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Partitioning the Intrinsic Order Graph for Complex Stochastic Boolean Systems *

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Abstract—**Many different problems in Engineering and Computer Science can be modeled by a complex system depending on a certain number n of stochastic Boolean variables: the so-called complex stochastic Boolean system (CSBS). The most useful graphical representation of a CSBS is the intrinsic order graph (IOG). This is a symmetric, self-dual diagram on 2^n nodes (denoted by I_n) that displays all the binary n-tuples in decreasing order of their occurrence probabilities. In this paper, two different ways of partitioning the IOG –with applications to the analysis of CSBSs– are presented. The first one is based on the successive bisections of this graph into smaller and smaller equal-sized subgraphs. The second one consists of decomposing the graph I_n into totally ordered subsets (chains) of the set $\{0,1\}^n$ of all binary n-tuples.**

Keywords: complex stochastic Boolean system, intrinsic order, intrinsic order graph, graph bisection, chain cover

1 Introduction

This paper deals with the analysis of those complex systems depending on an arbitrary number n of random Boolean variables. That is, the n basic variables $x_1, \ldots, x_n$ of the system are assumed to be stochastic, and they only take two possible values, either 0 or 1, with basic probabilities

$$\Pr\{x_i = 1\} = p_i, \;\; \Pr\{x_i = 0\} = 1 - p_i \quad (1 \le i \le n).$$

We call such a system a *complex stochastic Boolean system* (CSBS). Each one of the 2^n possible elementary states associated to a CSBS is given by a binary n-tuple $u = (u_1, \ldots, u_n) \in \{0,1\}^n$ of 0s and 1s, and it has its own occurrence probability $\Pr\{(u_1, \ldots, u_n)\}$.

Throughout this paper we assume that the n Boolean variables x_i are statistically independent, so that the occurrence probability of a given binary string u of length n can be easily computed as

$$\Pr\{u\} = \prod_{i=1}^{n} p_i^{u_i} (1 - p_i)^{1-u_i} \quad \text{for all } u \in \{0,1\}^n, \quad (1.1)$$

that is, $\Pr\{u\}$ is the product of factors p_i if $u_i = 1$, $1 - p_i$ if $u_i = 0$.

Example 1.1 Let $n = 4$ and $u = (0,1,1,0) \in \{0,1\}^4$. Suppose that $p_1 = 0.1,\; p_2 = 0.2,\; p_3 = 0.3,\; p_4 = 0.4$. Then using (1.1), we have

$$\Pr\{(0,1,1,0)\} = (1 - p_1)\, p_2 p_3 (1 - p_4) = 0.0324.$$

The behavior of a CSBS is determined by the ordering between the current values of the 2^n associated binary n-tuple probabilities $\Pr\{u\}$. Computing all these 2^n probabilities –using (1.1)– and ordering them in decreasing or increasing order of their values is only possible in practice when the number n of basic variables is small. For large values of n, we need alternative procedures to compare the binary string probabilities. For this purpose, in [3, 4] we have established a simple positional criterion that allows one to compare two given elementary state probabilities, $\Pr\{u\}, \Pr\{v\}$, without computing them, simply looking at the positions of the 0s and 1s in the n-tuples u, v. We have called it the *intrinsic order criterion* (IOC), because it is independent of the basic probabilities p_i and it is exclusively determined by the positions of the 0s and 1s in the binary strings.

The intrinsic order relation on $\{0,1\}^n$ can be graphically illustrated through the *intrinsic order graph* (IOG). This is a fractal, symmetric diagram that scales all the 2^n binary n-tuples (associated to a CSBS with n basic variables) by decreasing order of their occurrence probabilities.

In this context, the aim of this paper is to present different partitions of the IOG that provide us with some significant information about the behavior of the CSBSs.

This paper has been organized as follows. In Section 2, we present some background on the intrinsic order relation and the IOG, in order to make this work self-contained. Section 3 describes a recursive bisection procedure of the IOG. Section 4 is devoted to decompose this graph into totally ordered subsets (chains) of the set $\{0,1\}^n$ of all binary n-tuples. Finally, in Section 5, some closing remarks

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describe the practical interest of the two different ways of partitioning the IOG, presented in the two previous sections.

