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Two-Dimensional PCBFs : Application to Nonlinear Volterra Integral Equations

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Abstract—**In this study, we present a direct method to solve nonlinear two-dimensional Volterra-Hammerestein integral equations in terms of two-dimensional piecewise constant block-pulse functions (2D-PCBFs). Properties of these functions and operational matrix of integration together with the product operational matrix are presented and used to transform the integral equation to a matrix equation which corresponds to a system of nonlinear algebraic equations with unknown block-pulse coefficients. An error analysis is given and numerical examples illustrate efficiency and accuracy of the proposed method.**

Keywords: ***Nonlinear integral equations; Two-dimensional Volterra integral equations; Two-dimensional block-pulse functions; Direct method.***

1 Introduction

Consider the following nonlinear two-dimensional Volterra integral equation of the form

$$u(t_1,t_2)=f(t_1,t_2)+$$

$$\int_0^{t_1}\int_0^{t_2} K(t_1,t_2,s_1,s_2)g(s_1,s_2,u(s_1,s_2))\,ds_1ds_2, \quad (1)$$

where $(t_1,t_2)\in D=[0,T_1)\times[0,T_2)$, $u(t_1,t_2)$ is an unknown function and the functions $f(t_1,t_2)$ and $K(t_1,t_2,s_1,s_2)$ are given continuous functions defined, respectively, on D and

$$W=\Big\{(t_1,t_2,s_1,s_2):0\le s_1\le t_1\le T_1, 0\le s_2\le t_2\le T_2\Big\}.$$

The existence, uniqueness, and stability of solutions to (1) is given in [1, 2].
In this work, we apply two-dimensional block-pulse functions (2D-BPFs), constructed on D to solve Eq. (1). Our method consists of reducing (1) to a set of algebraic equations by expanding unknown function as 2D-BPFs with unknown coefficients. The representation error analysis is worked out and method is tested with the aid of the some numerical examples.

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2 Two-dimensional BPFs

Block-pulse functions are a set of orthogonal functions with piecewise constant values and are usually applied as a useful tool in the analysis, synthesis, identification and other problems of control and systems science. This set of functions was first introduced to electrical engineers by Harmuth in 1969, and have been extensively applied -due to their simple and easy operations- for one dimensional problems [3, 4, 5, 6]. A complete details for 1D block-pulse functions is given in [3, 4]. These discussions can also be extended to the 2D block-pulse functions.

2.1 Definition and properties

An (m_1m_2)-set of 2D block-pulse functions $\phi_{i_1,i_2}(t_1,t_2)$ ($i_1=1,2,\dots,m_1$;
$i_2=1,2,\dots,m_2$) is defined in the region of $t_1\in[0,T_1)$ and $t_2\in[0,T_2)$ as:

$$\phi_{i_1,i_2}(t_1,t_2)=\begin{cases}1\,, & (i_1-1)h_1\le t_1<i_1h_1 \text{ and } (i_2-1)h_2\le t_2<i_2h_2,\\ 0, & \text{otherwise.}\end{cases} \quad (2)$$

where m_1, m_2 are arbitrary positive integers, and $h_1=\frac{T_1}{m_1}, h_2=\frac{T_2}{m_2}$.
Similar to the 1D case, There are some properties for 2D-BPFs, the most important properties are disjointness, orthogonality, and completeness.
The 2D block pulse functions are disjoined with each other:

$$\phi_{i_1,i_2}(t_1,t_2)\phi_{j_1,j_2}(t_1,t_2)=\begin{cases}\phi_{i_1,i_2}(t_1,t_2)\,, & \text{if } i_1=j_1 \text{ and } i_2=j_2,\\ 0, & \text{otherwise}\end{cases} \quad (3)$$

and are orthogonal with each other:

$$\int_0^{T_1}\int_0^{T_2}\phi_{i_1,i_2}(t_1,t_2)\phi_{j_1,j_2}(t_1,t_2)dt_2dt_1 = \begin{cases}h_1h_2\,, & \text{for } i_1=j_1 \text{ and } i_2=j_2,\\ 0, & \text{otherwise}\end{cases} \quad (4)$$

