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Analytical Solution of Pressurized Rotating Composite Disk under Thermal Loading

Mohammadi F. [1,*], Hadadian A.[2] , Singh Grewal J. [3]

***Abstract*— In current work, a composite rotating disk under internal and external pressure subjected to temperature distribution is analyzed. The modified Tsai-Wu FC is used in this study. It has been shown that FC is strongly depends on temperature distribution. Polynomial function is selected for thermal distribution. For a limited range of the temperature the coefficient of the polynomial has a great effect on failure criterion. In a case study, the best coefficient of the function is derived somehow to make the failure criterion as uniform as possible.**

***Index Terms*— Rotating disk; Temperature distribution; Failure criterion; Thermoplastic composite.**

I. INTRODUCTION

Composite materials are implemented where high stress and low weight are required. Rotating disk has many applications in industry. Using composite material in rotating disk leads to increase of the specific strength. There have been some studies dealing with thermal stresses in the basic structural components of FGMs. Kolakowski [1] has presented a modification of the Tsai-Wu criterion, needed in the case of the multi-criterion optimal design of thin-walled composite structure and a proposal of the evaluation of the load carrying capacity of multi-layered composites with respect to their failure mode. Asghari and Ghafoori [2] have presented a semi-analytical three-dimensional elasticity solution for rotating functionally graded disks. Their solution includes the responses of both of the hollow and solid disks and is a generalization of the two-dimensional plane-stress solution. Vivio and Vullo [3,4], have introduced an analytical procedure for evaluation of elastic stresses and strains in rotating conical disks and in non-linear variable thickness rotating disks, either solid or annular, subjected to thermal load, and having a fictitious density variation along the radius. Hasan Çallioglu [5] has examined the stress analysis on orthotropic rotating annular disks subjected to various temperature distributions, such as uniform, linearly increasing and decreasing with radius temperatures. Hosseini Kordkheili and Naghdabadi [6] have presented a semi-analytical thermoelasticity solution for hollow and solid rotating axisymmetric disks made of functionally graded materials and investigated on the effects of the radial gradation of constitutive components on stress, strain and displacement components of the functionally graded disk for both centrifugal force and uniform thermal loadings. Bayat [7] have studied a rotating functionally graded disk with variable thickness under a steady temperature field [8] and have also conducted Elastic solutions for axis-symmetric rotating disks made of functionally graded material with variable thickness.

In this study analytical thermo elastic solution is developed to find stress distribution and numerical procedure is used to increase uniformity in the rotating disk. A polynomial function has been chosen for temperature distribution and through numerical experiments the optimum ratio of polynomial coefficients are achieved.

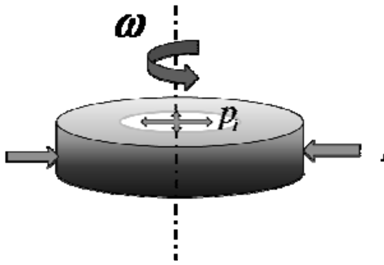

Fig. 1. Pressurized rotating disk under thermal loading

II. FORMULATION OF THE PROBLEM

For the plane stress problem, the constitutive Equations in polar coordinate system is given as

$$\varepsilon_r = \frac{1}{E_r}\sigma_r - \frac{\upsilon_{\theta r}}{E_\theta}\sigma_\theta + \alpha_1 T \tag{1}$$

$$\varepsilon_\theta = -\frac{\upsilon_{r\theta}}{E_r}\sigma_r + \frac{1}{E_\theta}\sigma_\theta + \alpha_2 T \tag{2}$$

In which E_r, σ_r, ε_r and α_1 are respectively modulus of elasticity, stress, strain and thermal coefficient in radial direction and E_θ ,σ_θ, ε_θ and α_2 are respectively modulus of elasticity, stress, strain and thermal coefficient in tangential direction. $\nu_{r\theta}$ and $\nu_{\theta r}$ are Poisson's ratios for orthotropic materials. T is temperature distribution. As, all the loads including thermal, body force and pressure inside as well as outside are function of radius only, the problem can be considered as axis-symmetric problem which means there is no variation in circumferential direction.