2 Some Previous Results

Let us fix some basic concepts and notations, that we shall use in the rest of the paper.

Definition 2.1 *Let $n \geq 1$ and let $u = (u_1, \ldots, u_n)$ be a binary n-tuple. Then*

(i) We indistinctly denote the n-tuple $u \in \{0,1\}^n$ by its binary representation $(u_1, \ldots, u_n)$ or by its decimal representation $u_{(10}$, and we use the symbol "$\equiv$" to indicate the conversion between these two numbering systems, i.e.,

$$(u_1, \ldots, u_n) \equiv u_{(10} = \sum_{i=1}^{n} 2^{n-i} u_i.$$

(ii) The Hamming weight (or simply weight) $w_H(u)$ of u is the number of 1-bits in u, i.e., $w_H(u) = \sum_{i=1}^{n} u_i$.

(iii) Let $u \neq (0, \ldots, 0)$, i.e., $w_H(u) = m > 0$. The vector of positions of 1s in u is the vector of positions of its m 1-bits, with the convention that these positions are arranged in increasing order from the right-most possible position 1 to the left-most possible position n. This vector will be denoted by $[i_1, i_2, \ldots, i_m]_n$ $(1 \leq i_1 < i_2 < \cdots < i_m \leq n)$.

(iv) Let $0 < m < n$ be arbitrary but fixed. The lexicographic order between the vectors of positions of 1s of all n-tuples with the same weight m is the usual alphabetic order. That is, $[i_1, i_2, \ldots, i_m]_n$ precedes (in the lexicographic order) $[j_1, j_2, \ldots, j_m]_n$ iff the first index $p \in \{1, 2, \ldots, m\}$ for which $i_p \neq j_p$, satisfies $i_p < j_p$.

(v) The complementary n-tuple u^c of u is the n-tuple obtained by changing all its 0s into 1s and vice versa, i.e.,

$$u^c = (u_1, \ldots, u_n)^c = (1 - u_1, \ldots, 1 - u_n),$$

$$(u_1, \ldots, u_n) + (u_1, \ldots, u_n)^c = (1, \ldots, 1) \equiv 2^n - 1, \quad (2.1)$$

i.e., to obtain $(u^c)_{(10}$, just subtract $u_{(10}$ from $2^n - 1$.

(vi) The complementary set S^c of a subset $S \subseteq \{0,1\}^n$ is the set of complementary n-tuples of all the n-tuples of S, i.e.,

$$S^c = \{u^c \mid u \in S\}.$$

Example 2.1 Let $n = 5$ and $u = (0,1,1,0,1) \in \{0,1\}^5$.

$$\begin{array}{ll}
(i) & (0,1,1,0,1) \equiv 2^0 + 2^2 + 2^3 = 13 = u_{(10}. \\
(ii) & w_H((0,1,1,0,1)) = 3. \\
(iii) & (0,1,1,0,1) = [i_1, i_2, i_3]_5 = [1,3,4]_5. \\
(iv) & [1,3,4]_5 \text{ precedes (lexicographically) } [1,3,5]_5. \\
(v) & 13^c \equiv (0,1,1,0,1)^c = (1,0,0,1,0) \equiv 18. \\
(vi) & \{4,6,10,13\}^c = \{27,25,21,18\}.
\end{array}$$

2.1 The Intrinsic Order

As mentioned in Section 1, in [3, 4] we have established a quite simple criterion that allows us to compare two given binary string probabilities, $\Pr\{u\}$ and $\Pr\{v\}$, without computing them. This criterion is rigorously established by the following characterization theorem.

