Induced Aggregation Operators in the Generalized Adequacy Coefficient

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***Abstract*—We analyze the use of induced aggregation operators in the generalized adequacy coefficient. We introduce the induced generalized ordered weighted averaging adequacy coefficient (IGOWAAC) operator. It is an extension of the adequacy coefficient by using OWA operators, generalized means and order inducing variables. We study some of its main properties and we see that the IGOWAAC operator can also be seen as an extension of the Minkowski distance. The main advantage is that it provides a more complete generalization that includes a wide range of situations. We further generalize the IGOWAAC operator by using quasi-arithmetic means. The result is the Quasi-IOWAAC operator.**

***Index Terms*—OWA operator; Generalized means; Induced aggregation operators; Adequacy coefficient.**

I. Introduction

The adequacy coefficient [4] is a method for calculating the differences between two sets, fuzzy sets, interval-valued fuzzy sets, etc. It is very similar to the Hamming distance with the difference that it establishes a threshold from which the results are always the same. In [9], Merigó and A.M. Gil-Lafuente suggested a generalization of the adequacy coefficient by using generalized means and ordered weighted averaging (OWA) operators [1-3,5-6,8-15]. Thus, they provided a more general form of the adequacy coefficient that included a wide range of particular cases.

An interesting extension of the OWA operator is the induced OWA (IOWA) operator [14]. It is an extension that uses order inducing variables in the reordering process of the aggregation. Thus, it is able to deal with complex environments where it is not easy to establish the attitudinal character of the decision maker. In [10] it has been suggested a generalization of the IOWA operator that includes a wide range of particular cases by using generalized means (induced generalized OWA (IGOWA) operator) and quasi-arithmetic means (Quasi-IOWA operator).

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The aim of this paper is to present a more complete generalization of the adequacy coefficient by using the IGOWA operator in the aggregation process. We present the IGOWA adequacy coefficient (IGOWAAC) operator. It gives a very general formulation that includes a wide range of aggregation operators including the adequacy coefficient, the OWA operator, the IGOWA operator and the Minkowski distance. We study some of its main properties and we see different families of IGOWAAC operators. We further generalize the IGOWAAC by using quasi-arithmetic means (Quasi-IOWAAC). We also see that the applicability of these new aggregation operators is very similar to the previous ones.

This paper is organized as follows. In Section 2, we briefly review some basic concepts about the induced aggregation operators and the adequacy coefficient. In Section 3, we present the IGOWAAC operator. Section 4 analyses different families of IGOWAAC operators and in Section 5 we end the paper with the conclusions.

II. Preliminaries

A. Induced Aggregation Operators

The IOWA operator was introduced by Yager and Filev [14] and it represents an extension of the OWA operator. The main difference is that the reordering step of the IOWA is carried out with order-inducing variables, rather than depending on the values of the arguments a_i. The IOWA operator also includes the maximum, the minimum and the average operators, as special cases. It can be defined as follows.

Definition 1. An IOWA operator of dimension n is a mapping IOWA: $R^n \to R$ defined by an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0,1]$, and a set of order-inducing variables u_i, by a formula of the following form:

$$\text{IOWA}(\langle u_1,a_1\rangle, \langle u_2,a_2\rangle \ldots, \langle u_n,a_n\rangle) = \sum_{j=1}^{n} w_j b_j \tag{1}$$

where $(b_1, \ldots, b_n)$ is simply $(a_1, a_2, \ldots, a_n)$ reordered in decreasing order of the values of the u_i, u_i is the order-inducing variable and a_i is the argument variable.

The IOWA operator can be generalized by using generalized and quasi-arithmetic means. The result is the IGOWA and the Quasi-IOWA operator. For example, the Quasi-IOWA operator can be defined as follows.

Definition 2. A Quasi-IOWA operator of dimension n is a mapping QIOWA: $R^n \to R$ that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0, 1]$, according to the following formula:

$$\text{QIOWA}(\langle u_1,a_1\rangle, \langle u_2,a_2\rangle\ldots, \langle u_n,a_n\rangle) = g^{-1}\left(\sum_{j=1}^{n} w_j\, g\left(b_{(j)}\right)\right) \quad (2)$$

where $(b_1, \ldots, b_n)$ is simply $(a_1, a_2, \ldots, a_n)$ reordered in decreasing order of the values of the u_i, u_i is the order-inducing variable, a_i is the argument variable, and g is a strictly continuous monotonic function.

