Spatial Disorder Of Coupled Discrete Nonlinear Schrödinger Equations

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Abstract—**In this paper, we study the spatial disorder of coupled discrete nonlinear Schrödinger (CDNLS) equations with piecewise-monotone nonlinearities. By the construction of horseshoes, we show that the CDNLS equation possesses a hyperbolic invariant Cantor set on which it is topological conjugate to the full shift on N symbols. The CDNLS equation exhibits spatial disorder, resulting from the strong amplitudes and stiffness of the nonlinearities in the system. The complexity of the disorder is determined by the oscillations of the nonlinearities.**

Keywords:coupled discrete nonlinear Schrödinger equations, horseshoe, localized solution, spatial disorder

1 Introduction

In this paper, we study solitary wave solutions of the time-dependent coupled discrete nonlinear Schrödinger (CDNLS) equation

$$\begin{cases} \iota \frac{d}{dt}\phi_n^{(i)} = -\phi_{n+1}^{(i)} + 2\phi_n^{(i)} - \phi_{n-1}^{(i)} + \tilde{f}_i(|\phi_n^{(i)}|)\phi_n^{(i)} \\ \qquad + \sum_{j=1}^m \beta_{ij}|\phi_n^{(j)}|^2\phi_n^{(i)}, \\ n \in \mathbb{Z},\ i = 1, \dots, m, \end{cases} \tag{1.1}$$

where $\iota = \sqrt{-1}$, and $\tilde{f}_i \in C^1$ is piecewise-monotone. This means that $\tilde{f}_i$ has a finite number of turning points. Equation (1.1) is a discretization of the coupled nonlinear Schrödinger (CNLS) equation

$$\iota\frac{\partial}{\partial t}\phi_i = -\triangle\phi_i + \tilde{f}_i(|\phi_i|)\phi_i + \sum_{j=1}^m \beta_{ij}|\phi_j|^2\phi_i, \quad i = 1, \dots, m.$$

The connection with the CNLS equations is clearer from the alternative form of (1.1):

$$\begin{cases} \iota \frac{d}{dt}\phi_n^{(i)} = \frac{-1}{h^2}(\phi_{n+1}^{(i)} - 2\phi_n^{(i)} + \phi_{n-1}^{(i)}) + \tilde{f}_i(|\phi_n^{(i)}|)\phi_n^{(i)} \\ \qquad + \sum_{j=1}^m \beta_{ij}|\phi_n^{(j)}|^2\phi_n^{(i)}, \\ n \in \mathbb{Z},\ i = 1, \dots, m. \end{cases}$$

Systems of CNLS equations arise in many fields of physics, including condensed matter, hydrodynamics, optics, plasmas, and Bose-Einstein condensates (BECs) (see e.g. [1, 4, 8, 15]). The coupling constants β_{ij} are the interaction between the i-th and the j-th component of the system. The interaction is attractive if $\beta_{ij} < 0$ and repulsive if $\beta_{ij} > 0$. In the presence of strong periodic trapped potentials, a CNLS equation can be approximated by a CDNLS equation. Equation (1.1) describes a large class of discrete nonlinear systems such as optical fibers [5, 6], small molecules such as benzene [7], and, more recently, dilute BECs trapped in a multiwell periodic potential [2, 3, 18, 17].

The interplay between disorder and nonlinearity is a central topic of nonlinear science. This raises a number of mathematical questions related to the behavior of many physical systems. Our principal focus is to study the spatial disorder of solitary wave solutions of the CDNLS equation (1.1). To obtain such solitary wave solutions, we set $\phi_n^{(i)}(t) = e^{-\iota\omega_i t}u_n^{(i)}$ and transform (1.1) into the time-independent coupled discrete nonlinear Schrödinger equation

$$\begin{cases} -u_{n+1}^{(i)} + f_i(u_n^{(i)})u_n^{(i)} - u_{n-1}^{(i)} + \sum_{j=1}^m \beta_{ij}(u_n^{(j)})^2 u_n^{(i)} = 0, \\ i = 1, \dots, m, \end{cases} \tag{1.2}$$