Theorem 2.1 (The intrinsic order theorem) *Let $n \geq 1$. Let $x_1, \ldots, x_n$ be n pairwise statistically independent stochastic Boolean variables, whose basic probabilities $p_i = \Pr\{x_i = 1\}$ satisfy*

$$0 < p_1 \leq p_2 \leq \cdots \leq p_n \leq 0.5. \quad (2.2)$$

Then the occurrence probability of the binary n-tuple v, i.e., $v = (v_1, \ldots, v_n) \in \{0,1\}^n$, is intrinsically less than or equal to the occurrence probability of the binary n-tuple u, i.e., $u = (u_1, \ldots, u_n) \in \{0,1\}^n$, (that is, for all set $\{p_i\}_{i=1}^n$ satisfying (2.2)) if and only if the matrix

$$M_v^u := \begin{pmatrix} u_1 & \ldots & u_n \\ v_1 & \ldots & v_n \end{pmatrix}$$

either has no $\binom{1}{0}$ columns, or for each $\binom{1}{0}$ column in M_v^u there exists (at least) one corresponding preceding $\binom{0}{1}$ column (IOC).

Remark 2.1 (i) In the following, we assume that the probabilities p_i always satisfy condition (2.2). Fortunately, this hypothesis is not restrictive for practical applications. (ii) The $\binom{0}{1}$ column preceding each $\binom{1}{0}$ column is not required to be necessarily placed at the immediately previous position, but just at previous position. (iii) The term *corresponding,* used in Theorem 2.1, has the following meaning: for each two $\binom{1}{0}$ columns in matrix M_v^u, there must exist (at least) two *different* $\binom{0}{1}$ columns preceding each other.

The matrix condition IOC, stated by Theorem 2.1, is called the *intrinsic order criterion*, because it is independent of the basic probabilities p_i and it *intrinsically* depends on the relative positions of the 0s and 1s in the binary n-tuples u, v to be compared. Theorem 2.1 naturally leads to the following partial order relation on the set $\{0,1\}^n$ [3]. The so-called intrinsic order will be denoted by "$\preceq$", and we shall write $v \preceq u$ to indicate that v is intrinsically less than or equal to u. Of course, $u \succeq v$ means that u is intrinsically greater than or equal to v.

Definition 2.2 *For all $u, v \in \{0,1\}^n$*

$$\begin{array}{c} v \preceq u \text{ iff } \Pr\{v\} \leq \Pr\{u\} \text{ for all set } \{p_i\}_{i=1}^n \text{ s.t. (2.2)} \\ \text{iff } M_v^u \text{ satisfies IOC.} \end{array}$$

From now on, the partially ordered set (poset, for short) $(\{0,1\}^n, \preceq)$ will be denoted by I_n.

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2.2 The Intrinsic Order Graph

To finish this section, we present the graphical representation of the poset I_n. The usual representation of a poset is its Hasse diagram (see, e.g., [9] for more details about posets and Hasse diagrams). This is a directed graph (digraph, for short) whose vertices are the binary n-tuples of 0s and 1s, and whose edges go downward from u to v whenever u covers v (denoted by $u \rhd v$), that is, whenever u is intrinsically greater than v with no other elements between them, i.e.,

$$u \rhd v \quad \text{iff} \quad u \succ v \text{ and there is no } w \in \{0,1\}^n \text{ s.t. } u \succ w \succ v.$$

In [5], we have stated and demonstrated the following simple matrix description of the covering relation associated to the intrinsic order.

Theorem 2.2 (Covering relation in I_n) *Let $n \geq 1$ and $u, v \in \{0,1\}^n$. Then $u \rhd v$ if and only if the only columns of matrix M_v^u different from $\binom{0}{0}$ and $\binom{1}{1}$ are either its last column $\binom{0}{1}$ or just two columns, namely one $\binom{1}{0}$ column immediately preceded by one $\binom{0}{1}$ column, i.e., either*

$$M_v^u = \begin{pmatrix} u_1 & \dots & u_{n-1} & 0 \\ u_1 & \dots & u_{n-1} & 1 \end{pmatrix} \text{ or} \tag{2.3}$$

$$M_v^u = \begin{pmatrix} u_1 & \dots & u_{i-2} & 0 & 1 & u_{i+1} & \dots & u_n \\ u_1 & \dots & u_{i-2} & 1 & 0 & u_{i+1} & \dots & u_n \end{pmatrix}. \tag{2.4}$$

The Hasse diagram of the poset I_n will be also called the IOG for n variables, denoted as well by I_n. The IOG I_n has a downward path (edge, respectively) from u to v if and only if $u \succ v$ ($u \rhd v$, respectively). For all $n \geq 2$, in [5] we have developed the following algorithm for iteratively building up the digraph of I_n from the digraph $\begin{array}{c}0\\|\\1\end{array}$ of I_1. Note that I_1 has a downward edge from 0 to 1 because, clearly, $0 \rhd 1$, since matrix $\binom{0}{1}$ has the pattern (2.3)!

