Dynamics of Nonlinear Beam on Elastic Foundation

Salih N Akour[1]

Abstract —Simply supported nonlinear beam resting on linear elastic foundation and subjected to harmonic loading is investigated. Parametric study is carried out in the view of the linear model of the problem. Hamilton's principle is utilized in deriving the governing equations. Well known forced duffing oscillator equation is obtained. The equation is analyzed numerically using Runk-Kutta technique. Three main parameters are investigated: the damping coefficient, the natural frequency, and the coefficient of the nonlinear term. Stability regions are unveiled.

***Index Terms*— Elastic Foundation, Nonlinear Beam, Parametric Study.**

I. Introduction

There are many applications for beam on elastic foundation mainly in mechanical and civil engineering e.g. disc brake pad, shafts supported on ball, roller, or journal bearings, vibrating machines on elastic foundations, network of beams in the construction of floor systems for ships, buildings, and bridges, submerged floating tunnels, buried pipelines, railroad tracks etc. The elastic foundation for the beam part is supplied by the resilience of the adjoining portions of a continuous elastic structure. More details of the applications of this concept are discussed by Hetenyi [1]. Beams on elastic foundations received great attention of researches due to its wide applications in engineering. Hetenyi [1] and Timoshenko [2] presented an analytical solution for beams on elastic supports using classical differential equation approach, and considering several loading and boundary conditions.

It is well known in engineering that a beam supported by discrete elastic supports spaced at equal intervals acts analogously to a beam on an elastic foundation and that the appropriateness of that analogy depends on the flexural rigidity of the beam as well as the stiffness and spacing of the supports. Ellington investigated conditions under which a beam on discrete elastic supports could be treated as equivalent to a beam on elastic foundation [3].

Beams resting on elastic foundations have been studied extensively over the years due to the wide application of this system in engineering. This system according to the literature can be divided at least into three categories. The first category is "linear beam on linear elastic foundation". Example of this type can be found in references [4]-[18]. The applications in this category include but not limited to Euler - Bernoulli beam, Timoshenko beam, Winkler foundation, Pasternak foundation, tensionless foundation, single parameter or two parameter foundation, static loading, harmonic loading and moving loading.

The second category is "linear beam on nonlinear elastic foundation" [18]-[24]. In this category the foundation is considered to have nonlinear stiffness. Also this type includes different boundary and loading conditions according to the engineering application.

The third category is nonlinear beam on linear elastic foundation [25]-[37]. Usually the beam nonlinearity means large deflections. Most of the studies related to this category have analyzed the system either using boundary element method or boundary integral equation method. Similar to the above two categories, there is wide variety of boundary and loading conditions being applied to such system according to the application.

Nonlinear beam subjected to harmonic distributed load resting on linear elastic foundation is investigated in this research. The study is carried out in the view of the linearized model of the system. Well known duffing equation is obtained using Hamilton's principle. Three main parameters are investigated: the damping coefficient, the natural frequency, and the coefficient of the nonlinear term. The effect of these parameters on the system stability is unveiled. Up to the author's knowledge, this work is not published in the literature.

II. Problem Statement

Nonlinear beam resting on elastic foundation that is shown in Fig. 1 is subjected to the following conditions:

1. The beam material properties are linear.
2. The damping (μ) and stiffness (k_f) of the foundation are linear.
3. The beam is slender and prismatic.
4. The beam is simply supported (pin-pin ends)
5. The load applied is harmonic and distributed over the length of the beam.

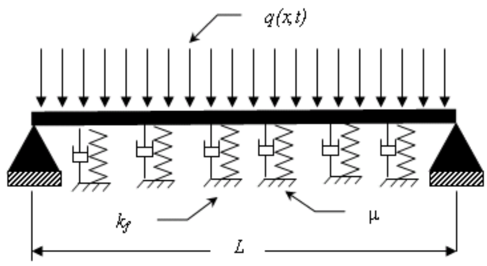

Fig. 1: schematic drawing of the beam on elastic foundation.

III. Mathematical Formulation

A. Kinetic Energy

The rotary inertia of the beam will be neglected since the beam is slender.