B. *The Adequacy Coefficient*

The normalized adequacy coefficient [4] is an index used for calculating the differences between two elements, two sets, etc. In fuzzy set theory, it can be useful, for example, for the calculation of distances between fuzzy sets and interval-valued fuzzy sets. It is very similar to the Hamming distance with the difference that it neutralizes the result when the comparison shows that the real element is higher than the ideal one. In [8], they proposed a new version of the adequacy coefficient that uses the OWA operator in the aggregation. They called it the OWAAC operator. It can be defined as follows for two sets P and P_k.

Definition 3. An OWAAC operator of dimension n is a mapping OWAAC: $[0, 1]^n \times [0, 1]^n \rightarrow [0, 1]$ that has an associated weighting vector W, with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, such that

$$\text{OWAAC}(\langle \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle \mu_n, \mu_n^{(k)}\rangle) = \sum_{j=1}^{n} w_j K_j \quad (3)$$

where K_j represents the jth largest of $[1 \wedge (1 - \mu_i + \mu_i^{(k)})]$, $\mu_i \in [0, 1]$, for the ith characteristic of the ideal P, $\mu_i^{(k)} \in [0, 1]$, for the ith characteristic of the kth alternative under consideration and $k = 1, 2, \ldots, m$.

The OWAAC operator can be generalized by using generalized and quasi-arithmetic means. The result is the generalized OWAAC (GOWAAC) and the Quasi-OWAAC operator [9]. The GOWAAC operator can be defined as follows.

Definition 4. An OWAAC operator of dimension n is a mapping OWAAC: $[0, 1]^n \times [0, 1]^n \rightarrow [0, 1]$ that has an associated weighting vector W, with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, such that

$$\text{GOWAAC}(\langle \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle \mu_n, \mu_n^{(k)}\rangle) = \left(\sum_{j=1}^{n} w_j K_j^{\lambda}\right)^{1/\lambda} \quad (4)$$

where K_j represents the jth largest of $[1 \wedge (1 - \mu_i + \mu_i^{(k)})]$, $\mu_i \in [0, 1]$, for the ith characteristic of the ideal financial product P, $\mu_i^{(k)} \in [0, 1]$, for the ith characteristic of the kth financial product under consideration and $k = 1, 2, \ldots, m$, and λ is a parameter such that $\lambda \in (-\infty, \infty)$.

III. The Induced Generalized Ordered Weighted Averaging Adequacy Coefficient

In this Section, we present the IGOWAAC operator. It is a new aggregation operator that uses induced aggregation operators, generalized means and the adequacy coefficient in the OWA operator. The main advantage is that it provides a complete generalization of the adequacy coefficient that includes a wide range of particular cases. It can be defined as follows.

Definition 5. An IGOWAAC operator of dimension n is a mapping IGOWAAC: $[0, 1]^n \times [0, 1]^n \rightarrow [0, 1]$ that has an associated weighting vector W, with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, such that

$$f(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle) = \left(\sum_{j=1}^{n} w_j K_j^{\lambda}\right)^{1/\lambda} \quad (5)$$

where K_j is the $[1 \wedge (1 - \mu_i + \mu_i^{(k)})]$ value of the IGOWAAC pair $\langle u_i, a_i\rangle$ having the jth largest u_i, u_i is the order inducing variable, $\mu_i \in [0, 1]$, for the ith characteristic of the ideal, $\mu_i^{(k)} \in [0, 1]$, for the ith characteristic of the kth alternative, $k = 1, 2, \ldots, m$, and λ is a parameter such that $\lambda \in (-\infty, \infty)$.

Note that it is possible to distinguish the descending induced generalized OWAAC (DIGOWAAC) operator and the ascending induced generalized OWAAC (AIGOWAAC) operator by using $w_j = w^*_{n+1-j}$, where w_j is the jth weight of the DIGOWAAC operator and w^*_{n+1-j} the jth weight of the AIGOWAAC operator.

If B is the vector consisting of the ordered arguments K_j^{λ}, and W^T is the transpose of the weighting vector, then the IGOWAAC operator can be expressed as

$$\text{IGOWAAC}(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle) = \left(W^T B\right)^{1/\lambda} \quad (6)$$

Note that if the weighting vector is not normalized, i.e., $W = \sum_{j=1}^{n} w_j \neq 1$, then, the IGOWAAC operator can be expressed as

$$f(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle) = \left(\frac{1}{W}\sum_{j=1}^{n} w_j K_j^{\lambda}\right)^{1/\lambda} \quad (7)$$

The IGOWAAC operator is a mean or averaging operator. This is a reflection of the fact that the operator is commutative, monotonic, bounded and idempotent.

