Solving Nonlinear Hammerstein Integral Equations by Using B-spline Scaling Functions

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Abstract—**In this article, quadratic semiorthogonal B-spline scaling functions are developed to approximate the solutions of nonlinear Fredholm-Hammerstein integral equations. First, the quadratic B-spline scaling functions and their properties are presented; these properties are used to reduce the computation of integral equations to algebraic equations.**

Keywords: Semiorthogonal, Scaling function, Integral equation, Quadratic B-spline, Fredholm-Hammerstein

1 Introduction

Several numerical methods for approximating the solution of Hammerstein integral equations are known. In the present paper, we apply compactly supported quadratic semiorthogonal (SO) B-spline scaling functions to solve the nonlinear Fredholm- Hammerstein integral equations of the form

$$y(x) = f(x) + \int_0^1 k(x,t)g[t,y(t)]dt, \qquad (1)$$

where $0 \le x \le 1$ and f, g and k are given continuous functions, with $g(t,y)$ nonlinear in y.

Our method consist of reducing (1) to a set of algebraic equations by expanding the unknown function as quadratic B-spline scaling functions, with unknown coefficients. To evaluate the unknown coefficients, the properties of the quadratic B-spline scaling functions are then utilized.

This paper organized as follows. In section 2, we present some properties of general order B-spline functions. Also, we describe the formulation of the quadratic B-spline scaling functions on $[0,1]$ required for our subsequent development. In section 3, we illustrate the function approximation with quadratic B-spline scaling functions, then, the proposed method is used o approximate the solution of nonlinear Fredholm-Hamerstein integral equation. Also, we demonstrate the accuracy of the proposed numerical scheme by considering a numerical example.

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2 General order B-spline functions

In this section we review (from [1, 2]) some properties about B-spline scaling functions. For these examples, scaling functions on real line are the m-th order (degree $m-1$) B-splines $N_m(x)$. We first introduce general order B-splines $N_m(x)$ with a brief summery of some nice properties.

There are several ways to define B-splines. Typically, the m-th order B-splines N_m is defined recursively by convolution:

$$N_1(x) = \begin{cases} 1, & 0 \le x < 1; \\ 0, & otherwise. \end{cases} \qquad (2)$$

$$N_m(x) = \int_{-\infty}^{+\infty} N_{m-1}N_1(t)dt = \int_0^1 N_{m-1}(x-t)dt \qquad (3)$$

Note that the 1st B-spline $N_1(x)$ is the Haar scaling function.

The two-scale relation for B-spline scaling functions of general order m is written as:

$$N_m(x) = \sum_{k=0}^{m} p_k N_m(2x-k) \qquad (4)$$

where the two-scale sequence $\{p_k\}$ for B-spline scaling functions are given by:

$$p_k = 2^{-m+1}\binom{m}{k}, \quad 0 \le k \le m. \qquad (5)$$

2.1 Quadratic B-spline

The quadratic B-spline scaling function is defined as:

$$N_3(x) = \phi(x) = \begin{cases} \frac{1}{2}(x)^2, & 0 \le x < 1; \\ \frac{3}{4} - (x-\frac{3}{2})^2, & 1 \le x \le 2; \\ \frac{1}{2}((x_j-k)-3)^2, & 2 \le x \le 3; \\ 0, & \text{otherwise.} \end{cases} \qquad (6)$$

Functions $\phi(2x-k)$ in V_1 space are expressed as

$$\phi(2x-k) = \begin{cases} \frac{1}{2}(2x-k)^2, & \frac{k}{2} \le x \le \frac{k}{2}+\frac{1}{2}; \\ \frac{3}{4} - (2x-k-\frac{3}{2})^2, & \frac{k}{2}+\frac{1}{2} \le x \le \frac{k}{2}+1; \\ \frac{1}{2}(2x-k-3)^2, & \frac{k}{2}+1 \le x < \frac{k}{2}+\frac{3}{2}; \\ 0, & \text{otherwise.} \end{cases} \qquad (7)$$

ISBN: 978-988-17012-5-1

The two scale relation for quadratic B-Spline $N_3(x)$ is:

$$\phi(x) = \frac{1}{4}\phi(2x) + \frac{3}{4}\phi(2x-1) + \frac{3}{4}\phi(2x-2) + \frac{1}{4}\phi(2x-3). \tag{8}$$

2.2 B-spline scaling functions on $[0, 1]$

Scaling functions can be used to expand any function in $L^2(\mathbb{R})$. These functions are defined on the entire real lines, so that they could be outside of the domain of the problem (see [3, 4]). In order to avoid this, compactly supported spline scaling functions, constructed for the bounded interval $[0, 1]$, have been taken into account in this article.