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ISBN: 978-988-18210-7-2
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$$T = \frac{1}{2}\int_0^L \iint_A \left(\frac{\partial w}{\partial t}\right)^2 dydzdx = \frac{\rho A}{2}\int_0^L \left(\frac{\partial w}{\partial t}\right)^2 dx \qquad (1)$$

Where ρ: material density, *A*: beam cross sectional area, *L*: beam length, *w=w(x,t):* beam transverse displacement (in y-direction).

B. Potential Energy:

The potential energy due to bending can be calculated as the following:

$$U_{bend} = \frac{1}{2}\int_0^L \iint_A E\left(-z\frac{\partial^2 w}{\partial x^2}\right)^2 dydzdx = \frac{EI}{2}\int_0^L \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx \qquad (2)$$

Where

$$I = \iint_A z^2 dydz$$

The formulation of the due to stretching potential energy can be casted as the following [38]:

$$U_{stretch} = \frac{EA}{2L}\left(\frac{1}{2}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx\right)^2 \qquad (3)$$

The elastic foundation is assumed to have constant linear spring modulus. This results in,

$$U_{foundation} = \frac{1}{2}\int_0^L k_f w^2\, dx \qquad (4)$$

The load is uniform along the length of the beam and varies harmonically with respect to time. Therefore,

$$U_{load} = -\int_0^L q(x,t).w(x,t)\, dx = -\int_0^L P.sin(\omega_e.t).w(x,t)\, dx \qquad (5)$$

Where *P*: amplitude of excitation and ω_e: excitation frequency

C. Derivation of governing equation

The lagrangian is defined as the following:

$$L = T - U_{bend} - U_{strech} - U_{foundation} - U_{load}$$

$$L = \frac{1}{2}\int_0^L \left(\rho A\dot{w}^2 - EI(w'')^2 - k_f w^2 + 2\,P\,w\sin(\omega_e t)\right) dx - \frac{AE}{2L}\left[\frac{1}{2}\int_0^L (w')^2 dx\right]^2$$

By applying Hamilton's principle

$$\delta^{(1)}\left(\int_{t_1}^{t_2} L dt\right) = 0$$

$$\delta^{(1)}\left(\int_{t_1}^{t_2}\int_0^L \frac{1}{2}\left(\rho A\dot{w}^2 - EI(w'')^2 - k_f w^2 + 2\,P\,w\sin(\omega_e t)\right) dxdt - \int_{t_1}^{t_2}\frac{AE}{2L}\left[\frac{1}{2}\int_0^L (w')^2 dx\right]^2 dt\right) = 0$$

Denote the first and the second integral by F_1 and F_2 respectively. This gives

$$\int_{t_1}^{t_2}\int_0^L \left(\frac{\partial F_1}{\partial \dot{w}}\delta\dot{w} + \frac{\partial F_1}{\partial w''}\delta w'' + \frac{\partial F_1}{\partial w}\delta w\right) dxdt - \delta^{(1)}F_2 = 0$$

Integrating the first and the second term by parts with respect to *x* the result is the following equation:

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_0^L \Bigg[-\frac{\partial^2 F_1}{\partial t\,\partial \dot{w}}\left(\frac{\rho A}{2}\dot{w}^2\right) - \frac{\partial^2}{\partial x^2}\frac{\partial}{\partial w''}\left(\frac{EI}{2}(w'')^2\right) + \frac{\partial}{\partial w}\left(\frac{-k_f}{2}w^2 + P\,w\sin(\omega_e t)\right) + \frac{\partial}{\partial x}\left(\frac{AE}{2L}(w')^3\right)\Bigg]\delta w dxdt \\
&+ \int_0^L \left[\frac{\partial}{\partial \dot{w}}\left(\frac{\rho A}{2}\dot{w}^2\right)\delta w\right]_{t_1}^{t_2} dx - \int_{t_1}^{t_2}\left[\frac{\partial}{\partial w''}\left(\frac{EI}{2}(w'')^2\right)\delta w'\right]_0^L dt \\
&+ \int_{t_1}^{t_2}\left[\frac{\partial}{\partial x}\left(\frac{\partial}{\partial w''}\left(\frac{EI}{2}(w'')^2\right)\right)\delta w\right]_0^L dt - \int_{t_1}^{t_2}\left(\frac{AE}{2L}(w')^3\delta w\right)_0^L dt = 0
\end{aligned} \qquad (6)$$