Strain-displacement relations in axis-symmetric polar coordinate can be written as follows

$$\gamma_{r\theta} = 0 \tag{3}$$

$$\varepsilon_r = \frac{\partial u}{\partial r} \tag{4}$$

$$\varepsilon_\theta = \frac{u}{r} \tag{5}$$

Mohammadi F, Department of Mechanical and Industrial Engineering, Concordia University, Montreal, Quebec, CANADA (e-mail: far_moh@encs.concordia.ca)
Hadadian A. Department of Mechanical and Industrial Engineering, Concordia University, Montreal, Quebec, CANADA (e-mail: a_hadad@encs.concordia.ca).
Singh Grewal J. Department of Mechanical and Industrial Engineering, Concordia University, Montreal, Quebec, CANADA (e-mail: jasrobin@gmail.com).

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In which u is displacement in radial direction. $\gamma_{r\theta}$ is shear strain that is equal zero in axiymmetric problems. Therefore, according to the above equation, the compatibility equation becomes

$$\varepsilon_r = \frac{d}{dr}(r\varepsilon_\theta) \tag{6}$$

Equilibrium equation in polar coordinate for plane stress in r direction is

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + \rho\omega^2 r = 0 \tag{7}$$

In which ω is angular velocity of disk. One may write Eq. (1) and (2) as following

$$\varepsilon_r - \alpha_1 T = \frac{1}{E_r}\sigma_r - \frac{\upsilon_{\theta r}}{E_\theta}\sigma_\theta \tag{8}$$

$$\varepsilon_\theta - \alpha_2 T = -\frac{\upsilon_{\theta r}}{E_r}\sigma_r + \frac{1}{E_\theta}\sigma_\theta \tag{9}$$

Solving Eq. (8) and (9) for σ_r and σ_θ gives

$$\sigma_r = C_1\varepsilon_r + C_2\varepsilon_\theta + C_3 T \tag{10}$$

$$\sigma_\theta = C_4\varepsilon_r + C_5\varepsilon_\theta + C_6 T \tag{11}$$

where

$$C_1 = \frac{E_r}{1 - v_{\theta r}v_{r\theta}}$$

$$C_2 = \frac{v_{\theta r}E_r}{1 - v_{\theta r}v_{r\theta}}$$

$$C_3 = -\frac{E_r(\alpha_1 + \alpha_2 v_{\theta r})}{1 - v_{\theta r}v_{r\theta}}$$

$$C_4 = \frac{v_{r\theta}E_\theta}{1 - v_{\theta r}v_{r\theta}}$$

$$C_5 = \frac{E_\theta}{1 - v_{\theta r}v_{r\theta}}$$

$$C_6 = -\frac{E_\theta(\alpha_2 + \alpha_2 v_{r\theta})}{1 - v_{\theta r}v_{r\theta}}$$

Substituting Eq. (10) and (11) into Equation (7) gives

$$C_1\frac{\partial\varepsilon_r}{\partial r} + C_2\frac{\partial\varepsilon_\theta}{\partial r} + C_3\frac{\partial T}{\partial r} + \frac{(C_1 - C_4)\varepsilon_r + (C_2 - C_5)\varepsilon_\theta + (C_3 - C_6)T}{r} + \rho w^2 r = 0 \tag{12}$$

Substituting ε_r and ε_θ from Eq. (4) and (5) into Eq. (12) gives

$$C_1\frac{\partial^2 u}{\partial r^2} + C_2\left(-\frac{u}{r} + \frac{1}{r}\frac{\partial u}{\partial r}\right) + C_3\frac{\partial T}{\partial r} + (C_1 - C_4)\frac{1}{r}\frac{\partial u}{\partial r} + (C_2 - C_5)\frac{u}{r^2} + (C_3 - C_6)\frac{T}{r} + \rho w^2 r = 0 \tag{13}$$

or

$$d_1\frac{\partial^2 u}{\partial r^2} + \frac{d_3}{r}\frac{\partial u}{\partial r} + \frac{d_2}{r^2}u = d_4\frac{dT}{dr} + d_5\frac{T}{r} + \rho w^2 r \tag{14}$$