Analogously to the IGOWAAC operator, we can suggest a removal index that is the dual of the IGOWAAC operator, because $Q(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle) = 1 - K(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle)$. We will call it the IGOWADAC operator.

Another interesting issue to consider is that the IGOWAAC operator becomes the induced Minkowski OWA distance (IMOWAD) operator [6] under certain conditions. As it is explained in [7], the adequacy coefficient and the Hamming distance (and also further generalizations by using generalized and quasi-arithmetic means) become the same measure when the adequacy coefficient fulfils the following theorem.

Theorem 1. *Assume* IMOWAD($\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle$) *is the IMOWAD operator, and* IGOWADAC($\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle$) *is the IGOWADAC operator. If* $\mu_i \geq \mu_i^{(k)}$ *for all i, then*:

$$\text{IMOWAD}(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle) = \text{IGOWADAC}(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle) \quad (8)$$

Proof. Let

$$\text{IGOWADAC}(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle) = \left(\sum_{j=1}^{n} w_j [0 \vee (\mu_i - \mu_i^{(k)})]^{\lambda}\right)^{1/\lambda} \quad (9)$$

$$\text{IMOWAD}(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle) = \left(\sum_{j=1}^{n} w_j \,|\, \mu_i - \mu_i^{(k)} \,|^{\lambda}\right)^{1/\lambda} \quad (10)$$

Since $\mu_i \geq \mu_i^{(k)}$ for all i, $[0 \vee (\mu_i - \mu_i^{(k)})] = (\mu_i - \mu_i^{(k)})$ for all i, then

$$\text{IMOWAD}(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle) = \text{IGOWADAC}(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle) \quad \blacksquare$$

Another interesting issue to analyze is the different measures used to characterize the weighting vector of the IGOWAAC operator. For example, we could consider the entropy of dispersion and the divergence of *W*.

The dispersion is a measure that provides the type of information being used. It can be defined as follows.

$$\text{H}(W) = -\sum_{j=1}^{n} w_j \ln(w_j) \quad (11)$$

For example, if $w_j = 1$ for some j, then $H(W) = 0$, and the least amount of information is used. If $w_j = 1/n$ for all j, then, the amount of information used is maximum.

The divergence can be defined as follows.

$$\text{Div}(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1} - \alpha(W)\right)^2 \quad (12)$$

Another interesting issue is the problem of ties in the reordering step. To solve this problem, we recommend following the method developed by Yager and Filev [14] where they replace each argument of the tied IOWA pair by its average. For the IGOWAAC operator, instead of using the arithmetic mean, we replace each argument of the tied IGOWA pair the generalized adequacy coefficient depending on the parameter of λ.

The IGOWAAC operator is an extension of the adequacy coefficient and the OWA operator. Therefore, it is applicable in a wide range of situations already considered with these two methods. Moreover, it is also applicable to other situations such as different problems in statistics, mathematics, economics, etc.

Note that it is possible to further generalize the IGOWAAC operator by using quasi-arithmetic means. The result is the Quasi-IOWAAC operator.

Definition 6. A Quasi-IOWAAC operator of dimension n is a mapping QIOWAAC: $[0, 1]^n \times [0, 1]^n \rightarrow [0, 1]$ that has an associated weighting vector W, with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, such that

$$f(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle) = g^{-1}\left(\sum_{j=1}^{n} w_j \, g\big(b_{(j)}\big)\right) \quad (13)$$

where K_j is the $[1 \wedge (1 - \mu_i + \mu_i^{(k)})]$ value of the Quasi-IOWAAC pair $\langle u_i, a_i\rangle$ having the jth largest u_i, u_i is the order inducing variable, $\mu_i \in [0, 1]$, for the ith characteristic of the ideal, $\mu_i^{(k)} \in [0, 1]$, for the ith characteristic of the kth alternative, $k = 1, 2, \ldots, m$, and g is a strictly continuous monotonic function.

IV. Families of IGOWAAC Operators

Different types of IGOWAAC operators may be used in the aggregation process. Mainly, we can distinguish between those families found in the weighting vector *W* and those found in the parameter λ.

Remark 1. By looking to the parameter λ, we find the following particular cases:

- The IOWAAC operator if λ = 1 (arithmetic).
- The IOWGAC operator if λ approaches to 0 (geometric).
- The IOWQAAC operator if λ = 2 (quadratic).
- The IOWHAAC operator if λ = –1 (harmonic).
- Etc.