Since δw is arbitrary, the following can be concluded from the above equation:
The governing equation comes from setting the expression within the brackets in (6) equal to zero. Upon carrying out the indicated differentiations, the governing can be rewritten as

$$\ddot{w} + \alpha w^{iv} + kw - \beta\frac{\partial (w')^3}{\partial x} = p\sin(\omega_e\, t) \qquad (7)$$

where $\alpha = \frac{EI}{\rho A}$, $k = \frac{k_f}{\rho A}$ *and* $\beta = \frac{E}{2\rho L}$

It is obvious that (7) is the duffing oscillator equation. This equation is going to be recasted into a more familiar form in the next section. The boundary and initial conditions can be obtained from the remaining terms in (6).
The boundary conditions at *x=0* and *x=L* are

Either EIw'' is zero or w' is prescribed (8a)

Either EIw'' is zero or w is prescribed (8b)

Either $\frac{EA}{2L}(w')^3$ is zero or w is prescribed (8c)

Boundary conditions (8a) correspond to end moments and slopes respectively. In (8b), *w* corresponds to end displacement, and in (8c) the first condition corresponds to pre-stretching. For the pinned ends, the boundary conditions are:

$$w(0) = w(L) = 0$$
$$EIw''(0) = EIw''(L) = 0$$

These boundary conditions must be satisfied by the mode shapes of the system. This fact will be used in the following sections as the criteria for selecting the form of the mode shape equation.

Finally the initial conditions for $t = t_1$ and $t = t_2$ are

Either $\frac{\rho A}{2}(\dot{w})^2$ is zero or w is prescribed

In this case, it will be assumed that the system starts from rest i.e. the initial displacement and velocity is zero.

D. Discretization and linearization

The following expression is used for *w(x, t)* in order to discretize the problem

$$w(x,t) = \sum_{n=1}^{N} w_n(t) sin\left(\frac{n\pi x}{L}\right) \tag{9}$$

For simplicity the limits of the above summation, the subscript of *w*, and the time dependence of *w* will be implied in the equations that follow. It is evident from (9) that the pinned ends boundary condition (8a) are satisfied since transverse displacements at 0 and L are zero, and the end slopes are free (implying zero bending moments at the ends). Equation (9) represents series summation of N modes each has time dependent amplitude response, $w_n(t)$ with spatial sine function. Substituting (9) into the original integral expressions for the kinetic and potential energy of (1) through (5) then applying the Lagrangian and utilizing the orthogonality, the following equation comes out:

$$L = T - U = \frac{\rho A}{4}\sum \dot{w}^2 - \frac{EI\pi^4}{4L^3}\sum n^4 w^2 - \frac{\pi^4 AE}{32L^3}\left[\sum n^2 w^2\right]^2 - \frac{k_f L}{4}\sum w^2 \tag{10}$$

Lagrangian's equation for each mode can be written as the following:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{w}_n}\right) - \frac{\partial L}{\partial w_n} = 0 \; for \; n = 1,2,\ldots\ldots, N \tag{11}$$

Substituting (10) in (11) and carry out the differentiation yields,

$$\frac{\rho AL}{2}\ddot{w}_n + \frac{EI\pi^4 n^4}{2L^3}w_n + \frac{k_f L}{2}w_n + \frac{EA\pi^4 n^2}{8L^3}\left(\sum_{m=1}^{N} m^2 w_m^2\right) w_n = 0 \tag{12}$$

A simplified form of (12) results after rearranging the coefficients and defining some new coefficients. The concise form and the coefficient definitions are

$$\ddot{w}_n + \omega_0^2 n^4\left[1 + \frac{\zeta^2}{n^4} + \frac{1}{4\eta^2 n^2}\sum_{m=1}^{N} m^2 w_m^2\right] w_n = 0 \tag{13}$$

Where

$$\omega_0^2 = \frac{EI\pi^4}{\rho AL^4} \qquad \zeta^2 = \frac{\omega_f^2}{\omega_0^2} \qquad \omega_f^2 = \frac{k_f}{\rho A} \qquad \eta = \sqrt{\frac{I}{A}}$$

