Oscillations of a Spacecraft with a Vertical Tether

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***Abstract*— The motion about a centre of mass of a spacecraft with a vertical tethered system under the action of the gravitational moment and small periodic tethered force at a circular orbit is studied. The paper contains bifurcation analysis, phase space research, and analytic solutions for separatrixes. The considered mechanical system performs chaotic motion near separatrixes under the influence small disturbances. Melnikov method gives a criterion for homo/heteroclinic chaos in terms of system parameters. Results of the research can be useful for the analysis of gravitational stabilization systems with space tethers and for studying the behavior of a spacecraft with a deployed tether.**

Index Terms—Chaos, Oscillation, Spacecraft, Tethered system.

I. Introduction

Space tethers have been proposed for a wide range of useful applications, including payload delivery from the Earth orbit and Earth monitoring, using surveillance equipment on the lower end of a vertical tether. In the majority of publications devoted to the analysis of space tethered systems, the object of investigation is the tether and the payload, the spacecraft is regarded as a point mass [1]-[4]. In this paper, we investigate the oscillations of a spacecraft as a rigid body under the action of the tethered force and gravitational moment in the case of the spacecraft with elastic tether deployed on a local vertical. The oscillations of the end mass initiate small periodic disturbances, affecting the spacecraft. On the other hand, depending on the ratio between the spacecraft's moments of inertia and tethered system parameters, points of unstable equilibrium can appear in a phase space. These two factors lead to chaos and irregular behavior of the spacecraft in its motion about a centre of mass [5], [6]. The aim of this paper is to research the influence of elastic fluctuations of the tether on chaotic behavior of the spacecraft.

II. Equation of Spacecraft Motion and Tethered Payload Oscillations

Consider a mechanical system (Fig. 1), consisting of a spacecraft with the center of mass located in the point O, a tether P_1P_2 and an end mass P_2. The spacecraft moves in a circular orbit (dashed curve in Fig. 1). Coordinate system Ox_1y_1 lays in the orbital plane and Ox_1 axis coincides with the local vertical. $Oxyz$ is fixed in the spacecraft frame of reference. The plane Oxy coincides with the orbital plane Ox_1y_1. Equation describing the motion of the spacecraft about of centre of mass, can be written as [7]

$$C\ddot{\alpha} = -3\mu p^{-3}(B-A)\sin\alpha\cos\alpha - T\Delta\sin(\alpha-\varphi), \quad (1)$$

where α is the angle between the axis Ox and the local vertical, A, B, C are the principal moments of inertia, T is the tether force, φ is the angle between the line of action of the tether force and the local vertical, $\Delta = OP_1$ (Fig.1), μ is the gravitation constant, p is the orbit parameter. The motion of the payload about the spacecraft is described by the equations in polar coordinates (r,φ) [1]

$$\begin{gathered}\ddot{\varphi} + \dot{\omega} + 2r^{-1}\dot{r}(\dot{\varphi}+\omega) + 3\omega^2\sin\varphi\cos\varphi = 0,\\ \ddot{r} - r\left[(\dot{\varphi}+\omega)^2 + \omega^2(3\cos^2\varphi - 1)\right] + Tm^{-1} = 0,\end{gathered} \quad (2)$$

where $\omega = \mu^{1/2}p^{-3/2}$, r is the tether length, m - the payload mass.

System (2) has two pairs of stationary solutions, one of them corresponds to the stable (vertical) position of the tether

$$\varphi = 0, \pi;\ T = 3\omega^2 mr. \quad (3)$$

If $\varphi = 0$ then the end mass is below the spacecraft and if $\varphi = \pi$ then the payload is above it. Let us consider an elastic tether

$$T = E(rl^{-1} - 1), \quad (4)$$

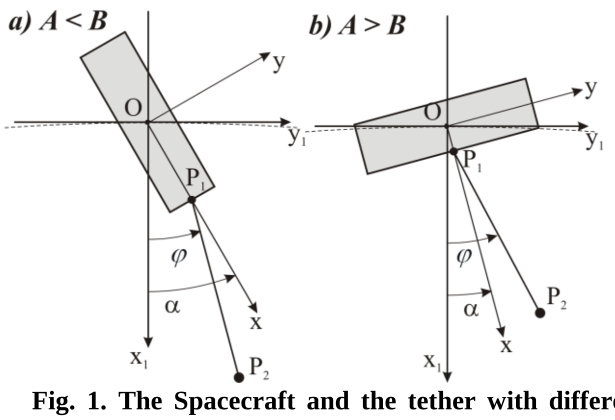

Fig. 1. The Spacecraft and the tether with different moments of inertia.