where

$$d_1 = C_1$$

$$d_2 = -C_5$$

$$d_3 = C_2 + C_1 - C_4$$

$$d_4 = -C_3$$

$$d_5 = C_6 - C_3$$

Eq. (14) is an ordinary differential equation in form of Euler equation. Homogenous part of the solution is

$$u = r^m \tag{15}$$

$$d_1 m(m-1)r^{m-2} + d_3 m r^{m-2} + d_2 r^{m-2} = 0 \tag{16-a}$$

$$m_{1,2} = \frac{d_1 - d_3 \pm \sqrt{(d_3 - d_1)^2 - 4d_1 d_2}}{2d_1} \tag{16-b}$$

$$u = B_0 r^{m_2} + B_1 r^{m_1} \tag{17}$$

In order to find particular solution a temperature function should be assumed. Second and third term of cubic polynomial function is selected because it gives more variety of thermal distribution

$$T = q_1 r^2 + q_2 r^3 \tag{18}$$

The particular solution will be in the form of

$$u_p = K_1 r^3 + K_2 r^4 \tag{19}$$

Therefore the complete solution of the Eq. (14) is obtained as

$$u = K_1 r^3 + K_2 r^4 + B_0 r^{m_1} + B_1 r^{m_2} \tag{20}$$

where

$$K_1 = \frac{2q_1 d_4 + q_2 d_5 - \rho w^2}{6d_1 + 3d_3 + d_2} \tag{21-a}$$

$$K_2 = \frac{3q_2 d_4 + q_2 d_5}{12d_1 + 4d_3 + d_2} \tag{21-b}$$

By substituting Eq. (20) into Eq. (4) and (5), one may reach strain components as following

$$\varepsilon_r = 3K_1 r^2 + K_2 r^3 + m_1 B_0 r^{m_1 - 1} + m_2 B_1 r^{m_2 - 1} \tag{22}$$

$$\varepsilon_\theta = K_1 r^2 + K_2 r^3 + B_0 r^{m_1 - 1} + B_1 r^{m_2 - 1} \tag{23}$$

Substituting Eq. (22) and (23) in Eq. (10) gives stress components which σ_r and σ_θ as

$$\sigma_r = (3C_1 + C_2)K_1 r^2 + (C_1 + C_2)K_2 r^3 + (C_1 m_1 + C_2)B_0 r^{m_1 - 1} + (C_1 m_2 + C_2)B_1 r^{m_2 - 1} + C_3(q_1 r^2 + q_2 r^3) \tag{24}$$

$$\sigma_\theta = (3C_4 + C_5)K_1 r^2 + (C_4 + C_5)K_2 r^3 + (C_4 m_1 + C_5)B_0 r^{m_1 - 1} + (C_4 m_2 + C_5)B_1 r^{m_2 - 1} + C_6(q_1 r^2 + q_2 r^3) \tag{25}$$

From Fig. 1 one can find B0 and B1 from the following boundary conditions

$$\sigma_r\big|_{r=r_i} = -p_i$$

$$\sigma_r\big|_{r=r_0} = -p_o$$

Using modified Tsai-Wu failure criterion (FC) from [1], the criterion for failure in an orthotropic material in axiysemmtric problem is derived as follows

$$f_1(\sigma_k) = \left(\frac{1}{2}X + \frac{1}{2}\sqrt{X^2 + 4Y^2}\right)^2 \le 1 \tag{26-a}$$

where

$$X = k_1\sigma_r + k_2\sigma_\theta \tag{26-b}$$

$$Y = \sqrt{k_{11}\sigma_r^{\,2} + k_{22}\sigma_\theta^{\,2} - k_{12}\sigma_r\sigma_{\theta p_0}} \tag{26-c}$$

where

$$k_1 = \frac{1}{LT} - \frac{1}{LC}$$

$$k_{11} = \frac{1}{(LT)(LC)}$$

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$$k_2 = \frac{1}{TT} - \frac{1}{TC}$$

$$k_{22} = -\frac{1}{(TT)(TC)}$$

$$k_{12} = \pm\sqrt{\frac{1}{(LT)(LC)}\frac{1}{(TT)(TC)}}$$

where
LT= Longitudinal Tension
LC= Longitudinal Compression
TT= Transverse Tension
TC= Transverse Compression
In choosing q1 and q2 for temperature function, there is limitation to meet $f(\sigma) \le 1$

III. Results and discussion

A thermoplastic composite material is made of Nylon 6 as resin containing 45wt% carbon fiber and its mechanical properties are given in Table 1.