Writing (13) for a single mode and inserting the linear damping term gives,

$$\ddot{w}_n + + 2\mu\dot{w}_n + \omega_0^2 n^4\left[1 + \frac{\zeta^2}{n^4}\right] w_n + + \frac{1}{4\eta^2} w_n^3 = 0 \tag{14}$$

Where μ is the damping coefficient

This makes it clear that the above equation represent unforced damped duffing oscillator. Recasting (14) into the following:

$$\ddot{w} + + 2\mu\dot{w} + \omega^2 w + \alpha w^3 = 0 \tag{15}$$

Where $\omega^2 = \omega_0^2 n^4\left[1 + \frac{\zeta^2}{n^4}\right] \qquad \alpha = \frac{1}{4\eta^2}$

In order to linearize the system for the first mode (*n=1*) the system is converted into first order ordinary differential equations by the following substitution

$$x = w \rightarrow \dot{x} = \dot{w}$$
$$y = \dot{w} \rightarrow \dot{y} = \ddot{w}$$

Applying this to (15)

$$\dot{x} = y, \qquad \dot{y} = -2\mu y - \omega^2 x - \alpha x^3$$
$$y = 0, \qquad -2\mu y - \omega^2 x - \alpha x^3 = 0$$

From the above equations it is obvious that (0, 0) is the only critical point for the system. So the equivalent linear system is obtained by expanding the above equation using Taylor series about (0, 0), so the remaining linear terms are

$$\dot{x} = y, \qquad \dot{y} = -2\mu y - \omega^2 x$$

The corresponding Jacobi matrix is

$$[J] = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\mu \end{bmatrix}$$

So the Eigenvalues of J are

$$\lambda_{1,2} = -\mu \pm \sqrt{\mu^2 - \omega^2}$$
$$\lambda^2 - p\lambda + q = 0, \qquad where\; p = -2\mu \quad and \quad q = \omega^2$$

$$\Delta = \mu^2 - \omega^2$$

The following can be said about (0, 0):

- Stable and attractive as long as $\mu > 0$.
- Stable if $\mu \geq 0$.
- Unstable if $\mu < 0$.
- A node as long as $\Delta \geq 0$
- A spiral point if $\Delta < 0$
- A center if $\mu = 0$

The general solution of the linearized unforced system is

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$
$$x(t) = e^{-\mu t}\left(C_1 e^{-\sqrt{\Delta}t} + C_2 e^{\sqrt{\Delta}t}\right)$$

Applying the initial conditions $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$ the constants of integration are going to be as the following:

$$C_1 = \frac{\left(\sqrt{\Delta} - \mu\right) x_0 - \dot{x}_0}{2\sqrt{\Delta}}$$
$$C_2 = \frac{\left(\sqrt{\Delta} + \mu\right) x_0 + \dot{x}_0}{2\sqrt{\Delta}}$$

E. Simulation of the nonlinear system

$$\ddot{w} + 2\mu\dot{w} + \omega^2 w + \alpha w^3 = P \sin(\omega_e t) \tag{16}$$

where

$$\omega^2 = \omega_0^2 n^4\left[1 + \frac{\zeta^2}{n^4}\right] \qquad \zeta^2 = \frac{\omega_f^2}{\omega_0^2}, \qquad \alpha = \frac{1}{4\eta^2}$$
$$\omega_0^2 = \frac{EI\pi^4}{\rho AL^4}, \qquad n = 1, \qquad \omega_f^2 = \frac{k_f}{\rho A},$$
$$\eta = \sqrt{\frac{I}{A}}$$

It is obvious that the strength of the nonlinearity is inversely proportional to the square of the radius of gyration of the beam. This indicates that the nonlinearity remains weak as long as the beam is relatively slender as assumed in this study. Finally, the frequency equation can be simplified to

$$\omega^2 = \omega_0^2 + \omega_f^2$$

ISBN: 978-988-18210-7-2
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

The apparent natural frequency of the system ω is the square root of the sum of the squares of the natural frequencies of the beam and the elastic foundation.
The nonlinear second order ordinary differential equation is converted into a system of first order ordinary differential equations. This is suitable for numerical study using Runge-Kutta Techniques.