in the region of $t_1\in[0,T_1)$ and $t_2\in[0,T_2)$, where $i_1,j_1=1,2,\dots,m_1$ and $i_2,j_2=1,2,\dots,m_2$.
The other property is completeness. For every $f\in\mathcal{L}^2([0,T_1)\times[0,T_2))$ when m_1 and m_2 approaches to the infinity, Parseval's identity holds:

$$\int_0^{T_1}\int_0^{T_2} f^2(t_1,t_2)dt_1dt_2=\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty} f^2_{i_1,i_2}\|\phi_{i_1,i_2}(t_1,t_2)\|^2, \quad (5)$$

where

$$f_{i_1,i_2} = \frac{1}{h_1 h_2} \int_0^{T_1} \int_0^{T_2} f(t_1,t_2)\phi_{i_1,i_2}(t_1,t_2) dt_1 dt_2. \quad (6)$$

The set of 2D block-pulse functions may be written as a vector $\Phi(t_1,t_2)$ of dimension $m_1 m_2$:

$$\Phi(t_1,t_2) = [\phi_{1,1}(t_1,t_2), \ldots, \phi_{m_1,m_2}(t_1,t_2)]^{\mathrm{T}} \quad (7)$$

where $(t_1,t_2) \in [0,T_1) \times [0,T_2)$.
From the above representation and disjointness property, it follows:

$$\Phi(t_1,t_2)\Phi^{\mathrm{T}}(t_1,t_2) = \begin{pmatrix} \phi_{1,1}(t_1,t_2) & 0 & \cdots & 0 \\ 0 & \phi_{1,2}(t_1,t_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_{m_1,m_2}(t_1,t_2) \end{pmatrix}, \quad (8)$$

$$\Phi^{\mathrm{T}}(t_1,t_2)\Phi(t_1,t_2) = 1 \quad (9)$$

and

$$\Phi(t_1,t_2)\Phi^{\mathrm{T}}(t_1,t_2)V = \widetilde{V}\Phi(t_1,t_2) \quad (10)$$

where V is an $m_1 m_2$-vector and $\widetilde{V} = diag(V)$. Moreover, it can be clearly concluded that for every $(m_1 m_2) \times (m_1 m_2)$ matrix A:

$$\Phi^{\mathrm{T}}(t_1,t_2)A\Phi(t_1,t_2) = \widehat{A}^{\mathrm{T}}\Phi(t_1,t_2), \quad (11)$$

where $\widehat{A}$ is an $m_1 m_2$-vector with elements equal to the diagonal entries of matrix A.

2.2 2D-BPFs expansions

A function $f(t_1,t_2)$ defined over $[0,T_1) \times [0,T_2)$ may be expanded by the 2D block-pulse functions as

$$f(t_1,t_2) \simeq \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} f_{i_1,i_2}\phi_{i_1,i_2}(t_1,t_2) = F^{\mathrm{T}}\Phi(t_1,t_2), \quad (12)$$

where F is an $(m_1 m_2) \times 1$ vector given by

$$F = [f_{1,1}, \ldots, f_{1,m_2}, \ldots, f_{m_1,1}, \ldots, f_{m_1,m_2}]^{\mathrm{T}}, \quad (13)$$

and $\Phi(t_1,t_2)$ is defined in (7).
The block-pulse coefficients, f_{i_1,i_2}, are obtained as

$$f_{i_1,i_2} = \frac{1}{h_1 h_2} \int_{(i_1-1)h_1}^{i_1 h_1} \int_{(i_2-1)h_2}^{i_2 h_2} f(t_1,t_2)\, dt_2 dt_1. \quad (14)$$

Similarly a function of four variables, $k(t_1,t_2,s_1,s_2)$, on $([0,T_1) \times [0,T_2) \times [0,T_3) \times [0,T_4))$ may be approximated with respect to BPFs such as:

$$k(t_1,t_2,s_1,s_2) \simeq \Phi^{\mathrm{T}}(t_1,t_2)K\Psi(s_1,s_2) \quad (15)$$

where $\Phi(t_1,t_2)$ and $\Psi(s_1,s_2)$ are 2D-BPF vectors of dimension $m_1 m_2$ and $m_3 m_4$ respectively, and K is the $(m_1 m_2) \times (m_3 m_4)$ 2D block-pulse coefficient matrix.

