is concluded that the mean-reverting dynamics of a leverage ratio should be a critical factor in modelling credit risk. In addition, the empirical studies by Duffie *et al.* (2007) and Löffler and Maurer (2008) come to the same conclusion that incorporating the dynamics of leverage into default prediction enhances predictive accuracy.

As the time-dependent target leverage ratios used in Hui *et al.* (2006) are rather ad hoc, here comes the key question: "How does the target leverage ratio actually evolve in time? Or, what is the dynamics of the target leverage ratio?" Thus, the purpose of this communication is to try to suggest an answer to the question.

In the aforementioned empirical studies the mean-reverting dynamics of leverage is modelled by a target adjustment model. The target adjustment model states that changes in the leverage ratio are explained by deviations of the current ratio from the target:

$$\begin{aligned} R_{t+1} - R_t &= k_0 + k_1 \left(R^*_{t+1} - R_t \right) + \epsilon_t \\ &= k_1 \left(\beta R^*_{t+1} - R_t \right) + \epsilon_t \end{aligned} \quad , \quad (1)$$

where $\beta = 1 + k_0 / \left(k_1 R^*_{t+1} \right)$, R_t denotes the leverage ratio observed at time t, and R^*_{t+1} is the target leverage ratio for the next period. The major difficulty encountered in these empirical studies is that the target leverage ratio R^* is unobservable and a proxy is needed. A natural candidate is the historical mean of the actual leverage ratio for a firm (Taggart, 1977; Marsh, 1982; Jalilvand and Harris, 1984; Shyam-Sunder and Myers, 1999). The use of the historical mean of leverage has the advantage of minimising the effects of transient variations in time due to business cycles, flotation costs and companies' lagged adjustments towards their target leverage ratios. An alternative specification employs a rolling target for each firm using only historical information and an adjustment process with lags of more than one year. Other more sophisticated choices like estimating the target leverage ratio as a function of observable covariates are available too (Hovakimian *et al.*, 2001; Bontempi, 2001; Fama and French, 2002).

In order to incorporate a time-varying target into a structural credit risk model, we propose a simple time-dependent function for the target ratio with features consistent with some empirical evidence. Based on 111,106 firm-year observations between the years 1965-2001, Flannery and Rangan (2006) conclude that firms adjust rapidly towards time-varying target leverage ratios, which depend on plausible firm features. They find that firms narrow the gaps between their current leverage ratios and target ratios rapidly (34.4% of the gaps per year). Such a fast adjustment speed suggests that a firm's current leverage should not be too far away from its near-term optimal leverage, otherwise it is not feasible for the firm to narrow a wide gap by adjusting its capital structure in a short period of time. Therefore, it is reasonable to assume that the time-varying target leverage ratio moves from a short-term value close to the current value towards a long-term target ratio over time. Flannery and Rangan (2006) also find that firms with high absolute leverage move towards their targets more quickly than those with low absolute leverage, suggesting that deviations from targets are more costly for more highly-leveraged firms.

As the target leverage ratio could continuously be adjusted according to the instantaneous value of the leverage ratio, this implies that the target ratios of highly-leveraged firms may simultaneously move towards the long-term target ratios at a faster pace over time. Delianedis and Geske (1999) find that highly-leveraged firms are usually associated with higher firm value volatility. A firm with more volatile firm value (i.e. leverage ratio) will thus adjust its target leverage ratio faster towards the long-term target ratio. By assuming the time evolution of the target leverage ratio to follow the first moment of the probability density function of the leverage ratio, we will then show how the current leverage ratio and long-term target ratio of a firm and its firm value (leverage) volatility enter into the function of the time-varying target.