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where E is modulus of elasticity, l is the length of the unstrained tether. We can suppose that the payload moves strictly along the local vertical in the vicinity of the position of equilibrium (3). The second equation of system (2) subject to (3) takes the form

$$\ddot{r}+\Omega^2 r = Em^{-1}, \tag{5}$$

where $\Omega^2 = Em^{-1}l^{-1}-3\omega^2>0$ for the materials implemented in orbital tether systems [1], [4]. The payload equilibrium is defined by the expression

$$r_0 = E(E-3m\omega^2 l)^{-1}l. \tag{6}$$

For the initial conditions:

$$t_0 = 0:\ r = r_0,\ \dot{r} = V_0, \tag{7}$$

solution of equation (5) is given in the form

$$r = r_0 + V_0\Omega^{-1}\sin\Omega t. \tag{8}$$

In order for the tether to remain constantly stretched, it is essential that the initial speed of the payload (7) is less than the following value

$$V_0 \le 3\omega^2 m^{1/2} l^{3/2}\left(E-3m\omega^2 l\right)^{-1/2}.$$

Using expressions (4) and (8) we can derive a harmonic function for the tether tension

$$T = T_0 + T_m \sin\Omega t, \tag{9}$$

where

$$T_0 = 3m\omega^2 lE\left(E-3m\omega^2 l\right)^{-1},$$
$$T_m = V_0 E m^{1/2} l^{-1/2}\left(E-3\omega^2 ml\right)^{-1/2}.$$

III. Bifurcation analysis and undisturbed solutions

If the end mass oscillates on the vertical tether ($\varphi = 0$), then motion of the spacecraft about the centre of mass subject to (1) and (9) is described with the following equation

$$\ddot{\alpha} = -a\sin\alpha - c\sin\alpha\cos\alpha - \varepsilon\sin\alpha\sin\Omega t, \tag{10}$$

where

$$a = \Delta T_0 C^{-1}, c = 3\mu p^{-3}(B-A)/C, \varepsilon = \Delta T_m C^{-1}. \tag{11}$$

If $\varepsilon = 0$ then periodic force is absent, the system remains conservative and describes the motion of the undisturbed biharmonic oscillator as [8]

$$\ddot{\alpha} = -a\sin\alpha - c\sin\alpha\cos\alpha. \tag{12}$$

The coefficient $a>0$ is always greater than zero, and the sign of c depends on the ratio between the moments of inertia A and B in compliance with (11). Positions of equilibrium of the undisturbed system (12) are defined as roots of the following equation

$$\sin\alpha\left(1+\gamma\cos\alpha\right) = 0, \tag{13}$$

where $\gamma = ca^{-1}$.

Expression (13) provides solutions for two constant positions of equilibrium: $\alpha^* = 0,\pi$, and third position $\alpha^* \in (0,\pi)$ existing only under the condition of

$$|\gamma| > 1. \tag{14}$$

If this condition is not satisfied, then there are only two positions of equilibrium. Point $\alpha^* = 0$ is always a center, and $\alpha^* = \pi$ - a saddle.

On the other hand, if $\gamma < -1$ then both $\alpha^* = 0$ and $\alpha^* = \pi$ are saddles, and the intermediate position of equilibrium

$$\alpha^* = \pm\arccos\left(-\gamma^{-1}\right)$$

is a centre (the left branch of the diagram, fig. 2).

If $\gamma > 1$, we see the opposite picture (the right branch of the diagram, fig. 2). The diagram of bifurcations for negative values of α looks like a mirrored transformation relatively to an abscissa axis.

Hyperbolic points (saddles) exist, when the condition (14) is satisfied. In such cases, the action of external periodic force $\varepsilon\sin\alpha\sin\Omega t$ in the disturbed system (10) may lead to chaos and homo/heteroclinic intersections [5]. Chaotic transitions can appear near separatrixes, dividing characteristic areas of motion and connecting hyperbolic points.