Table1. Mechanical properties of the thermoplastic material

Radial elasticity modulus (MPa)	E_r	20000
Tangential elasticity modulus (MPa)	E_θ	12000
Shear modulus (MPa)	$G_{r\theta}$	8000
Poisson's ratio	$v_{r\theta}$	0.35
Radial thermal expansion coefficients ($1/^0C$)	α_1	9×10^{-6}
Tangential thermal expansion coefficients ($1/^0C$)	α_2	114×10^{-6}
Longitudinal Tension (MPa)	LT	1034
Longitudinal Compression (MPa)	LC	689
Transverse Tension (MPa)	TT	411
Longitudinal Compression (MPa)	TC	117

The internal and external pressures are 6 Mpa and 8Mpa, respectively. ω is 100 rad/s and inner and outer radius of the disk are 0.15 m and 0.2 m, respectively. In Fig. 1 and 2 the FC is plotted versus ratio of temperature coefficient and radial distance, it shows that the minimum amount of FC occurs when the ratio of temperature distribution coefficients is around 0.8.

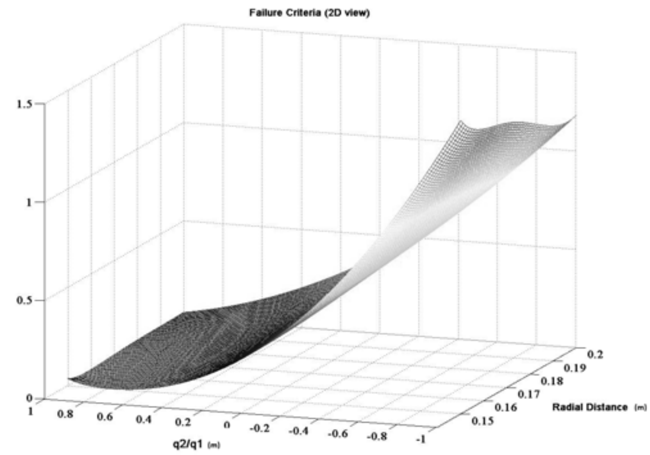

Fig. 2. FC versus radial distance and temperature coefficients ratio; 3D view of the figure.

With the q2/q1=0.8, the radial stress and tangential stress are shown in Fig. 3 and 4, respectively.