where $f_i(u) = (2 - \omega_i)u + \tilde{f}_i(|u|)u$. To be more precise, we observe that (1.2) can be written as an iteration of the map $(\bar{\mathbf{u}}, \bar{\mathbf{v}}) = \mathbf{F}(\mathbf{u}, \mathbf{v})$, $\mathbf{u}$ and $\mathbf{v} \in \mathbb{R}^m$, defined by

$$\mathbf{F}: \begin{cases} \bar{u}_i = f_i(u_i) - v_i + \sum_{j=1}^m \beta_{ij}u_j^2 u_i, \\ \bar{v}_i = u_i, \end{cases} \tag{1.3}$$

for $i = 1, \dots, m$, or equivalently, in the vector form

$$\begin{cases} \bar{\mathbf{u}} = \mathbf{f}(\mathbf{u}) - \mathbf{v} + \mathrm{diag}(\mathbf{u})\mathbf{B}\mathbf{u}^{②}, \\ \bar{\mathbf{v}} = \mathbf{u}, \end{cases} \tag{1.4}$$

where $\mathbf{f}(\mathbf{u}) = (f_1(u_1), \dots, f_m(u_m))$, $\mathbf{B} = (\beta_{ij}) \in \mathbb{R}^{m\times m}$ and $\mathbf{u}^{②} = \mathbf{u} \circ \mathbf{u}$. Here $\circ$ denotes the Hadamard product (the elementwise product). We further assume the CDNLS equations (1.2) and (1.3) satisfy the following assumptions:

($\mathcal{A}1$) Denote $c_1^{(i)} < c_2^{(i)} < \cdots < c_{t_i}^{(i)}$ the turning points of f_i. Assume there exist $b > 0$, and closed intervals $I_j^{(i)}$, for $i = 1, \dots, m$ and $j = 0, \dots, t_i$, such that

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$-b < I_0^{(i)} < c_1^{(i)} < I_1^{(i)} < c_2^{(i)} < \cdots < c_{t_i}^{(i)} < I_{t_i}^{(i)} < b,$ and

$$(f_i \pm b^2\|\mathbf{B}\|_\infty)(I_j^{(i)}) \supset [-2b, 2b],$$

for all i, j. Here $(f_i \pm b^2\|\mathbf{B}\|_\infty)(u) = f_i(u) \pm b^2\|\mathbf{B}\|_\infty u$. Also, by $c_j^{(i)} < I_j^{(i)} < c_{j+1}^{(i)}$ with $I_j^{(i)} = [d_j^{(i)}, e_j^{(i)}]$, we mean that $c_j^{(i)} < d_j^{(i)}$ and $e_j^{(i)} < c_{j+1}^{(i)}$.

($\mathcal{A}2$) Let $a = \min_{1\le i\le m} \left\{ |f_i'(u)| \;\middle|\; u \in \bigcup_{j=0}^{t_i} I_j^{(i)} \right\}$. Assume

$$a - 3b^2\|\mathbf{B}\|_\infty \ge \sqrt{5}.$$

Our first theorem concerns the topological conjugacy of $\mathbf{F}$ to the full shifts.

Theorem 1.1. *Suppose assumptions* ($\mathcal{A}1$) *and* ($\mathcal{A}2$) *hold. The map* $\mathbf{F}$ *introduced in* (1.3) *possesses a hyperbolic invariant Cantor set on which* $\mathbf{F}$ *is topological conjugate to the full shift on* $\prod_{i=1}^m (t_i + 1)$ *symbols.*