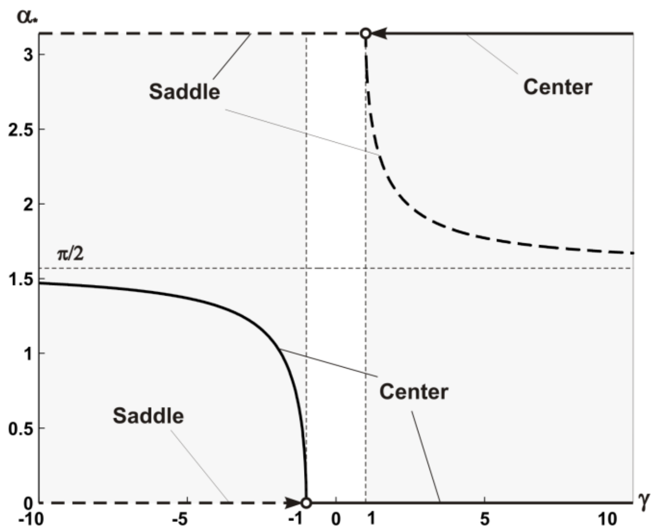

Fig. 2. Bifurcation diagram of the undisturbed system (12).

Now let us refer to the analysis of the disturbed system (10) by means of Melnikov method [9]. The Melnikov method is an analytical tool used to define the existence of homo/heteroclinic intersections and as a result the chaotic behavior. The method provides a necessary condition for the existence of chaos. In order to apply the Melnikov method we need know the analytical solutions of the equation of undisturbed motion at separatrixes.

Solution form varies depending on the values of initial conditions and parameter $\gamma = ca^{-1}$.

We consider two cases: $\gamma > 1$ (Fig 1.a) and $\gamma < 1$ (Fig. 1.b).

Case 1. If

$$\gamma = ca^{-1} > 1 \text{ (Fig 1.a)}, \quad (15)$$

A case like this results in two unstable – saddle type points

$$\alpha_S = \pm \arccos\left(-\gamma^{-1}\right) \quad (16)$$

and two stable points – centers

$$\alpha_C = 0, \pm\pi .$$

We can notice, that the center $\alpha_C = -\pi$ coincides with the center $\alpha_C = \pi$. At the points $\alpha \to -\pi$ and at $\alpha \to \pi$ the speeds $\dot{\alpha}$ coincide, therefore we can say, that phase trajectories are closed on a cylindrical phase space. From now on, we can consider the evolution of the cylindrical space in the range of $\alpha \in [-\pi, \pi]$. The phase space can be divided into the areas A_0 and A_1, separated by the saddles s_1 and s_{-1} (Fig. 3). It is necessary to note, that the region A_1 has a discontinuity point $\alpha = \pi, -\pi$. From the expression (16) it follows, that the saddle s_1 belongs to the interval: $\alpha_S \in (\pi/2, \pi)$, and at negative values of $a < 0$ the saddle s_1 belongs to the interval: $\theta_* \in (\pi/2, \pi)$. At $\gamma \to \infty$ we receive $\alpha_S \to \pi/2$.

The following energy integral corresponds to the equation (12):

$$\dot{\alpha}^2/2 + W(\alpha) = E , \quad (17)$$

where $W(\alpha) = -a\cos\alpha - (c/2)\cos^2\alpha$ is the potential energy and E is the total energy. The shape of the phase portrait depends on the potential energy $W(\alpha)$. The centers α_C correspond to the minimums of the potential energy, and the saddles α_S - to the maximums. If $E > W_S$, where $W_S = W(\alpha_S)$, then the motion is possible in the outer regions (Fig. 3). In the opposite case ($E < W_S$) the motion can occur in any of the inner regions, depending on initial conditions. The equality $E = W_S$ corresponds to the motion along separatrixes. In this case, the two saddles s_1 and s_{-1} are connected by four heteroclinic trajectories.

First of all, we consider the separatrixes, limiting the region A_0. Separating the variables in the energy integral (17), the equation of motion on the separatrixes can be written in the integrated form

$$t = \int_{\alpha_0}^{\alpha} \left\{ 2\left[W(\alpha_S) + a\cos\alpha + (c/2)\cos^2\alpha \right] \right\}^{-1/2} d\alpha , \quad (18)$$

where $W(\alpha_S) = -a\cos\alpha_S - (c/2)\cos^2\alpha_S = a^2/(2c)$.