2.3 Operational matrix of integration

The integration of the vector $\Phi(t_1,t_2)$ defined in (7) can be approximately obtained as

$$\begin{aligned} \int_0^{t_1} \int_0^{t_2} \Phi(\tau_1,\tau_2)\, d\tau_1 d\tau_2 &\simeq P\,\Phi(t_1,t_2), \\ &= [E_{(m_1 \times m_1)} \otimes E_{(m_2 \times m_2)}]\Phi(t_1,t_2), \end{aligned} \quad (16)$$

where $t_1 \in [0,T_1), t_2 \in [0,T_2)$ and P is the $(m_1 m_2) \times (m_1 m_2)$ operational matrix of integration for 2D-BPFs where E is the operational matrix of 1D-BPFs defined over $[0,T)$ with $h = \frac{T}{m}$ and $T = T_1 = T_2$ as follows

$$E = \frac{h}{2} \begin{pmatrix} 1 & 2 & 2 & \ldots & 2 \\ 0 & 1 & 2 & \ldots & 2 \\ 0 & 0 & 1 & \ldots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}. \quad (17)$$

In (16), $\otimes$ denotes the Kronecker product defined as

$$A \otimes B = (a_{ij}B).$$

In the next sections, it is assumed that $T_1 = T_2 = 1$, so 2D-BPFs is defined over $[0,1) \times [0,1)$, and $h_1 = \frac{1}{m_1}, h_2 = \frac{1}{m_2}$.

3 Method of solution

In this section,we solve two-dimensional nonlinear Volterra-Hammerstein integral equations of the form in (1) using 2D block-pulse functions. For this purpose,we first assume

$$\mathcal{V}(t_1,t_2) = g(t_1,t_2,u(t_1,t_2)), \qquad (t_1,t_2) \in [0,1) \times [0,1). \quad (18)$$

Approximating functions $u(t_1,t_2), \mathcal{V}(t_1,t_2)$ and $f(t_1,t_2), K(t_1,t_2,s_1,s_2)$ with respect to 2D-BPFs by the way mentioned in section 2 gives

$$\begin{aligned} u(t_1,t_2) &= U^{\mathbf{T}}\Phi(t_1,t_2), \\ \mathcal{V}(t_1,t_2) &= \Phi^{\mathbf{T}}(t_1,t_2)\Lambda, \\ f(t_1,t_2) &= F^{\mathbf{T}}\Phi(t_1,t_2), \\ K(t_1,t_2,s_1,s_2) &= \Phi^{\mathbf{T}}(t_1,t_2)\Theta\Phi(s_1,s_2), \end{aligned} \quad (19)$$

where the vectors U, Λ, F, and matrix Θ are BPFs coefficients of $u(t_1,t_2)$, $\mathcal{V}(t_1,t_2)$, $f(t_1,t_2)$ and $K(t_1,t_2,s_1,s_2)$, respectively and $\Phi(t_1,t_2)$ is defined in (7). In (19), U and Λ are $(m_1 m_2 \times 1)$ unknown vectors.
To approximate the integral part in (1), from Eqs. (18) and (19) we get

$$\begin{aligned} &\int_0^{t_1} \int_0^{t_2} K(t_1,t_2,s_1,s_2)g(s_1,s_2,u(s_1,s_2))\, ds_1 ds_2 \\ &\simeq \int_0^{t_1} \int_0^{t_2} \Phi^{\mathbf{T}}(t_1,t_2)\Theta\Phi(s_1,s_2)\Phi^{\mathbf{T}}(s_1,s_2)\Lambda\, ds_1 ds_2, \\ &= \Phi^{\mathbf{T}}(t_1,t_2)\Theta\left(\int_0^{t_1} \int_0^{t_2} \Phi(s_1,s_2)\Phi^{\mathbf{T}}(s_1,s_2)\Lambda\right) ds_1 ds_2, \end{aligned}$$