According to Hui *et al.* (2006), the leverage ratio R of a firm is shown to obey the stochastic differential equation:

$$\frac{dR}{R} = \kappa \left\{ \ln \theta(t) - \ln R \right\} dt + \sigma dZ \tag{2}$$

where σ is the volatility associated with the Wiener process dZ, and the leverage ratio R is mean-reverting at speed κ towards the time-dependent target leverage ratio $\ln \theta(t)$. In terms of the new variable $x = \ln R$, the stochastic differential equation can be re-written as

$$dx = \left\{ \kappa \left[\ln \theta(t) - x \right] - \frac{1}{2}\sigma^2 \right\} dt + \sigma dZ \quad . \tag{3}$$

Inspired by the empirical target adjustment model, we propose to model the time-dependent target leverage ratio $\ln \theta(t)$ in Eq.(3) by the term $\lambda \langle x(t) \rangle$, where $\langle x(t) \rangle$ is the first moment of the probability density function $P(x,t)$:

$$\langle x(t) \rangle = \int x P(x,t)\, dx \tag{4}$$

and λ is a real parameter to be determined. As a result, we obtain a new stochastic differential equation of x:

$$dx = \left\{ \kappa \left[\lambda \langle x(t) \rangle - x \right] - \frac{1}{2}\sigma^2 \right\} dt + \sigma dZ \tag{5}$$

with finite memory in the drift term. This implies that given the initial data $\langle x(t_0) \rangle$, the target leverage ratio is then being continuously monitored according to the instantaneous expectation value of the leverage ratio at any time $t > t_0$. In other words, the time evolution of the target leverage ratio depends upon its history. The corresponding Fokker-Planck equation governing the probability density function $P(x,t)$ is given by

$$\begin{aligned} &\frac{\partial P(x,t)}{\partial t} \\ &= \frac{\partial^2}{\partial x^2} \left\{ \frac{1}{2}\sigma^2 P(x,t) \right\} - \\ &\quad \frac{\partial}{\partial x} \left\{ \left(\kappa \left[\lambda \langle x(t) \rangle - x \right] - \frac{1}{2}\sigma^2 \right) P(x,t) \right\} \end{aligned} \tag{6}$$

which is manifestly nonlinear and its solution enables us to derive the dynamics of the target leverage ratio analytically.

Following Lo (2005), the solution $P(x,t)$ of Eq.(6) can be easily found to be

$$P(x,t) = \int_{-\infty}^{\infty} K(x,t;x',0)\; P(x',0)\, dx' \tag{7}$$

where

$$\begin{aligned} &K(x,t;x',0) \\ &= \frac{1}{\sqrt{4\pi\eta(t)}} \times \\ &\quad \exp\left\{ -\frac{\left[x \exp(\kappa t) + \xi(t) - x' \right]^2}{4\eta(t)} + \kappa t \right\} \end{aligned} \tag{8}$$

with

$$\begin{aligned} \eta(t) &= \frac{\sigma^2}{4\kappa} \left\{ \exp(2\kappa t) - 1 \right\} \\ \xi(t) &= -\int_0^t \mu(t') \exp(\kappa t')\, dt' \\ \mu(t) &= \kappa\lambda \langle x(t) \rangle - \sigma^2/2 \quad . \end{aligned} \tag{9}$$

We suppose that the random variable x currently has the value x_0, *i.e.* $P(x,0) = \delta(x - x_0)$. Then, $P(x,t) = K(x,t;x_0,0)$, and

$$\begin{aligned} \langle x(t) \rangle &= \int_{-\infty}^{\infty} x\, K(x,t;x_0,0)\, dx \\ &= -\left\{ \xi(t) - x_0 \right\} \exp(-\kappa t) \end{aligned} \tag{10}$$

which in turn yields

$$\begin{aligned} \frac{d\xi(t)}{dt} &= -\mu(t) \exp(\kappa t) \\ &= \kappa\lambda \left\{ \xi(t) - x_0 \right\} + \frac{1}{2}\sigma^2 \exp(\kappa t) \quad . \end{aligned} \tag{11}$$

Eq.(11) can be easily solved to give

$$\xi(t) = x_0 \{1 - \exp(\kappa\lambda t)\} - \frac{\sigma^2}{2\kappa(1-\lambda)} \times \{\exp(\kappa\lambda t) - \exp(\kappa t)\} \quad (12)$$

$$\Longrightarrow \langle x(t) \rangle = \left(x_0 + \frac{\sigma^2}{2\kappa(1-\lambda)} \right) \times \exp\{-\kappa(1-\lambda)t\} - \frac{\sigma^2}{2\kappa(1-\lambda)} . \quad (13)$$