Theorem 2.3 (Building up I_n from I_1) *Let $n > 1$. The graph of $I_n = \{0, \dots, 2^n - 1\}$ can be drawn simply by adding to the graph of $I_{n-1} = \{0, \dots, 2^{n-1} - 1\}$ its isomorphic copy $2^{n-1} + I_{n-1} = \{2^{n-1}, \dots, 2^n - 1\}$. This addition must be performed placing the powers of 2 at consecutive levels of the Hasse diagram of I_n. Finally, the edges connecting one vertex u of I_{n-1} with the other vertex v of $2^{n-1} + I_{n-1}$ are given by the set of vertex pairs*

$$\left\{(u,v) \equiv \left(u_{(10}, 2^{n-2} + u_{(10}\right) \mid 2^{n-2} \leq u_{(10} \leq 2^{n-1} - 1\right\}.$$

In Fig. 1, the IOGs for $n = 1, 2, 3, 4$ are shown from left to right, using the decimal numbering for their 2^n nodes.

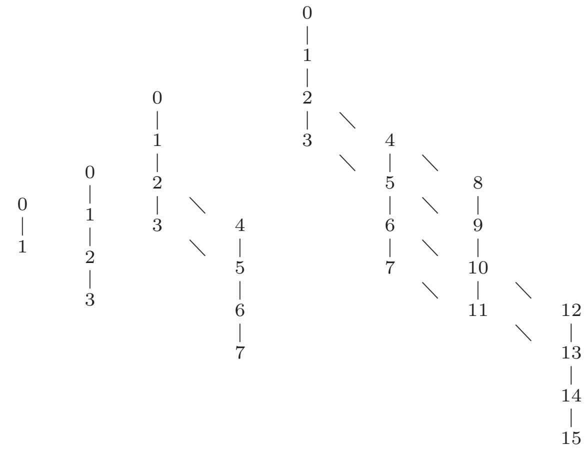

Figure 1: The intrinsic order graphs for $n = 1, 2, 3, 4$.

The edgeless graph for a given graph is obtained by removing all its edges, keeping all its nodes at the same positions. In Fig. 2, the edgeless IOG for $n = 5$ is depicted.

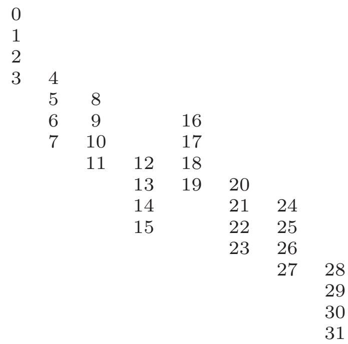

Figure 2: The edgeless intrinsic order graph for $n = 5$.

For further theoretical properties and practical applications of the intrinsic order and the IOG, we refer the reader to [2, 6, 7, 8].

3 Bisecting the Intrinsic Order Graph

A bisection of a graph is a partition of its vertex set into two (disjoint) subsets with half the vertices each. So, Theorem 2.3 suggests us a natural bisection of the (edgeless) graph I_n into its two isomorphic (edgeless) subgraphs I_{n-1} and $2^{n-1} + I_{n-1}$.

Of course, this bisection process of the graph I_n can be reiterated by successively partitioning each one of the obtained subgraphs into its top and bottom halves. This iterative bisection process finishes when we have partitioned I_n into 2^n singleton subgraphs (with 1 vertex each), i.e., into its 2^n nodes. Moreover, this process shows that the poset I_n has a "fractal structure": the whole graph has the same "shape" that each one of its two

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halves, and the same happens with each one of them, and so on. In other words, the poset I_n has the self-similarity property. Figs. 1 and 2 illustrate this fact.