Using Eq. (10) follows:

$$= \Phi^{\mathbf{T}}(t_1,t_2)\Theta \int_0^{t_1}\int_0^{t_2} \widetilde{\Lambda}\Phi(s_1,s_2)\, ds_1 ds_2,$$

$$= \Phi^{\mathbf{T}}(t_1,t_2)\Theta\widetilde{\Lambda} \int_0^{t_1}\int_0^{t_2} \Phi(s_1,s_2)\, ds_1 ds_2,$$

Using operational matrix P in Eq. (16) gives

$$\int_0^{t_1}\int_0^{t_2} K(t_1,t_2,s_1,s_2)g(s_1,s_2,u(s_1,s_2))\, ds_1 ds_2 \simeq \Phi^{\mathbf{T}}(t_1,t_2)\Theta\widetilde{\Lambda}P\Phi(t_1,t_2),$$

in which $\Theta\widetilde{\Lambda}P$ is an $(m_1m_2)\times(m_1m_2)$ matrix. Eq. (11) follows:

$$\int_0^{t_1}\int_0^{t_2} K(t_1,t_2,s_1,s_2)g(s_1,s_2,u)\, ds_1 ds_2 \simeq \widehat{\Lambda}^{\mathrm{T}}\Phi(t_1,t_2) \tag{20}$$

where $\widehat{\Lambda}$ is an (m_1m_2)-vector with components equal to the diagonal entries of matrix $\Theta\widetilde{\Lambda}P$.
Applying (18)-(20) in (1), we get

$$U^{\mathbf{T}}\Phi(t_1,t_2) - \widehat{\Lambda}^{\mathrm{T}}\Phi(t_1,t_2) \simeq F^{\mathbf{T}}\Phi(t_1,t_2) \tag{21}$$

Replacing $\simeq$ with $=$, Eq. (21) gives

$$U - \widehat{\Lambda} = F. \tag{22}$$

Equation (22) generates a set of m_1m_2 algebraic equations. Since the total number of unknowns for vectors U and Λ in (19) is $2(m_1m_2)$, we collocate the following equation in $(t_{1_i}, t_{2_i}) = (\frac{i-\frac{1}{2}}{m_1m_2}, \frac{i-\frac{1}{2}}{m_1m_2})$, $i = 1,2,\ldots,m_1m_2$,

$$g(t_1,t_2,U^{\mathbf{T}}\Phi(t_1,t_2)) = \Lambda^{\mathbf{T}}\Phi(t_1,t_2). \tag{23}$$

The resulting equations (22) and (23) generate a system of $2(m_1m_2)$ nonlinear equations which can be solved using Newton's iterative method.

4 Error analysis

In this section, we analyse the representation error (or the residual error) when a differentiable function $f(t_1,t_2)$ is represented in a series of 2D-BPFs over the region $D = [0,1)\times[0,1)$. For convenience, we put $m_1 = m_2 = m$, so $h_1 = h_2 = \frac{1}{m}$. We need the following theorem.

Theorem 1 *Suppose that f maps a convex open set $D \subset \mathcal{R}^2$ into $\mathcal{R}$, f is differentiable in D, and there is a real number M such that*

$$\|f'(\mathbf{t})\| \le M$$

for every $\mathbf{t} \in D$. Then

$$|f(\mathbf{b}) - f(\mathbf{a})| \le M|\mathbf{b} - \mathbf{a}|$$

for all $\mathbf{a}\in D, \mathbf{b}\in D$ (See [7]).