Changing the variable

$$x = \tan\alpha/2 , \quad (19)$$

simplifies the integral (18) and we obtain the following expression [10]:

$$t = 2P^{-1/2}\int_{x_0}^{x}\left(x_1^2 - x^2\right)^{-1} dx = 2P^{-1/2}\ln\left|(x_1 + x)/(x_1 - x)\right|\Big|_{x_0}^{x} ,$$

where $x_1 = \tan(\alpha_S/2)$ and $P = c(c-a)^2 > 0$.

Finally, the solution of the equation (12) for the heteroclinic orbits in the region A_0 (Fig. 3), can be written as [8]

$$\begin{aligned} &\alpha_\pm(t) = \pm 2\arctan\left[\tan(\alpha_S/2)\tanh(\lambda_1 t/2)\right], \\ &\sigma_\pm(t) = (\dot{\alpha})_\pm = \pm\lambda_1 \sin\alpha_S\left(\cosh\lambda_1 t + \cos\alpha_S\right)^{-1} \end{aligned} \quad (20)$$

where $\lambda_1 = \left(c^2 - a^2\right)^{1/2} c^{-1/2}$ is real, if the condition (14) is satisfied.

For the area A_1 heteroclinic trajectories have a similar form [8]

$$\begin{aligned} &\alpha_\pm(t) = \pi \pm 2\arctan\left[\cot(\alpha_S/2)\tanh(\lambda_1 t/2)\right], \\ &\sigma_\pm(t) = (\dot{\alpha})_\pm = \lambda_1 \sin\alpha_S\left(\cosh\lambda_1 t - \cos\alpha_S\right)^{-1} \end{aligned} \quad (21)$$

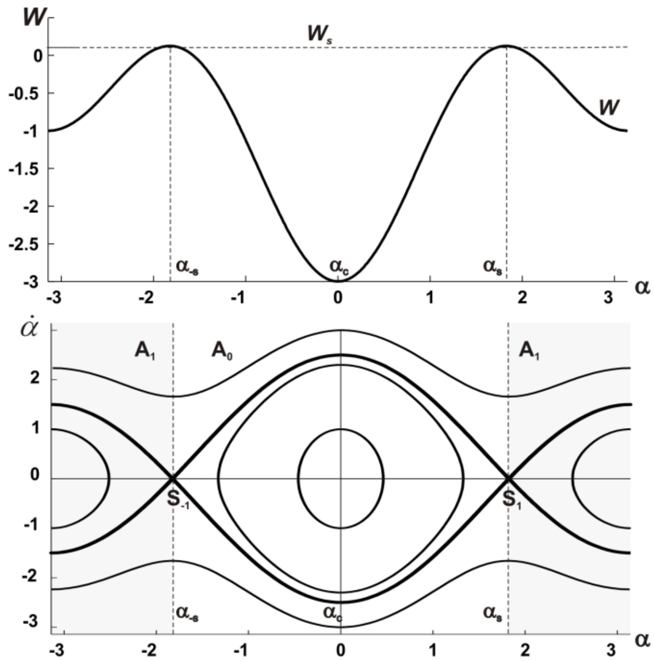

Fig. 3. Phase space and potential energy plots for the undisturbed system (12) at a=1, c=4.

Case 2. If

$$\gamma = ca^{-1} < -1 \text{ (Fig. 1.b)}, \tag{22}$$

then there are two stable centers

$$\alpha_C = \pm \arccos\left(-\gamma^{-1}\right)$$

and one unstable saddle (Fig. 4)

$$\alpha_S = 0 .$$

In this case the substitution of variables (19) simplifies the integral (18) and results in the following expression

$$\begin{aligned} t &= a^{-1/2} \int_{x_0}^{x} x^{-1} \left(x_2^2 - x^2\right)^{-1/2} dx \\ &= -\lambda_2^{-2} \ln \left\| \left[x_2 + \left(x_2^2 - x^2\right)^{1/2} \right] / x \right\|_{x_0}^{x} , \end{aligned} \tag{23}$$

where $\lambda_2 = \left(-a-c\right)^{1/2}$, $x_2 = \left[-\left(a+c\right)/a\right]^{1/2}$.