Let us set an adequate notation for this iterative bisection process. For all $n \geq 1$, for all $1 \leq k \leq n$, and for all k fixed binary digits $\bar{u}_1, \ldots, \bar{u}_k \in \{0,1\}$, from now on we denote by $I_n^{\bar{u}_1,\ldots,\bar{u}_k}$ the subset of binary n-tuples whose k first (or left-most) components are constant, namely $u_1 = \bar{u}_1, \ldots, u_k = \bar{u}_k$; while the $n-k$ last (or right-most) components, $u_{k+1}, \ldots, u_n$, take all possible values (0 or 1). More precisely, $I_n^{\bar{u}_1,\ldots,\bar{u}_k}$ is the set

$$\left\{(\bar{u}_1, \ldots, \bar{u}_k, u_{k+1}, \ldots, u_n) \;\middle|\; (u_{k+1}, \ldots, u_n) \in \{0,1\}^{n-k}\right\}$$

$$\equiv \left[(\bar{u}_1, \ldots, \bar{u}_k, 0, \ldots, 0)_{(10}, (\bar{u}_1, \ldots, \bar{u}_k, 1, \ldots, 1)_{(10}\right] \tag{3.1}$$

and, obviously, the cardinality of this subposet is

$$\left|I_n^{\bar{u}_1,\ldots,\bar{u}_k}\right| = \left|\{0,1\}^{n-k}\right| = 2^{n-k}. \tag{3.2}$$

Note that $I_n^{\bar{u}_1,\ldots,\bar{u}_k}$ can be graphically obtained after k successive bisections of I_n $(1 \leq k \leq n)$ simply by changing, from right to left, the "0" and "1" bits of the vector $(\bar{u}_1, \ldots, \bar{u}_k)$, by the words "first half" and "second half", respectively.

Example 3.1 For $n = 5$, $k = 3$ and for the upper indices $(\bar{u}_1, \bar{u}_2, \bar{u}_3) = (1,1,0)$, using (3.1), we get the subgraph

$$\begin{aligned} I_5^{1,1,0} &= \left\{(1,1,0,u_4,u_5) \;\middle|\; (u_4,u_5) \in \{0,1\}^2\right\} \\ &\equiv \left[(1,1,0,0,0)_{(10}, (1,1,0,1,1)_{(10}\right] \\ &= [24,27] = \{24,25,26,27\} \end{aligned}$$

and looking at Fig. 2, we confirm that the subposet $I_5^{1,1,0} = [24,27]$ is exactly the first half ($\bar{u}_3 = 0$) of the second half ($\bar{u}_2 = 1$) of the second half ($\bar{u}_1 = 1$) of the poset I_5. Moreover, in accordance with (3.2), we see that $I_5^{1,1,0}$ has exactly $2^{5-3} = 4$ elements.

The most important property of the above bisection procedure of the IOG is stated by the following Theorem.

Theorem 3.1 *Let $n \geq 1$ and $1 \leq k \leq n$. Bisect the edgeless graph I_n into its 2^k subgraphs $I_n^{\bar{u}_1,\ldots,\bar{u}_k}$ (i.e., make k successive bisections of I_n). Replace each subgraph $I_n^{\bar{u}_1,\ldots,\bar{u}_k}$ by an unique node labeled by its corresponding vector of upper indices $(\bar{u}_1, \ldots, \bar{u}_k)$ and weighted by its occurrence probability $\Pr\{(\bar{u}_1, \ldots, \bar{u}_k)\}$. Then the probability (weight) of each of these nodes $(\bar{u}_1, \ldots, \bar{u}_k)$ coincides with the sum of the occurrence probabilities of all the binary n-tuples u lying on the corresponding replaced subgraph $I_n^{\bar{u}_1,\ldots,\bar{u}_k}$. Moreover, the Hasse diagram obtained by sorting these 2^k new nodes in decreasing order of their weights is precisely the IOG I_k.*