We see in Theorem 1.1 that the strong amplitudes (Assumption ($\mathcal{A}1$)) and stiffness (Assumption ($\mathcal{A}2$)) of the nonlinearities in f_i lead the CDNLS equation (1.2) to exhibit spatial disorder. The complexity of this disorder is determined by the oscillations (number of turning points) of the nonlinearities. More precisely, the spatial entropy of the CDNLS equation (1.2) equals to $\sum_{i=1}^m \log(t_i + 1)$. In this paper, we also consider the CDNLS equation with the Kerr-like nonlinearity [1], that is,

$$-u_{n+1}^{(i)} - \omega_i u_n^{(i)} - u_{n-1}^{(i)} + \alpha_i (u_n^{(i)})^3 + \sum_{j=1}^m \beta_{ij} (u_n^{(j)})^2 u_n^{(i)} = 0, \tag{1.5}$$

for $i = 1, \ldots, m$. In the decoupled case (with $m = 1$), localized solutions (homoclinic/hetroclinic orbits) of (1.5) have been extensively studied by many researchers, especially the existence for localized solutions (see e.g., [14] and the references cited therein). The chaotic behavior of (1.5) when $m = 1$ as well as its synchronization is studied by [13]. For the two-coupled case, bifurcation analysis of (1.5) for the ground state solutions is studied by [11]. Recently, it is reported by [12] that the phase separation of the ground state solutions of the CDNLS equation in higher-dimensional lattices occurs as the coupling constants β_{ij} are sufficiently large. The construction of horseshoes of (1.2) is studied by [16]. Our second theorem concerns the spatial disorder and patterns of localized solutions for the CDNLS equation (1.5).

Theorem 1.2. *Let* $\gamma = b^2\|\mathbf{B}\|_\infty$. *Suppose*

$$\omega_i - 3\gamma > \sqrt{5}\ ,$$

$$\frac{\omega_i + 2 + \gamma}{b^2} < \alpha_i < \frac{\omega_i - 3\gamma - \sqrt{5}}{3} \left(\frac{2\omega_i + \sqrt{5}}{6b} \right)^2 ; \tag{1.6a}$$

or

$$\omega_i + 3\gamma < -\sqrt{5}\ ,$$

$$\frac{\omega_i - 2 - \gamma}{b^2} > \alpha_i > \frac{\omega_i + 3\gamma + \sqrt{5}}{3} \left(\frac{2\omega_i - \sqrt{5}}{6b} \right)^2 ; \tag{1.6b}$$

for all $i = 1, \ldots, m$. *Then CDNLS equation* (1.5) *possesses a hyperbolic invariant Cantor set on which it is topological conjugate to the full shift on* 3^m *symbols. Moreover, there exist disjoint intervals* I_{-1}, I_0, *and* I_1, *where* $0 \in I_0$, *such that for given finite sequences* $k_n^{(1)}, k_n^{(2)}, \ldots, k_n^{(m)} \in \{-1, 0, 1\}$, $|n| \le N$, *there is a unique localized solution to* (1.5) *satisfying*

$$u_n^{(i)} \in \begin{cases} I_{k_n^{(i)}}, & |n| \le N, \\ I_0, & |n| > N, \end{cases} \tag{1.7a}$$

and

$$\lim_{|n|\to\infty} u_n^{(i)} = 0 \text{ exponentially}, \tag{1.7b}$$

for all $i = 1, \ldots, m$.

2 Construction of horseshoe and its hyperbolicity

Let $\mathcal{B}$ denote the box $[-b, b] \times \cdots \times [-b, b]$ in $\mathbb{R}^m$.

Definition 2.1. A μ-horizontal surface is the graph of a differentiable function $\mathbf{v} = \mathbf{h}(\mathbf{u})$, $\mathbf{u} \in \mathcal{B}$, satisfying $\|D\mathbf{h}(\mathbf{u})\| \le \mu$. A μ-horizontal strip in $\mathcal{B} \times \mathcal{B}$ is the set

$$\mathcal{H} = \{(\mathbf{u}, \mathbf{v}) | \mathbf{h}_1(\mathbf{u}) \le \mathbf{v} \le \mathbf{h}_2(\mathbf{u}),\ \mathbf{u} \in \mathcal{B}\},$$

where $\mathbf{h}_1 < \mathbf{h}_2$ are μ-horizontal surfaces. Similarly, a μ-vertical surface is the graph of a differentiable function $\mathbf{u} = \mathbf{g}(\mathbf{v})$, $\mathbf{v} \in \mathcal{B}$, satisfying $\|D\mathbf{g}(\mathbf{v})\| \le \mu$. A μ-vertical strip in $\mathcal{B} \times \mathcal{B}$ is the set