$$\dot{w} = y$$
$$\dot{y} = \ddot{w} = P \sin(\omega_e t) - 2\mu\dot{w} - \omega^2 w - \alpha w^3$$

IV. Results and discussion

The results for simply supported beam on elastic foundation are presented in Tables I and II. Table I represents sample phase diagrams of the studied ranges. Table II shows the time response for toward chaos cases that are presented in Table I. The phase diagram and the time response are collected after long period of time to be sure that the system has passed the transient range. The duffing (16) is solved using MATLAB package by utilizing the Runga-Kutta ODE (Ordinary Differential Equation) solver. The equation which represents the system under investigation is of cubic nonlinearity with harmonic excitation.
The sample results present the effect of the damping when the system has weak, medium and strong nonlinearity for excitation frequencies below, at and above resonance. The whole study is considering weak nonlinearity that does not exceed α=0.1 and those levels of weak, medium and strong within that range. Only the first mode is considered in this study. The parameters range covered in this investigation are for μ=0.0 through 0.1, α= 0.001 through 0.1 and natural frequency ω= 0.7 through 1.4. It can be seen from Table I that when there is no damping the system is tending toward chaos however when little damping is applied the system is tending towards limit cycle.
It is obvious that the damping and the nonlinearity are the most effective parameters in controlling the chaotic behavior of the system. As long as the radius of gyration for the beam under consideration is large i.e., the beam is more towards slender, the nonlinearity is going to be weak. This means that the contribution of the stretching energy to the behavior of the system is going to be low. The damping system dissipates the oscillating energy and provides a control over the system behavior. For the linear system, as long as damping coefficient is positive, the transient response is going to decrease exponentially and the homogenous response (due to the forcing excitation) is bounded even at resonance. For undamped linear system homogenous response is not bounded and the response is increasing with time. For the nonlinear system, the response is tending toward chaos as it can be seen in table 1 for μ=0.0. However when the damping increases the system is transferring from chaos to limit cycle. For the resonance case of no damping i.e., the excitation frequency equals the natural frequency, the linear system has increasing amplitude response whereas the nonlinear system is tending toward chaos with bounded amplitude. Also for the nonlinear case (within the range of investigation) as long as there is no damping the system is tending toward chaos.

V. Conclusion

The behavior of nonlinear beam on elastic foundation is unveiled. It is found that the system is stable and controllable as long as the damping coefficient is non zero and positive. As the nonlinearity increases more damping is required to prevent it from moving towards chaos. For first mode shape the natural frequency could be calculated as square root of the sum of squares of both natural frequency of the beam and the foundation. The strength of the nonlinearity is inversely proportional to the square of the radius of gyration, i.e. as long as the beam more towards slender the nonlinearity is weaker. The stretching potential energy is responsible for generating the cubic nonlinearity in the system.