Remark 2. If we analyse the weighting vector *W*, then, we find the following cases:

- The maximum ($w_1 = 1$ and $w_j = 0$, for all $j \neq 1$).
- The minimum ($w_n = 1$ and $w_j = 0$, for all $j \neq n$).
- The generalized adequacy coefficient ($w_j = 1/n$, for all i).
- The weighted generalized adequacy coefficient (the ordered position of i is the same as the ordered position of u_i).
- The generalized Hurwicz adequacy coefficient criteria ($w_p = \alpha$, $w_q = 1 - \alpha$, $w_j = 0$, for all $j \neq p, q$, and $u_p = \text{Max}\{a_i\}$ and $u_q = \text{Min}\{a_i\}$).
- The GOWAAC operator (the ordered position of j is the same as the ordered position of u_i).
- The step-IGOWAAC ($w_k = 1$ and $w_j = 0$, for all $j \neq k$).
- The S-IGOWAAC ($w_p = (1/n)(1 - (\alpha + \beta)) + \alpha$, $w_q = (1/n)(1 - (\alpha + \beta)) + \beta$, $u_p = \text{Max}\{a_i\}$ and $u_q = \text{Min}\{a_i\}$, and $w_j = (1/n)(1 - (\alpha + \beta))$ for $j \neq p, q$, where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$).
- The centered-IGOWAAC (if it is symmetric, strongly decaying from the center to the maximum and the minimum, and inclusive).

- The olympic-IGOWAAC operator ($w_1 = w_n = 0$, and $w_j = 1/(n - 2)$ for all others).
- Etc.

Remark 3. We could develop a lot of other families of IGOWAAC weights in a similar way as it has been developed in a lot of studies such as [1-3,5-6,8-15].

V. Investment selection with the IGOWAAC operator

The IGOWAAC operator is applicable in a wide range of situations such as in decision making, statistics, engineering, economics, etc. In this paper, we will consider a decision making application in investment selection. The use of the IGOWAAC operator can be useful in a lot of situations, for example, when the board of directors of a company wants to take a decision. Obviously, the attitudinal character of the board of directors is very complex because it involves the decision of different persons with different interests.

The process to follow in the selection of investments with the IGOWAAC operator is similar to the process developed in [4-5], with the difference that now we are considering a financial management problem. The 5 steps of the decision process can be summarized as follows:

Step 1: Analysis and determination of the significant characteristics of the available investment strategies for the company. Theoretically, it is represented as: $C = \{C_1, C_2,\dots, C_i,\dots, C_n\}$, where C_i is the ith characteristic of the investment strategy and we suppose a limited number n of characteristics.

Step 2: Fixation of the ideal levels of each characteristic in order to form the ideal investment strategy.

Table 1: Ideal investment strategy

	C_1	C_2	…	C_i	…	C_n
$P =$	μ_1	μ_2	…	μ_i	…	μ_n

where P is the ideal investment strategy expressed by a fuzzy subset, C_i is the ith characteristic to consider and $\mu_i \in [0, 1]$; $i = 1, 2, \dots, n$, is a number between 0 and 1 for the ith characteristic.

Step 3: Fixation of the real level of each characteristic for all the investment strategies considered.

Table 2: Available alternatives

	C_1	C_2	…	C_i	…	C_n
$P_k =$	$\mu_1^{(k)}$	$\mu_2^{(k)}$	…	$\mu_i^{(k)}$	…	$\mu_n^{(k)}$

with $k = 1, 2, \dots, m$; where P_k is the kth investment strategy expressed by a fuzzy subset, C_i is the ith characteristic to consider and $\mu_i^{(k)} \in [0, 1]$; $i = 1, \dots, n$, is a number between 0 and 1 for the ith characteristic of the kth investment strategy.

Step 4: Comparison between the ideal investment strategy and the different alternatives considered using the IGOWAAC operator. In this step, the objective is to express numerically the removal between the ideal investment strategy and the different alternatives considered. Note that it is possible to consider a wide range of IGOWAAC operators such as those described in Section 3 and 4.

Step 5: Adoption of decisions according to the results found in the previous steps. Finally, we should take the decision about which investment strategy select. Obviously, our decision is to select the investment strategy with the best results according to the type of IGOWAAC operator used in the analysis.