When semiorthogonal B-spline scaling functions of order m used, the condition

$$2^{j_0} \geq m, \tag{9}$$

must be satisfied in order to have at least one complete inner scaling function. In this paper, we will use quadratic B-spline, $m = 3$ cardinal B-spline function. From (9), the third-order B-spline lowest level, which must be an integer, is determined to $j_0 = 2$.

The third-order B-spline scaling functions are given by

$$\phi_{j,k}(x) = \begin{cases} \frac{1}{2}(x_j-k)^2, & k \leq x_j \leq k+1; \\ \frac{3}{4} - ((x_j-k)-\frac{3}{2})^2, & k+1 \leq x_j \leq k+2; \\ \frac{1}{2}((x_j-k)-3)^2, & k+2 \leq x_j < k+3, \\ & k = 0, \ldots, 2^j-3; \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

Furthermore the third-order B-spline function have two left side boundary scaling functions. The first left side quadratic B-spline scaling function is:

$$\phi_{j,k}(x) = \begin{cases} \frac{1}{2}(x_j-k)^2, & 0 \leq x_j \leq 1; \quad k=-2 \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

The second left side quadratic B-spline scaling function is:

$$\phi_{j,k}(x) = \begin{cases} \frac{3}{4} - ((x_j-k)-\frac{3}{2})^2, & k+1 \leq x_j \leq k+2; \\ \frac{1}{2}((x_j-k)-3)^2, & k+2 \leq x_j \leq k+3, \\ & k=-1; \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

The third-order B-spline function have two right side boundary scaling functions. The first right side quadratic B-spline scaling function is:

$$\phi_{j,k}(x) = \begin{cases} \frac{1}{2}(x_j-k)^2, & k \leq x_j \leq k+1; \\ \frac{3}{4} - ((x_j-k)-\frac{3}{2})^2, & k+1 \leq x_j \leq k+2, \\ & k = 2^j-2; \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

The second right side quadratic B-spline scaling function is:

$$\phi_{j,k}(x) = \begin{cases} \frac{1}{2}((x_j-k)-3)^2, & k+2 \leq x_j \leq k+3, \\ & k = 2^j-1; \\ 0, & \text{otherwise.} \end{cases} \tag{14}$$

The actual coordinate position x is related to x_j according to $x_j = 2^j x$.

3 Function approximation

For any fixed positive integer M, a function $f(x)$ defined over $[0, 1]$ may be presented by B-spline scaling functions as

$$f(x) = \sum_{k=-2}^{2^M-1} s_k \phi_{M,k} = S^T \Phi_M \tag{15}$$

where

$$\begin{aligned} S &= [s_{-2}, s_{-1}, \ldots, s_{2^M-1}], \\ \Phi_M &= [\phi_{M,-2}, \phi_{M,-1}, \ldots, \phi_{M,2^M-1}], \end{aligned} \tag{16}$$

with

$$s_k = \int_0^1 f(x)\tilde{\phi}_{M,k}(x)dx, \quad k=-2,-1,\ldots,2^M-1, \tag{17}$$

where $\tilde{\phi}_{M,k}(x)$ are dual functions of $\phi_{M,k}(x)$. These can be obtained by linear combinations of $\phi_{M,k}(x)$, $k = -2, -1, \ldots, 2^M - 1$ as follows. Let $\tilde{\Phi}_M$ be the dual functions of Φ_M given by

$$\tilde{\Phi}_M = \left[\tilde{\phi}_{M,-2}, \tilde{\phi}_{M,-1}, \ldots, \tilde{\phi}_{M,2^M-1}\right]. \tag{18}$$

Using (16) and (18), we get

$$\int_0^1 \tilde{\Phi}_M \Phi_M^T dx = I_1, \tag{19}$$

where I_1 is $(2^M+2) \times (2^M+2)$ identity matrix. Let

$$P_M = \int_0^1 \Phi_M \Phi_M^T dx. \tag{20}$$

The entry $(P_M)_{i,j}$ of the matrix P_M in (20) is calculated from

$$\int_0^1 \phi_{M,i}(x)\phi_{M,j}(x)dx. \tag{21}$$

For example for $M = 2$, using (10)-(14) and (8) we get:

$$\int_0^1 \Phi\Phi^T dx = \frac{1}{960}\begin{bmatrix} 12 & 26 & 2 & 0 & 0 & 0 \\ 26 & 120 & 52 & 2 & 0 & 0 \\ 2 & 52 & 132 & 52 & 2 & 0 \\ 0 & 2 & 52 & 132 & 52 & 2 \\ 0 & 0 & 2 & 52 & 132 & 52 \\ 0 & 0 & 0 & 2 & 26 & 12 \end{bmatrix}. \tag{22}$$