$$\mathcal{V} = \{(\mathbf{u}, \mathbf{v}) | \mathbf{g}_1(\mathbf{v}) \le \mathbf{u} \le \mathbf{g}_2(\mathbf{v}),\ \mathbf{v} \in \mathcal{B}\},$$

where $\mathbf{g}_1 < \mathbf{g}_2$ are μ-horizontal surfaces. The widths of the horizontal and the vertical strips are defined, respectively, as

$$d(\mathcal{H}) = \max_{\mathbf{u}\in\mathcal{B}} ||\mathbf{h}_1(\mathbf{u}) - \mathbf{h}_2(\mathbf{u})||\ ,$$
$$d(\mathcal{V}) = \max_{\mathbf{v}\in\mathcal{B}} ||\mathbf{g}_1(\mathbf{v}) - \mathbf{g}_2(\mathbf{v})||.$$

Let $\mathbb{E} \subset \mathbb{Z}^m$ be the set of m-tuples defined by

$$\mathbb{E} = \{\mathbf{k} = (k_1, \ldots, k_m) |\ k_i = 0, \ldots, t_i\},$$

where t_i denotes the number of turning points for f_i. For a given $\mathbf{k} \in \mathbb{E}$, let $\mathcal{B}_\mathbf{k} = I_{k_1}^{(1)} \times \cdots \times I_{k_m}^{(m)}$. Here $I_{k_i}^{(i)}$ are the closed intervals given in ($\mathcal{A}1$). We define the horizontal and vertical strips

$$\mathcal{H}_\mathbf{k} = \mathcal{B} \times \mathcal{B}_\mathbf{k} = \{(u_1, \ldots, u_m, v_1, \ldots, v_m) \in (\mathcal{B} \times \mathcal{B}) |\ \mathbf{v} \in \mathcal{B}_\mathbf{k}\},$$
$$\mathcal{V}_\mathbf{k} = \mathcal{B}_\mathbf{k} \times \mathcal{B} = \{(u_1, \ldots, u_m, v_1, \ldots, v_m) \in (\mathcal{B} \times \mathcal{B}) |\ \mathbf{u} \in \mathcal{B}_\mathbf{k}\}.$$

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Theorem 2.1. *Let $\mu = (\tilde{a} - \sqrt{\tilde{a}^2 - 4})/2$ with $\tilde{a} = a - 3\|\mathbf{B}\|b^2$ and $\mathbf{k} \in \mathbb{E}$ be given. If $\mathcal{S}$ is a μ-horizontal surface then $\mathbf{F}(\mathcal{S} \cap \mathcal{V}_\mathbf{k}) \cap (\mathcal{B} \times \mathcal{B})$ is a μ-horizontal surface contained in $\mathcal{H}_\mathbf{k}$. If $\mathcal{S}$ is a μ-vertical surface, then $\mathbf{F}^{-1}(\mathcal{S} \cap \mathcal{H}_\mathbf{k}) \cap (\mathcal{B} \times \mathcal{B})$ is a μ-vertical surface contained in $\mathcal{V}_\mathbf{k}$.*

From Theorem 2.1, we see that $\mathbf{F}(\mathcal{V}_\mathbf{k}) \cap (\mathcal{B} \times \mathcal{B}) \subset \mathcal{H}_\mathbf{k}$ and $\mathbf{F}^{-1}(\mathcal{H}_\mathbf{k}) \cap (\mathcal{B} \times \mathcal{B}) \subset \mathcal{V}_\mathbf{k}$ form a μ-horizontal strip and a μ-vertical strip, respectively. Let

$$\mathcal{H}^*_\mathbf{k} = \mathbf{F}(\mathcal{V}_\mathbf{k}) \cap (\mathcal{B} \times \mathcal{B}) \ , \quad \mathcal{V}^*_\mathbf{k} = \mathbf{F}^{-1}(\mathcal{H}_\mathbf{k}) \cap (\mathcal{B} \times \mathcal{B}). \tag{2.1}$$