VI. Numerical example

In the following, we present a numerical example of the new approach in a decision making problem. We study a problem of investment selection where a decision maker is looking for the optimal strategy. Note that other decision-making applications could be developed such as in financial decision making [8], human resource management and strategic decision making.

We analyze different particular cases of the IGOWAAC operator such as the NAC, the WAC, the OWAAC and the IOWAAC. Note that with this analysis, we obtain "optimal" choices that depend on the aggregation operator used in each particular case. Then, we see that each aggregation operator may lead to different results and decisions. The main advantage of the IGOWAAC is that it includes a wide range of particular cases, reflecting different potential factors to be considered in the decision-making problem depending on the situation found in the analysis. Thus, the decision maker is able to consider a lot of possibilities and select the aggregation operator that is in closest accordance with his interests.

Assume that a company wants to invest some money in a region. Initially, they consider five possible alternatives.

- A_1 = Invest in the European market.
- A_2 = Invest in the American market.
- A_3 = Invest in the Asian market.
- A_4 = Invest in the African market.
- A_5 = Do not invest money.

In order to evaluate these investments, the investor has brought together a group of experts. This group considers that each investment alternative can be described with the following characteristics:

- C_1 = Benefits in the short term.
- C_2 = Benefits in the mid term.
- C_3 = Benefits in the long term.
- C_4 = Risk of the investment.
- C_5 = Other variables.

The experts establish the values of an ideal investment as follows.

Table 3: Ideal investment strategy

	C_1	C_2	C_3	C_4	C_5
I	0.7	0.9	0.8	0.9	1

The results of the available investment strategies, depending on the characteristic C_i and the alternative A_k that the decision maker chooses, are shown in Table 4.

Table 4: Available investment strategies

	C_1	C_2	C_3	C_4	C_5
A_1	0.6	0.7	0.9	0.8	0.9
A_2	0.9	0.9	0.2	1	0.7
A_3	0.6	0.8	0.7	0.7	0.6
A_4	0.9	0.5	0.8	1	0.7
A_5	0.6	0.7	0.8	0.9	0.8

In this problem, the experts assume the following weighting vector: W = (0.3, 0.2, 0.2, 0.2, 0.1). Due to the fact that the attitudinal character is very complex because it involves the opinion of different members of the board of directors, the experts use order-inducing variables to represent it. The results are shown in Table 5.

Table 5: Order inducing variables

	C_1	C_2	C_3	C_4	C_5
A_1	15	12	17	13	10
A_2	17	20	15	14	16
A_3	11	14	12	18	13
A_4	10	19	17	15	13
A_5	12	14	16	17	11

With this information, we can aggregate the expected results for each state of nature in order to make a decision. In Table 6, we present different results obtained by using different types of IGOWAAC operators such as the NAC, the WAC, the OWAAC and the IOWAAC operator.

Table 6: Aggregated results

	NAC	WAC	OWAAC	IOWAAC
A_1	0.9	0.9	0.92	0.91
A_2	0.82	0.85	0.88	0.82
A_3	0.82	0.85	0.85	0.81
A_4	0.86	0.89	0.90	0.82
A_5	0.9	0.91	0.92	0.92

If we establish an ordering of the alternatives, a typical situation if we want to consider more than one alternative, then we get the results shown in Table 7. Note that the first alternative in each ordering is the optimal choice.

Table 7: Ordering of the investment strategies

	Ordering
NAC	$A_1=A_5 \succ A_4 \succ A_2=A_3$
WAC	$A_5 \succ A_1 \succ A_4 \succ A_2=A_3$
OWAAC	$A_1=A_5 \succ A_4 \succ A_2 \succ A_3$
IOWAAC	$A_5 \succ A_1 \succ A_2=A_4 \succ A_3$

As we can see, depending on the aggregation operator used, the ordering of the investment strategies may be different. Therefore, the decision about which investment strategy select may be also different.

VII. CONCLUSION

We have presented the IGOWAAC operator. It is a new aggregation operator that generalizes a wide range of aggregation operators by using order inducing variables, generalized means, OWA operators and the adequacy coefficient. We have studied some of its main properties and we have seen that it is an extension of the Minkowski distance. We have analyzed a wide range of families of IGOWAAC operators such as the OWAAC, the OWQAAC, the step-IGOWAAC, the centered-IGOWAAC, etc. We have also presented a further generalization of the IGOWAAC operator by using quasi-arithmetic means (Quasi-IGOWAAC operator).

In future research, we expect to develop further extensions of this approach by using other extensions of the GOWA operator and applying it to different problems such as in decision making.

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