Proof. First, using (3.1), we get

$$\begin{aligned} &\sum_{u \in I_n^{\bar{u}_1,\ldots,\bar{u}_k}} \Pr\{u\} \\ &= \sum_{(u_{k+1},\ldots,u_n) \in \{0,1\}^{n-k}} \Pr\{(\bar{u}_1, \ldots, \bar{u}_k, u_{k+1}, \ldots, u_n)\} \\ &= \Pr\{(\bar{u}_1, \ldots, \bar{u}_k)\} \sum_{v \in \{0,1\}^{n-k}} \Pr\{v\} \\ &= \Pr\{(\bar{u}_1, \ldots, \bar{u}_k)\}. \end{aligned}$$

Second, sorting the 2^k vertices of the new condensed graph in decreasing order of their weights $\Pr\{(\bar{u}_1, \ldots, \bar{u}_k)\}$ is equivalent to ordering the 2^k binary k-tuples $(\bar{u}_1, \ldots, \bar{u}_k) \in \{0,1\}^k$ in decreasing order of their occurrence probabilities. Thus, the new condensed graph is, by definition, the IOG I_k. □

Example 3.2 Let $n = 5$ and $k = 2$. The $2^k = 2^2 = 4$ equal-sized subgraphs obtained after $k = 2$ successive bisections of I_5, are (see Fig. 2):

$$I_5^{0,0} = [0,7], I_5^{0,1} = [8,15], I_5^{1,0} = [16,23], I_5^{1,1} = [24,31].$$

Then the occurrence probability of each subgraph $I_5^{\bar{u}_1,\bar{u}_2}$ (i.e., the sum of the occurrence probabilities of all the eight nodes lying on it) coincides with the occurrence probability of its corresponding binary 2-tuple $(\bar{u}_1, \bar{u}_2)$. Moreover, the four subgraphs $I_5^{0,0}, I_5^{0,1}, I_5^{1,0}, I_5^{1,1}$ are replaced by their corresponding binary 2-tuples (upper indices), $(0,0) \equiv 0$, $(0,1) \equiv 1$, $(1,0) \equiv 2$, $(1,1) \equiv 3$, and displayed –in decreasing order of their occurrence probabilities– on the digraph of I_2 (the second one from the left in Fig. 1.). So, $k = 2$ successive bisections of I_5 lead to I_2, following a "nice fractal behavior".

4 Chains in the Intrinsic Order Graph

Two elements u, v of a poset $(P, \leq)$ are said to be comparable if either $u \leq v$ or $v \leq u$. A chain in a poset is a totally ordered subset, i.e., a subset of pairwise comparable elements. A chain $u = u^1 > u^2 > \cdots > u^l = v$ from u to v is said to have length $l-1$. A chain is said to be saturated when no further elements can be interpolated between its elements. In other words, all successive relations in a saturated chain are coverings [9].

In particular, a saturated chain of length $l-1$ in our poset I_n is a subset $\{u^1, u^2, \ldots, u^l\}$ of $\{0,1\}^n$, such that $u^1 \rhd u^2 \rhd \cdots \rhd u^l$, i.e., $u^1 \succ u^2 \succ \cdots \succ u^l$ with no other elements between them.

A chain decomposition of a poset P is a family of disjoint chains whose union is P. A chain cover of a poset P is

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a chain decomposition into saturated chains, i.e., a set of disjoint saturated chains covering all elements of P.

Let us mention that one can define many different chain covers of I_n. The most intuitively natural way for partitioning I_n into saturated chains is clearly suggested by Figs. 1 or 2. Just consider the 2^{n-2} "columns" obtained after $n-2$ successive bisections of I_n, containing four consecutive numbers beginning with a multiple $4k$ of 4, each. For instance, see the $2^3 = 8$ "columns" in I_5, depicted in Fig. 2. Next Corollary formalizes this chain cover.