Thus the resulting surfaces in Theorem 2.1, $\mathbf{F}(\mathcal{S} \cap \mathcal{V}_\mathbf{k}) \cap (\mathcal{B} \times \mathcal{B})$ and $\mathbf{F}^{-1}(\mathcal{S} \cap \mathcal{H}_\mathbf{k}) \cap (\mathcal{B} \times \mathcal{B})$, can be accordingly rewritten as $\mathbf{F}(\mathcal{S}) \cap \mathcal{H}^*_\mathbf{k}$ and $\mathbf{F}(\mathcal{S}) \cap \mathcal{V}^*_\mathbf{k}$, respectively. We have the following immediate consequence of Theorem 2.1.

Corollary 2.2. *Let μ be the constant given in Theorem 2.1 and $\mathbf{k} \in \mathbb{E}$ be given. If $\mathcal{H}$ is a μ-horizontal strip, then $\mathbf{F}(\mathcal{H}) \cap \mathcal{H}^*_\mathbf{k}$ is also a μ-horizontal strip. If $\mathcal{V}$ is a μ-vertical strip, then $\mathbf{F}^{-1}(\mathcal{V}) \cap \mathcal{V}^*_\mathbf{k}$ is also a μ-vertical strip.*

In Corollary 2.2, we see that $\mathbf{F}$ (resp., $\mathbf{F}^{-1}$) maps a μ-horizontal strip (resp., μ-vertical strip) to $\prod_{i=1}^m (t_i + 1)$ μ-horizontal strips (resp., μ-vertical strips). In the next theorem, we will see that every strip becomes thinner under the mapping by a factor less than 1.

Theorem 2.3. *Let μ be be the constant given in Theorem 2.1 and $\mathbf{k} \in \mathbb{E}$ be given. Suppose $\mathcal{H}$ is a μ-horizontal strip and $\mathcal{V}$ is a μ-vertical strip. If $\bar{\mathcal{H}} = \mathbf{F}(\mathcal{H}) \cap \mathcal{H}^*_\mathbf{k}$ and $\tilde{\mathcal{V}} = \mathbf{F}^{-1}(\mathcal{V}) \cap \mathcal{V}^*_\mathbf{k}$, then*

$$d(\bar{\mathcal{H}}) \le \frac{\mu}{1-\mu^2} d(\mathcal{H}) \ , \quad d(\tilde{\mathcal{V}}) \le \frac{\mu}{1-\mu^2} d(\mathcal{V}).$$

Proof of Theorem 1.1. Let $N = \prod_{i=1}^m (t_i + 1)$ and μ be the constant defined in Theorem 2.1. Define

$$\Lambda_{-1} = \bigcup_{\mathbf{k}_{-1} \in \mathbb{E}} \mathcal{H}^*_{\mathbf{k}_{-1}} \ , \quad \Lambda_0 = \bigcup_{\mathbf{k}_0 \in \mathbb{E}} \mathcal{V}^*_{\mathbf{k}_0},$$

where $\mathcal{H}^*_{\mathbf{k}_{-1}}$ and $\mathcal{V}^*_{\mathbf{k}_0}$ are defined in (2.1). By Corollary 2.2, an inductive argument shows that the sets

$$\Lambda_{-n-1} = \Lambda_{-1} \cap \mathbf{F}(\Lambda_{-1}) \cap \cdots \cap \mathbf{F}^n(\Lambda_{-1}),$$
$$\Lambda_n = \Lambda_0 \cap \mathbf{F}^{-1}(\Lambda_0) \cap \cdots \cap \mathbf{F}^{-n}(\Lambda_0),$$

respectively, consist of N^{n+1} μ-horizontal and N^{n+1} μ-horizontal strips. Hence, we may set