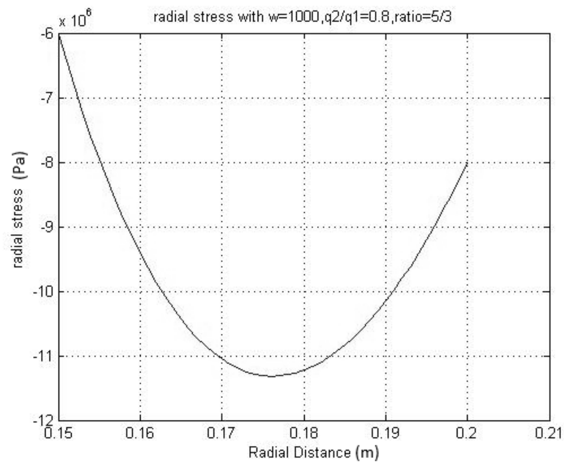

Fig. 3. Radial stress versus radial distance

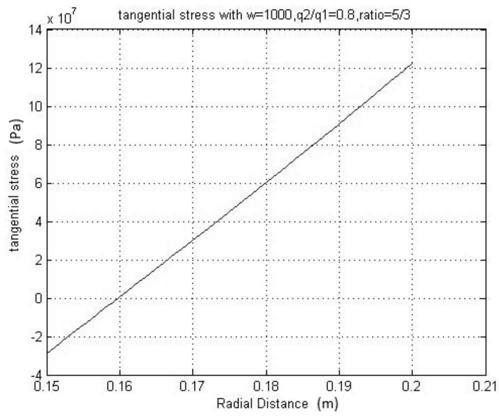

Fig. 4. Tangential stress versus radial distance

Fig.3 and 4 show that the critical region is at the outer side of the rotating disk. At the end, the FC is depicted versus of temperature coefficient (q1, q2). It is obvious that in some region the FC can be minimized. The critical situation happens when one of the temperature coefficient i.e. q1 or q2 is zero. The feasible region can be chosen for temperature coefficients so that the FC is smaller than one.

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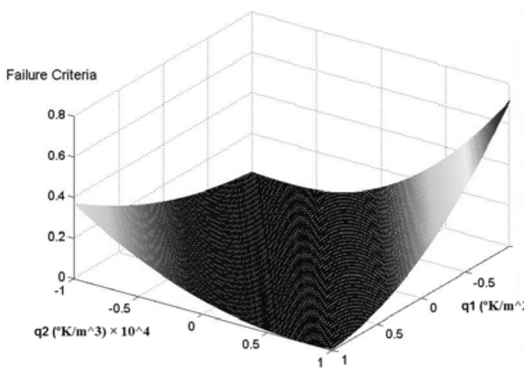

Fig. 5. The feasible region for temperature coefficients versus FC at critical radius

Therefore, q1 and q2 have significant effect on both the maximum FC as well as its distribution in the rotating disk. An efficient design for rotating disk is somehow that all material at any point fail uniformly so that all material are used before failure which causes less wasting of material. A numerical study has been done to obtain FC for different ratio=q1/q2 .

From Fig. 6 it is obvious the optimum value of ratio is around one which can maximize the uniformity of the FC along the radial distance.

IV. Conclusion

An orthotropic rotating disk under internal and external pressure and thermal loading was analyzed in order to obtain efficient temperature distribution. It was shown that if only one term for temperature distribution is used, the possibility of failure in disk increases. Using two terms in temperature distribution function, decrease the amount of the FC. If the two coefficients are equal then the distribution of the FC is more uniform. More terms for temperature polynomial function can be used in this analysis. Optimization procedure can be done in order to find the coefficient of these extra terms.

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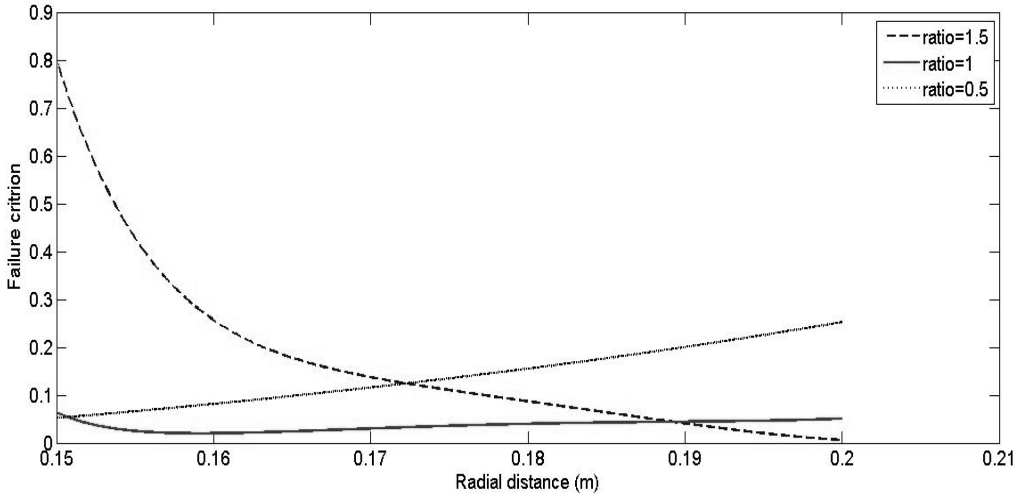

Fig. 6. FC versus radial distance for three values of ratio.

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