Two homoclinic trajectories are symmetrical to the right and to the left of the hyperbolic point $\alpha_S = 0$ (Fig. 4). Let us consider only the right orbit. Making a reversed substitution of variables (19) in the expression (23) we receive the following solutions

$$\begin{aligned} &\alpha_{\pm}(t) = \pm 2 \arctan\left(x_2 / \cosh \lambda_2 t\right), \\ &\sigma_{\pm}(t) = (\dot{\alpha})_{\pm} = \mp 2 \lambda_2 x_2 \sinh \lambda_2 t / \left[\left(\cosh \lambda_2 t\right)^2 + x_2^2\right]. \end{aligned} \tag{24}$$

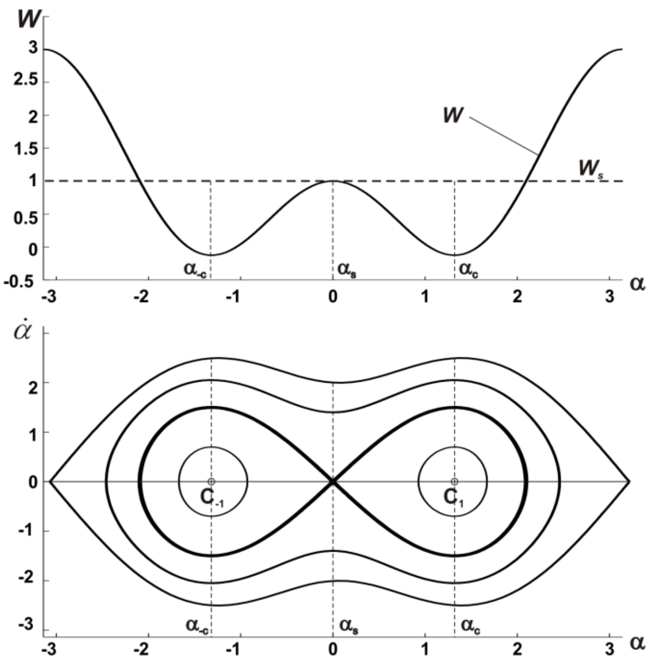

Fig. 4. Phase space and potential energy plots for the undisturbed system (12) at a=1, c=-4.

IV. The Melnikov function

Stable and unstable manifolds do not necessarily coincide and it is possible for them to cross transversally, leading to an infinite number of new heteroclinic points. Then, a heteroclinic tangle is generated. In such case, as a result of disturbance, the motion of the system (10) near the undisturbed separatrixes becomes chaotic. Inside this chaotic layer small isolated regions of regular motion with periodic orbits can also appear. The existence of heteroclinic intersections may be proved by means of the Melnikov method [9]. We present a more convenient way of using Melnikov method, applying it to a system of two first order differential equations instead of one disturbed equation of the second order (10).

$$\begin{aligned} \dot{\alpha} &= \sigma = f_1 + g_1, \\ \dot{\sigma} &= -a \sin\alpha - c \sin\alpha \cos\alpha - \varepsilon \sin\alpha \sin \Omega t \\ &= f_2 + g_2, \end{aligned} \tag{25}$$

where

$$\begin{aligned} &f_1 = \sigma,\ f_2 = -a \sin\alpha - c \sin\alpha \cos\alpha, \\ &g_1 = 0,\ g_2 = -\varepsilon \sin\alpha \sin(\omega t). \end{aligned} \tag{26}$$

The Melnikov function [9] for system (25) is given as

$$\begin{aligned} M^{\pm}(t_0) = \int_{-\infty}^{\infty} \{ & f_1[q_{\pm}^0(t)]\ g_2[q_{\pm}^0(t), \Omega(t+t_0)] - \\ & f_2[q_{\pm}^0(t)]\ g_1[q_{\pm}^0(t), \Omega(t+t_0)]\} dt \\ = \int_{-\infty}^{\infty} \{ & f_1[q_{\pm}^0(t)]\ g_2[q_{\pm}^0(t), \Omega(t+t_0)]\} dt, \end{aligned} \tag{27}$$

where $q_{\pm}^0(t) = [\alpha_{\pm}(t), \sigma_{\pm}(t)]$ are the solutions at the undisturbed homo/heteroclinic orbits (20), (21) or (24).

The Melnikov function (27) subject to expression (26) can be written as

$$\begin{aligned} M^{\pm}(t_0) &= -\varepsilon \int_{-\infty}^{\infty} \sigma_{\pm}\ [\sin\alpha_{\pm} \sin(\Omega t + \Omega t_0)] dt = \\ & - \varepsilon \cos \Omega t_0 \int_{-\infty}^{\infty} \sigma_{\pm} \sin\alpha_{\pm} \sin \Omega t dt. \end{aligned} \tag{28}$$

An integral from of the expression (28)

$$I = -\int_{-\infty}^{\infty} \sigma_{\pm} \sin\alpha_{\pm} \sin\Omega t dt \tag{29}$$

defines the amplitude of changes in the thickness of chaotic layer. We can calculate its absolute size for three orbits:

two heteroclinic orbits (20) and (21) - case 1,

one homoclinic orbit (24) - case 2.