$$\Lambda_{-n-1} = \bigcup_{\substack{\mathbf{k}_{-j} \in \mathbb{E} \\ j=1,\ldots,n+1}} \mathcal{H}^*_{\mathbf{k}_{-1},\mathbf{k}_{-2},\ldots,\mathbf{k}_{-n-1}} \text{ and}$$
$$\Lambda_n = \bigcup_{\substack{\mathbf{k}_j \in \mathbb{E} \\ j=0,\ldots,n}} \mathcal{V}^*_{\mathbf{k}_0,\mathbf{k}_1,\ldots,\mathbf{k}_n},$$

where

$$\mathcal{H}^*_{\mathbf{k}_{-1},\ldots,\mathbf{k}_{-n-1}} = \{(\mathbf{u},\mathbf{v}) \in \mathcal{B} \times \mathcal{B} |\ \mathbf{F}^{-j}(\mathbf{u},\mathbf{v}) \in \mathcal{H}^*_{\mathbf{k}_{-j-1}}, j = 0, \ldots, n\}$$

and

$$\mathcal{V}^*_{\mathbf{k}_0,\mathbf{k}_1,\ldots,\mathbf{k}_n} = \{(\mathbf{u},\mathbf{v}) \in \mathcal{B} \times \mathcal{B} |\ \mathbf{F}^j(\mathbf{u},\mathbf{v}) \in \mathcal{V}^*_{\mathbf{k}_j}, j = 0, \ldots, n\}.$$

It follows from Theorem 2.3 that

$$d\left(\mathcal{H}^*_{\mathbf{k}_{-1},\mathbf{k}_{-2},\ldots,\mathbf{k}_{-n-1}}\right) \le \left(\frac{\mu}{1-\mu^2}\right)^n d(\mathcal{H}^*_{\mathbf{k}_{-1}}) \ ,$$
$$d\left(\mathcal{V}^*_{\mathbf{k}_0,\mathbf{k}_1,\ldots,\mathbf{k}_n}\right) \le \left(\frac{\mu}{1-\mu^2}\right)^n d(\mathcal{V}^*_{\mathbf{k}_0}).$$

Hence, for any sequences $(\mathbf{k}_{-1}, \mathbf{k}_{-2} \ldots)$ and $(\mathbf{k}_0, \mathbf{k}_1, \ldots) \in \mathbb{E}^\mathbb{N}$,

$$\bigcap_{n=1}^\infty \mathcal{H}^*_{\mathbf{k}_{-1},\mathbf{k}_{-2},\ldots,\mathbf{k}_{-n}} \ , \quad \bigcap_{n=0}^\infty \mathcal{V}^*_{\mathbf{k}_0,\mathbf{k}_1,\ldots,\mathbf{k}_n} \tag{2.2}$$

are decreasing to m-dimensional surfaces, say $\mathcal{H}^*_{\mathbf{k}_{-1},\mathbf{k}_{-2},\ldots} = \{\mathbf{v} = \mathbf{h}(\mathbf{u})\}$ and $\mathcal{V}^*_{\mathbf{k}_0,\mathbf{k}_1,\ldots} = \{\mathbf{u} = \mathbf{g}(\mathbf{v})\}$. Here we note that $\mathbf{h}$ and $\mathbf{g}$ may be not differentiable. However, the uniform convergency of the upper and lower surfaces in (2.2) implies they satisfy a Lipschitz condition with Lipschitz constant μ; i.e., for any $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in \mathcal{B}$,

$$\|\mathbf{h}(\mathbf{u}_1) - \mathbf{h}(\mathbf{u}_2)\| \le \mu \|\mathbf{u}_1 - \mathbf{u}_2\| \ ,$$
$$\|\mathbf{g}(\mathbf{v}_1) - \mathbf{g}(\mathbf{v}_2)\| \le \mu \|\mathbf{v}_1 - \mathbf{v}_2\|.$$