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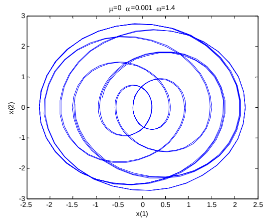
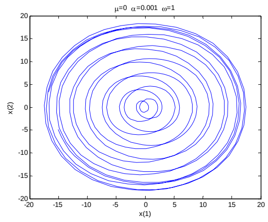
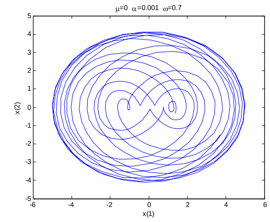
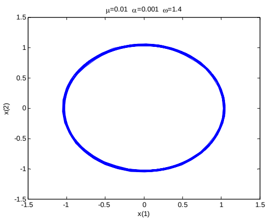
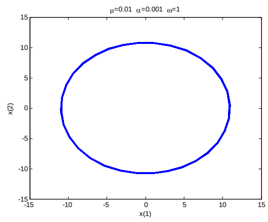
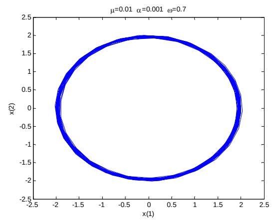
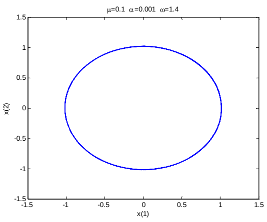
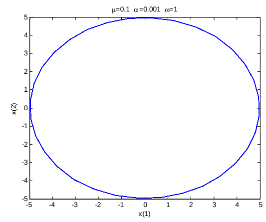
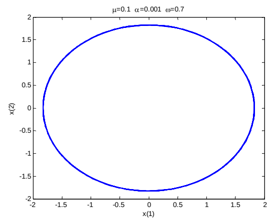
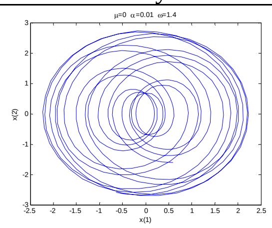
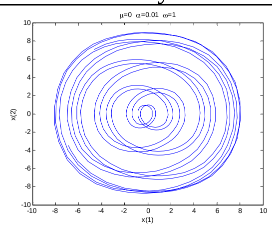
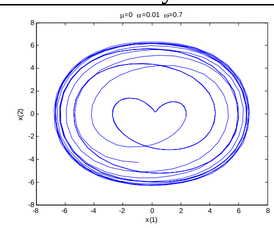
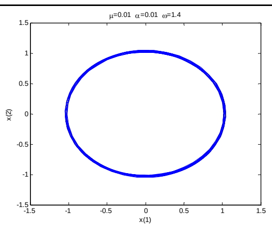
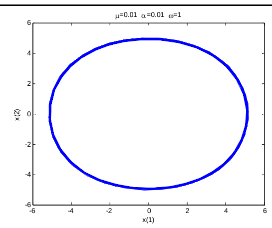
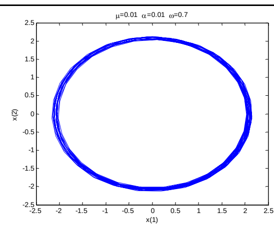
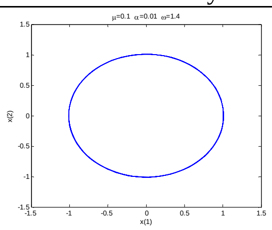
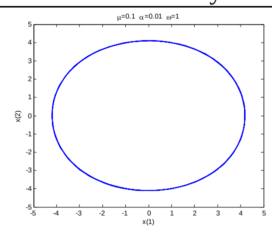
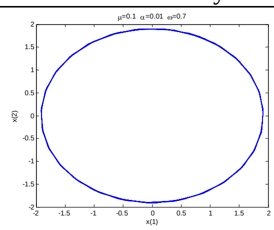
ISBN: 978-988-18210-7-2
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Table I: Presentation of sample phase trajectories for the parameters under investigation.

	$\acute{\omega} = 1.4\ \ \acute{\omega}_e = 1$	$\acute{\omega} = 1.0\ \ \acute{\omega}_e = 1.0$	$\acute{\omega} = 0.7\ \ \acute{\omega}_e = 1.0$
μ= 0.00 α=0.001 P=1			
	Toward Chaios	Toward Chaios	Toward Chaios
μ=0.01 α=0.001 P=1			
	Towards Limit Cycle	Towards Limit Cycle	Towards Limit Cycle
μ=0.1 α=0.001 P=1			
	Limit cycle	Limit cycle	Limit cycle
μ=0.0 α=0.01 P=1			
	Toward Chaios	Toward Chaios	Toward Chaios
μ=0.01 α=0.01 P=1			
	Towards Limit Cycle	Towards Limit Cycle	Towards Limit Cycle
μ=0.1 α=0.01 P=1			
	Limit cycle	Limit cycle	Limit cycle