Substituting the solutions (20), (21) and (24) into the integral (29) properly, we receive three integrals in the form of functions of a dimensionless frequency of the external disturbance

$$I_0\left(\Omega_1\right) = \sin^2 \alpha_S \int_{-\infty}^{\infty} \frac{\sinh \tau_1}{(\cosh \tau_1 + \cos \alpha_S)^2} \sin(\Omega_1 \tau_1) d\tau_1, \tag{30}$$

$$I_1\left(\Omega_1\right) = \sin^2 \alpha_S \int_{-\infty}^{\infty} \frac{\sinh \tau_1}{(\cosh \tau_1 - \cos \alpha_S)^2} \sin \Omega_1 \tau_1 d\tau_1, \tag{31}$$

$$I_2\left(\Omega_2\right) = 2 x_2^2 \int_{-\infty}^{\infty} \frac{\sinh(2\tau_2)}{\left(\cosh^2 \tau_2 + x_2^2\right)^2} \sin \Omega_2 \tau_2 d\tau_2, \tag{32}$$

where $\tau_1 = \lambda_1 t$, $\tau_2 = \lambda_2 t$ is the dimensionless time, $\Omega_1 = \Omega \lambda_1^{-1}$ and $\Omega_2 = \Omega \lambda_2^{-1}$ are the dimensionless frequencies.

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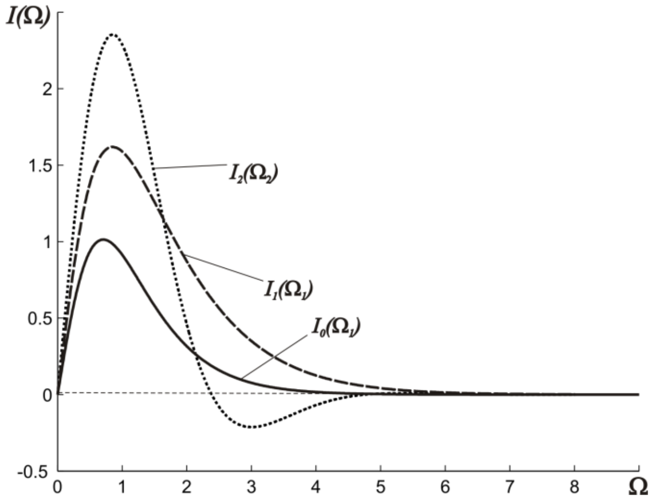

Fig. 5. Evolution of the maximum thickness of chaotic layers (30), (31) and (32) as the functions of dimensionless frequencies $\Omega_1 = \Omega\lambda_1^{-1}$ and $\Omega_2 = \Omega\lambda_2^{-1}$

We can notice, that according to (11), the natural frequencies

$$\lambda_1 = \left(c^2 - a^2\right)^{1/2} c^{-1/2} \text{ and } \lambda_2 = \left(-a - c\right)^{1/2}$$

depend on the parameters of the tethered system and on inertia moments of the spacecraft. We have analyzed the evolution of the maximum thickness of the chaotic layers for the homo/heteroclinic orbits (20), (21) and (24) as the functions of the dimensionless frequencies Ω_1 and Ω_2. Calculations, based on the numerical integration of (30)-(32) show, that at $\Omega_1, \Omega_2 > 6$ the thickness of chaotic layer tends to zero (Fig. 5), therefore the regular structure of a phase space of the disturbed system (10) is observed and trajectories have no homo/heteroclinic intersections (Fig. 5). It means, that at $\Omega_1, \Omega_2 > 6$ the external periodic force $\varepsilon \sin\alpha \sin\Omega t$ has no influence on the behavior of the disturbed system (10).

V. Conclusion

This work attempts to describe transient cases of motion of a spacecraft with an elastic tether deployed on a local vertical using the methods of chaotic mechanics, particularly, the Melnikov method. We have established the borders of homo/heteroclinic chaos using the Melnikov method, which allows us to choose tethered systems parameters that will ensure a regular behavior of the spacecraft with elastic tether in their motion around the centre of mass.

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