Now, we assume that $f(t_1,t_2)$ is a differentiable function on $D = [0,1)\times[0,1)$ such that

$$\|f'(t_1,t_2)\| \le M.$$

We define the representation error between $f(t_1,t_2)$ and its 2D BPFs expansion, $\overline{f}(t_1,t_2)$, over every subregion D_{i_1,i_2} as follows:

$$\begin{aligned} e_{i_1,i_2}(t_1,t_2) &= f_{i_1,i_2}\phi_{i_1,i_2}(t_1,t_2) - f(t_1,t_2), \\ &= f_{i_1,i_2} - f(t_1,t_2), \qquad (t_1,t_2)\in D_{i_1,i_2}, \end{aligned}$$

where

$$D_{i_1,i_2} = \left\{(t_1,t_2) : \frac{i_1-1}{m} \le t_1 < \frac{i_1}{m},\ \frac{i_2-1}{m} \le t_2 < \frac{i_2}{m}\right\}.$$

It can be shown that

$$\begin{aligned} \|e_{i_1,i_2}\|^2 &= \int_{\frac{i_1-1}{m}}^{\frac{i_1}{m}}\int_{\frac{i_2-1}{m}}^{\frac{i_2}{m}} e^2_{i_1,i_2}(t_1,t_2)\, dt_2 dt_1, \\ &= \int_{\frac{i_1-1}{m}}^{\frac{i_1}{m}}\int_{\frac{i_2-1}{m}}^{\frac{i_2}{m}} (f_{i_1,i_2} - f(t_1,t_2))^2 dt_2 dt_1, \\ &= \frac{(f_{i_1,i_2} - f(\eta_1,\eta_2))^2}{m^2},\ (\eta_1,\eta_2)\in D_{i_1,i_2} \end{aligned} \tag{24}$$

where we used mean value theorem for 2D integrals. Using Eq. (14) and the mean value theorem we have

$$\begin{aligned} f_{i_1,i_2} &= m^2 \int_{\frac{i_1-1}{m}}^{\frac{i_1}{m}}\int_{\frac{i_2-1}{m}}^{\frac{i_2}{m}} f(t_1,t_2)\, dt_2 dt_1, \\ &= m^2.\frac{1}{m^2} f(\xi_1,\xi_2), \\ &= f(\xi_1,\xi_2), \qquad (\xi_1,\xi_2)\in D_{i_1,i_2}. \end{aligned} \tag{25}$$

Substituting (25) into (24) and using theorem 1 we obtain:

$$\begin{aligned} \|e_{i_1,i_2}\|^2 &= \frac{1}{m^2}(f(\xi_1,\xi_2) - f(\eta_1,\eta_2))^2, \\ &\le \frac{1}{m^2}((\xi_1,\xi_2) - (\eta_1,\eta_2))^2, \\ &\le \frac{2}{m^4}M^2. \end{aligned} \tag{26}$$

This leads to

$$\begin{aligned} \|e(t_1,t_2)\|^2 &= \int_0^1\int_0^1 e^2(t_1,t_2)\, dt_1 dt_2, \\ &= \int_0^1\int_0^1 \Big(\sum_{i_1=1}^{m}\sum_{i_2=1}^{m} e_{i_1,i_2}(t_1,t_2)\Big)^2 dt_1 dt_2, \\ &= \int_0^1\int_0^1 \sum_{i_1=1}^{m}\sum_{i_2=1}^{m} e^2_{i_1,i_2}(t_1,t_2)\, dt_1 dt_2 + \\ &+2\sum_{i_1<j_1}\sum_{i_2<j_2}\int_0^1\int_0^1 e_{i_1,i_2}(t_1,t_2)\, e_{j_1,j_2}(t_1,t_2)\, dt_1 dt_2. \end{aligned}$$

Then

$$\begin{aligned} \|e(t_1,t_2)\|^2 &= \sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\int_0^1\int_0^1 e^2_{i_1,i_2}(t_1,t_2)\, dt_1 dt_2, \\ &= \sum_{i_1=1}^{m}\sum_{i_2=1}^{m} \|e_{i_1,i_2}\|^2, \\ &\le m^2\frac{2}{m^4}M^2, \end{aligned} \tag{27}$$

Table 1: Absolute values of error for Example 1.