ISBN: 978-988-18210-7-2
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

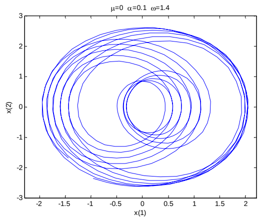
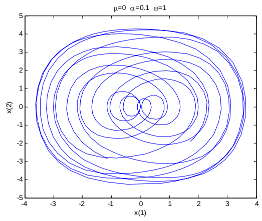
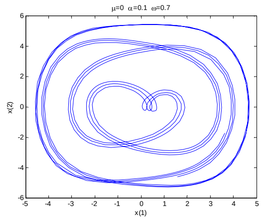
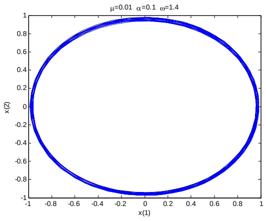
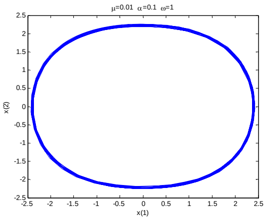
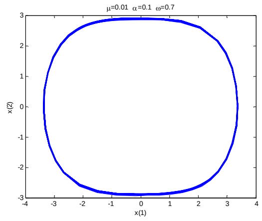
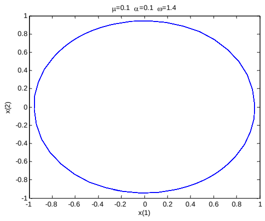
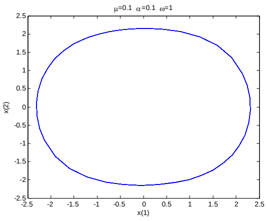
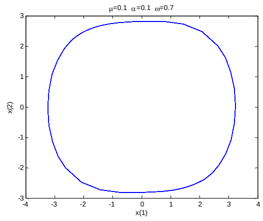
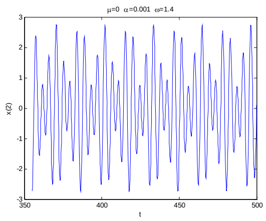
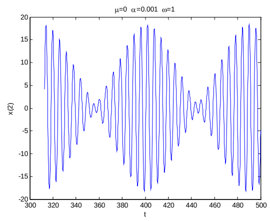
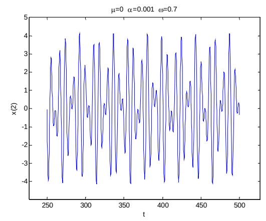
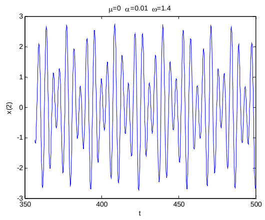
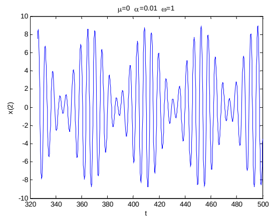
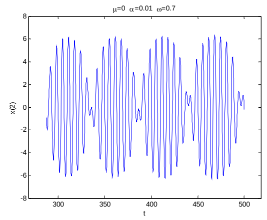
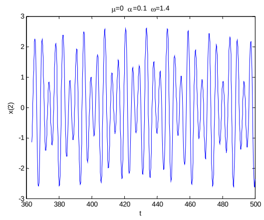
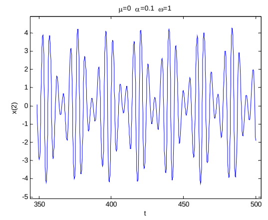
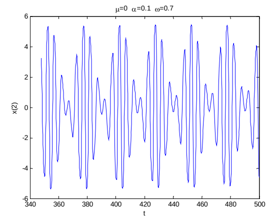
μ=0.00 **α**=0.1 P=1			
	Toward Chaios	Toward Chaios	Toward Chaios
μ=0.01 **α**=0.1 P=1			
	Towards Limit Cycle	Towards Limit Cycle	Towards Limit Cycle
μ=0.1 **α**=0.1 P=1			
	Limit cycle	Limit cycle	Limit cycle

Table II: Illustration of time response of the chaotic cases.

	$\acute{\omega} = 1.4 \ \ \acute{\omega}_e = 1$	$\acute{\omega} = 1.0 \ \acute{\omega}_e = 1.0$	$\acute{\omega} = 0.7 \ \ \acute{\omega}_e = 1.0$
μ= 0.00 **α**=0.001 P=1			
μ=0.0 **α**=0.01 P=1			
μ=0.00 **α**=0.1 P=1			

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