The probability density function $P(x,t)$ can then be expressed as

$$P(x,t) = \frac{1}{\sqrt{4\pi\Omega^2(t)}} \exp\left\{ -\frac{[x - \langle x(t) \rangle]^2}{4\Omega^2(t)} \right\} \quad (14)$$

where

$$\Omega(t) = \frac{\sigma}{2}\sqrt{\frac{1-\exp(-2\kappa t)}{\kappa}} \quad . \quad (15)$$

Obviously, provided that $\lambda < 1$, we have

$$\Omega(t) \longrightarrow \Omega_\infty \equiv \frac{\sigma}{2\sqrt{\kappa}} \quad (16)$$

$$\langle x(t) \rangle \longrightarrow x_\infty \equiv -\frac{\sigma^2}{2\kappa(1-\lambda)}$$

$$\Longleftrightarrow \lambda = 1 + \frac{\sigma^2}{2\kappa x_\infty} \quad (17)$$

as $t \longrightarrow \infty$. Since the long-term target ratio is empirically found to be below unity, x_∞ must assume a negative value and the requirement that $\lambda < 1$ is automatically satisfied. Substituting the definitions of both λ and x_∞ into Eq.(13) yields

$$\langle x(t) \rangle = (x_0 - x_\infty) \exp\left\{ \frac{\sigma^2}{2x_\infty} t \right\} + x_\infty . \quad (18)$$

Thus, the gap between the current target ratio and the long-term target ratio appears to decay exponentially, and the decay rate is dictated by two input parameters only, namely x_∞ and σ. Clearly, the more volatile the firm's leverage ratio is, the faster the self-adjustment of the target leverage ratio towards the long-term target ratio is. Moreover, the probability density function $P(x,t)$ will asymptotically approach the steady-state distribution $P_\infty(x)$:

$$P_\infty(x) \equiv \lim_{t \longrightarrow \infty} P(x,t) = \frac{1}{\sqrt{4\pi\Omega_\infty^2}} \exp\left\{ -\frac{(x - x_\infty)^2}{4\Omega_\infty^2} \right\} . (19)$$

Since the time path of target ratios given in Eq.(18) characterizes the default risk of individual firms according to the volatilities of their leverage ratios and the long-term target ratio, the time-varying target ratios may to some extent resolve the discrepancies between the higher actual default rates of low-leveraged firms (relative to the long-term target ratio) with relatively high volatilities and the default probabilities of the firms estimated by the structural models. For example, Leland (2004) and Hui *et al.* (2006) find that firms with investment grade ratings associated with low leverage ratios have actual default rates higher than the default probabilities generated by the structural models proposed by Longstaff and Schwartz (1995), Leland and Toft (1996) and Collin-Dufresne and Goldstein (2001). Faster time-varying target leverage ratios of these firms will adjust their leverage ratios upwards closer to the long-term target ratio. Such dynamics of the leverage ratios increases the default risks of the firms and may thus make them more consistent with the actual default rates.

Hui *et al.* (2006) report that the actual default rates of poorly rated (i.e. BB, B and CCC rated) firms, which have both high leverage ratios and high volatilities, are lower than the default probabilities estimated by the TDSL model with linear time-dependent target ratios. According to Eq.(18), higher volatilities of firms' leverage ratios would push the leverage ratios back to the long-term target ratio faster. Such dynamics of the leverage ratio reduces the default risk of a firm and may reconcile the discrepancies found in Hui *et al.* (2006). The dynamics is also consistent with the finding in Robert (2002) that firms gradually adjust their capital structure to time-varying targets due to firms' credit considerations of reducing their default risk.

As far as we know, no study has reported the actual behaviour of time-varying target ratios of individual firms. Using Eq.(18), the time-varying target ratios can be readily incorporated into the dynamics of the leverage ratios of firms. This resolves the problems of measuring the actual time path of the target ratios, that could be a very difficult empirical task. Furthermore, with the time-dependent target leverage ratios specified in Eq.(18), the TDSL model can be applied to generate the default probabilities of individual firms with no ad hoc input parameters and to assess the default risks of the firms.

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