From (19) and (20), we get

$$\tilde{\Phi}_M = (P_M)^{-1}\Phi_M. \tag{23}$$

3.1 Nonlinear Fredholm-Hammerstein integral equations

In this section, we solve nonlinear Fredholm-Hammerstein integral equations of the form (1) by using B-spline scaling functions. For this, we first assume

$$z(x) = g(x, y(x)), \ \ 0 \leq x \leq 1. \tag{24}$$

We now use (7) to approximate $y(x)$ and $z(x)$ as

$$y(x) = D^T\Phi(x), \ z(x) = E^T\Phi(x), \tag{25}$$

where $\Phi(x)$ is defined in (10), and D and E are $(2^{M+1}+2)\times 1$ unknown vectors defined similarly to S in (7). We also expand $f(x)$ and $k(x,t)$ by B-spline dual functions $\tilde{\Phi}$ defined as in (10) as

$$f(x) = F^T\tilde{\Phi}, \quad k(x,t) = \tilde{\Phi}^T(t)\Theta\tilde{\Phi}(x), \tag{26}$$

where

$$\Theta_{(i,j)} = \int_0^1 \left[\int_0^1 k(x,t)\Phi_i(t)dt\right]\Phi_j(x)dx. \tag{27}$$

From (25) and (26) we get

$$\begin{aligned}\int_0^1 k(x,t)g\big(t,y(t)\big)dt &= \int_0^1 E^T\Phi(t)\tilde{\Phi}^T(t)\Theta\tilde{\Phi}(x)dt\\ &= E^T\Big[\int_0^1 \Phi(t)\tilde{\Phi}^T(t)dt\Big]\Theta\tilde{\Phi}(x) \qquad (28)\\ &= E^T\Theta\tilde{\Phi}(x) \qquad (29)\end{aligned}$$

By applying (25)-(28) in equation (1) we have:

$$D^T\Phi(x) - F^T\tilde{\Phi}(x) - E^T\Theta\tilde{\Phi}(x) = 0 \tag{30}$$

By multiplying equation (30) in $\Phi^T(x)$ and integrating form 0 to 1 with respect to x we get:

$$D^TP - F^T - E^T\Theta = 0. \tag{31}$$

To find the solution $y(x)$ in (25), we first collocate the following equation in $x_i = \frac{i}{2M+1}, i = 0, 1, \ldots, 2^{M+1}$:

$$g\Big(x, D^T\Phi(x)\Big) = E^T\Phi(x). \tag{32}$$

Equation (31) generates a set of $2^{(M+1)}+1$ algebraic equations. The total number of unknowns for vectors D and E are $2[2^{(M+1)}+1]$. These can be obtained by using (31) and (32).

Notice that for calculating F^T in the equation $f(x) = F^T\tilde{\Phi}(x)$, multiply both side of $f(x) = F^T\tilde{\Phi}(x)$ in $\Phi(x)$. Now with integrating from 0 to 1 we have:

$$\int_0^1 f(x)\Phi(x)dx = \int_0^1 F^T\tilde{\Phi}(x)\Phi(x)dx$$

so we have

$$F^T = \int_0^1 f(x)\Phi(x)dx.$$

Example 3.1 [5] *Consider the equation*

$$y(x) = 1+3\sin^2(x)+\int_0^1 k(x,t)y^2(t)dt, \ \ 0 \leq x \leq 1 \tag{33}$$

where

$$k(x,t) = \begin{cases} -3\sin(x-t), & 0 \leq t \leq x; \\ 0, & x < t \leq 1. \end{cases} \tag{34}$$

The computational results for $M = 2$ and $M = 4$ together with the exact solution $y(x) = \cos(x)$ are given in Table 1.

x	*App.* $M = 2$	*App.* $M = 4$	*Exact*
0.1	0.995675	0.995079	0.995004
0.2	0.983153	0.983158	0.983095
0.3	0.955624	0.955356	0.955336
0.4	0.921328	0.921123	0.921061
0.5	0.877259	0.877519	0.877583
0.6	0.825486	0.825377	0.825336
0.7	0.764444	0.764831	0.764842
0.8	0.696137	0.696876	0.696707
0.9	0.621397	0.621589	0.621619
1.0	0.540359	0.540378	0.540302

Table 1: Exact and approximate values with $M = 2, 4$.

4 Conclusions

In the present work, a technique has been developed for solving nonlinear Fredholm-Hammerstein integral equations. The method is based upon compactly supported linear semiorthogonal B-spline scaling functions. The dual functions for these B-spline scaling functions were also given. The problem has been reduced to solving a system of nonlinear algebraic equations.

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