Since $|\mu| < 1$, by the contraction mapping theorem, the equation $\{\mathbf{v} = \mathbf{h}(\mathbf{u}), \ \mathbf{u} = \mathbf{g}(\mathbf{v})\}$ has a unique solution in $\mathcal{B} \times \mathcal{B}$. This means $\mathcal{H}^*_{\mathbf{k}_{-1},\mathbf{k}_{-2},\ldots} = \{\mathbf{v} = \mathbf{h}(\mathbf{u})\}$ and $\mathcal{V}^*_{\mathbf{k}_0,\mathbf{k}_1,\ldots} = \{\mathbf{u} = \mathbf{g}(\mathbf{v})\}$ have a unique intersection. Hence, the invariant set $\Lambda = \Lambda_{-\infty} \cap \Lambda_\infty$ is a Cantor set. To see $\mathbf{F}|_\Lambda$ is topological conjugate to the full shift, σ, on N symbols, we define the function

$$\phi(\mathbf{p}) = (\ldots, \mathbf{k}_{-1} | \mathbf{k}_0, \mathbf{k}_1, \ldots),$$

where $\mathbf{p} = \mathcal{H}_{\mathbf{k}_{-1},\mathbf{k}_{-2},\ldots} \cap \mathcal{V}_{\mathbf{k}_0,\mathbf{k}_1,\ldots}$. It is easy to see that ϕ is a homeomorphism from Λ to Σ_N. We only need to show that $\phi(\mathbf{F}(\mathbf{p})) = \sigma(\phi(\mathbf{p}))$. From the construction of $\mathcal{V}^*_{\mathbf{k}_0,\mathbf{k}_1,\ldots}$, we have

$$\mathbf{F}(\mathcal{V}^*_{\mathbf{k}_0,\mathbf{k}_1,\ldots}) = \mathcal{V}^*_{\mathbf{k}_1,\mathbf{k}_2,\ldots}. \tag{2.3}$$

On the other hand, $\mathbf{p} \in \mathcal{H}^*_{\mathbf{k}_{-1},\mathbf{k}_{-2},\ldots} \cap \mathcal{V}^*_{\mathbf{k}_0} \subset \mathcal{H}^*_{\mathbf{k}_{-1},\mathbf{k}_{-2},\ldots} \cap \mathcal{V}_{\mathbf{k}_0}$. From Theorem 2.1, it implies $\mathbf{F}(\mathbf{p}) \in \mathcal{H}^*_{\mathbf{k}_0,\mathbf{k}_{-1},\mathbf{k}_{-2},\ldots}$. Together with (2.3), this shows

$$\phi(\mathbf{F}(\mathbf{p})) = \phi(\mathcal{H}^*_{\mathbf{k}_0,\mathbf{k}_{-1},\mathbf{k}_{-2},\ldots} \cap \mathcal{V}^*_{\mathbf{k}_1,\mathbf{k}_2,\ldots})$$
$$= (\ldots, \mathbf{k}_{-1}, \mathbf{k}_0 | \mathbf{k}_1, \ldots) = \sigma(\phi(\mathbf{p})).$$

□

We also show that the map $\mathbf{F}$ satisfies the hyperbolicity. Before proving the hyperbolicity of Λ, we shall adopt the following theorem in [10, p. 266].

Theorem 2.4. *A compact* $\mathbf{F}$*-invariant set* Λ *is hyperbolic if there exist* $\nu > 1$ *such that for every* $\mathbf{p} \in \Lambda$ *there is a decomposition* $T_{\mathbf{p}}\mathcal{M} = S_{\mathbf{p}} \oplus T_{\mathbf{p}}$ *(in general, not* $D\mathbf{F}$ *invariant), a family of the horizontal cones* $H_{\mathbf{p}} \supset S_{\mathbf{p}}$*, and a family of vertical cones* $V_{\mathbf{p}} \supset T_{\mathbf{p}}$ *associated with the decomposition such that*

$$D\mathbf{F}(\mathbf{p})H_{\mathbf{p}} \subset Int\ H_{\mathbf{F}(\mathbf{p})}, \qquad D\mathbf{F}^{-1}(\mathbf{p})V_{\mathbf{p}} \subset Int\ V_{\mathbf{F}(\mathbf{p})}, \quad (2.4)$$

and

$$\|D\mathbf{F}(\mathbf{p})\boldsymbol{\zeta}\| \geq \nu\|\boldsymbol{\zeta}\| \text{ for } \boldsymbol{\zeta} \in H_{\mathbf{p}}, \qquad \|D\mathbf{F}^{-1}(\mathbf{p})\boldsymbol{\zeta}\| \geq \nu\|\boldsymbol{\zeta}\| \text{ for } \boldsymbol{\zeta} \in V_{\mathbf{F}(\mathbf{p})}. \quad (2.5)$$