	Errors with		
$(t_1,t_2)=2^{-i}$	$m=16$	$m=32$	$m=64$
$i=1$	$4.4D{-}1$	$2.1D{-}1$	$1.1D{-}1$
$i=2$	$2.1D{-}1$	$1.0D{-}1$	$6.4D{-}3$
$i=3$	$1.4D{-}1$	$7.0D{-}2$	$2.5D{-}2$
$i=4$	$1.2D{-}1$	$5.8D{-}2$	$3.5D{-}2$
$i=5$	$8.7D{-}4$	$5.3D{-}2$	$3.5D{-}2$
$i=6$	$5.1D{-}2$	$2.1D{-}4$	$4.3D{-}4$

Table 2: Absolute values of error for Example 2.

	Errors with		
$(t_1,t_2)=2^{-i}$	$m=8$	$m=16$	$m=32$
$i=1$	$1.3D{-}1$	$2.1D{-}2$	$1.1D{-}2$
$i=2$	$7.3D{-}2$	$1.0D{-}2$	$6.4D{-}4$
$i=3$	$4.1D{-}2$	$7.0D{-}3$	$2.5D{-}3$
$i=4$	$2.6D{-}3$	$5.8D{-}3$	$3.5D{-}3$
$i=5$	$8.4D{-}3$	$5.3D{-}4$	$3.5D{-}4$
$i=6$	$9.9D{-}3$	$2.1D{-}3$	$4.3D{-}5$

hence, $\|e(t_1,t_2)\| = \mathcal{O}(\frac{1}{m})$, where

$$e(t_1,t_2) = \overline{f}(t_1,t_2) - f(t_1,t_2).$$

5 Numerical examples

In this section, we applied the method presented in this paper for solving some test problems that are selected from different references. The numerical experiments are carried out for the selected grid points which are proposed as $(2^{-i}, i = 1,2,3,4,5,6)$ and m^2 terms of the 2D-BPFs series.

Example 1. Consider the linear two-dimensional integral equation

$$u(t_1,t_2) = e^{3t_1+2t_2} + e^{t_1+t_2} - e^{3t_1+t_2} - \int_0^{t_1}\int_0^{t_2} 2e^{t_1+t_2} u(s_1,s_2)\, ds_1 ds_2,$$

where $(t_1,t_2) \in [0,1) \times [0,1)$.
Exact solution of this problem is $u(t_1,t_2) = e^{t_1+2t_2}$. Table 1 presents the absolute errors for the selected grid points using the present method. As shown in table 1, the computational error decreases as the number of 2D-BPFs increases.

Example 2. As the second example, consider the following nonlinear two-dimensional integral equation

$$u(t_1,t_2) = f(t_1,t_2) + \int_0^{t_1}\int_0^{t_2} u^2(s_1,s_2)\, ds_1 ds_2,$$

$$(t_1,t_2) \in [0,1) \times [0,1)$$

with

$$f(t_1,t_2) = t_1^2 + t_2^2 - \frac{1}{45} t_1 t_2 (9t_1^4 + 10 t_1^2 t_2^2 + 9 t_2^4),$$

and the exact solution $u(t_1,t_2) = t_1^2 + t_2^2$. Table 2 illustrates the numerical results for Example 2.

6 Conclusion

Two-dimensional integral equations are usually difficult to solve analytically. In many cases, it is required to obtain the approximate solutions, for this purpose the presented method can be proposed. In this paper, the method based on 2D-BPFs and its operational matrix has been used for the approximate solution of 2D integral equations. This approach transformed a nonlinear 2D Volterra integral equation to a matrix equation which corresponds to a system of nonlinear equations with unknown coefficients. Finally, by using this system, we find the approximate solution of the 2D integral equations. This method can be easily extended and applied to 2D Volterra integral equations of the first kind and Fredholm 2D integral equations of the first and second kind.

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