Proof of Theorem 1.1: The hyperbolicity of Λ*.* We shall prove the hyperbolicity by verifying the conditions in Theorem 2.4. First, let

$$S_{\mathbf{p}} = \{\begin{bmatrix} \mathbf{0} \\ \boldsymbol{\eta} \end{bmatrix} \in \mathbb{R}^{2m} |\ \boldsymbol{\eta} \in \mathbb{R}^m\}, \qquad T_{\mathbf{p}} = \{\begin{bmatrix} \boldsymbol{\xi} \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{2m} |\ \boldsymbol{\xi} \in \mathbb{R}^m\},$$

and

$$H_{\mathbf{p}} = \{\begin{bmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{bmatrix} \in \mathbb{R}^{2m} |\ \|\boldsymbol{\eta}\| \leq \mu\|\boldsymbol{\xi}\|\}, \qquad V_{\mathbf{p}} = \{\begin{bmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{bmatrix} \in \mathbb{R}^{2m} |\ \|\boldsymbol{\xi}\| \leq \mu\|\boldsymbol{\eta}\|\}.$$

It is easy to see that $S_{\mathbf{p}} \subset H_{\mathbf{p}}$ and $T_{\mathbf{p}} \subset V_{\mathbf{p}}$. Now, let $\mathbf{p} = (\mathbf{u}, \mathbf{v}) \in \Lambda$ and $\boldsymbol{\zeta} = \begin{bmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{bmatrix} \in S_{\mathbf{p}}$ be given. Hence, $\mathbf{p} \in \mathcal{V}_{\mathbf{k}}$ for some $\mathbf{k} \in \mathbb{E}$. Moreover, there exists a μ-horizontal surface $\mathcal{S} = \{\mathbf{v} = \mathbf{h}(\mathbf{u})\}$ containing $\mathbf{p}$ such that $\boldsymbol{\zeta}$ is a tangent vector to $\mathcal{S}$ at $\mathbf{p}$, i.e. $\boldsymbol{\eta} = D\mathbf{h}(\mathbf{p})\boldsymbol{\xi}$. Since $\mathbf{F}(\mathbf{p}) \in \Lambda \subset \mathcal{B} \times \mathcal{B}$, it follows from Theorem 2.1 that the connected component of $\mathbf{F}(\mathcal{S}) \cap (\mathcal{B} \times \mathcal{B})$ containing $\mathbf{F}(\mathbf{p})$, denoted by $\bar{\mathcal{S}}$, is also a μ-horizontal surface. Suppose $\bar{\mathcal{S}}$ is the graph of $\bar{\mathbf{v}} = \bar{\mathbf{h}}(\bar{\mathbf{u}})$. Consequently, $\bar{\boldsymbol{\zeta}} = \begin{bmatrix} \bar{\boldsymbol{\xi}} \\ \bar{\boldsymbol{\eta}} \end{bmatrix} := D\mathbf{F}(\mathbf{p})\boldsymbol{\zeta}$ is a tangent vector to $\bar{\mathcal{S}}$ at $\mathbf{F}(\mathbf{p})$. From the result of Step 3 in the proof of Theorem 2.1, we conclude that

$$\|\bar{\boldsymbol{\eta}}\| = \|D\bar{\mathbf{h}}(\bar{\mathbf{u}})\bar{\boldsymbol{\xi}}\| < \mu\|\bar{\boldsymbol{\xi}}\|.$$

This proves the first invariance condition in (2.4). The second can be similarly obtained. Letting $\nu = 1/\mu$, the Contraction and Expansion condition (2.5) follows from the previous argument directly. This completes the proof. $\square$

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