Corollary 4.1 *For all $n \geq 2$ the poset I_n can be partitioned into the following 2^{n-2} saturated chains of length 3, that we call "congruence chains* (mod 4)*":*

$$4k \rhd 4k+1 \rhd 4k+2 \rhd 4k+3 \quad \left(0 \leq k \leq 2^{n-2}-1\right).$$

Proof. Clearly, for all $k \in \left[0, 2^{n-2}-1\right]$, that is, for all $k \equiv (u_1, \ldots, u_{n-2}) \in \{0,1\}^{n-2}$, the matrices

$$\begin{aligned} M_{4k+1}^{4k} &= \begin{pmatrix} u_1 & \ldots & u_{n-2} & 0 & 0 \\ u_1 & \ldots & u_{n-2} & 0 & 1 \end{pmatrix}, \\ M_{4k+2}^{4k+1} &= \begin{pmatrix} u_1 & \ldots & u_{n-2} & 0 & 1 \\ u_1 & \ldots & u_{n-2} & 1 & 0 \end{pmatrix}, \\ M_{4k+3}^{4k+2} &= \begin{pmatrix} u_1 & \ldots & u_{n-2} & 1 & 0 \\ u_1 & \ldots & u_{n-2} & 1 & 1 \end{pmatrix} \end{aligned}$$

have either the pattern (2.3) or the pattern (2.4). Finally, since all these saturated chains are pairwise disjoint, and they completely cover I_n, i.e.,

$$\bigcup_{0 \leq k \leq 2^{n-2}-1} \{4k, 4k+1, 4k+2, 4k+3\} = [0, 2^n - 1],$$

the proof is concluded. $\square$

The rest of this section is devoted to present a second chain cover of the poset I_n, far less intuitive than the congruence chains (mod. 4), but more relevant to the analysis of CSBSs. Basically, we shall use "diagonals" instead of "columns", for partitioning I_n.

Lemma 4.1 (A symmetry property) *For all $n \geq 2$, the complementary set C^c of a saturated chain C of I_n is also a saturated chain, and the elements of C^c are the complementary of the elements of C in reverse order, i.e.,*

$$u^1 \rhd u^2 \rhd \cdots \rhd u^l \iff \left(u^l\right)^c \rhd \cdots \rhd \left(u^2\right)^c \rhd \left(u^1\right)^c.$$

Proof. Clearly, the $\binom{0}{0}$, $\binom{1}{1}$, $\binom{0}{1}$ and $\binom{1}{0}$ columns in M_v^u, respectively become $\binom{1}{1}$, $\binom{0}{0}$, $\binom{0}{1}$ and $\binom{1}{0}$ columns in $M_{u^c}^{v^c}$. Hence, using Theorem 2.2, we have that $u \rhd v$ iff M_v^u has either the pattern (2.3) or the pattern (2.4) iff $M_{u^c}^{v^c}$ respectively has either the pattern (2.3) or the pattern (2.4) iff $v^c \rhd u^c$. $\square$

Next theorem establishes the chain cover of the poset I_n into "diagonal" saturated chains.

Theorem 4.1 *For all $n \geq 3$ the poset I_n can be partitioned into the "diagonal" saturated chains displayed from top to bottom* (*and also from right to left*) *in its IOG.*

Proof. We classify the set of "diagonal" saturated chains for covering I_n into the following three kinds (see Definition 2.1-(iii)&(iv) for the notation and nomenclature):

(i) The chain containing all n-tuples of weights 0 and 1. This is the right-most (and also the top-most) "diagonal" chain in the digraph of I_n:

$$0 \rhd [1]_n = 1 \rhd [2]_n = 2 \rhd [3]_n = 2^2 \rhd \cdots \rhd [n]_n = 2^{n-1}. \quad (4.1)$$

Indeed, $0 \rhd 1$, and $2^i \rhd 2^{i+1}$ $(0 \leq i \leq n-2)$ because

$$M_{0,\ldots,0,1}^{0,\ldots,0,0} \quad \text{and} \quad M_{0,\ldots,0,1,0,0,\ldots,0}^{0,\ldots,0,0,1,0,\ldots,0}$$

have the pattern (2.3) and (2.4), respectively.

(ii) The chains containing the following n-tuples:

$$\begin{cases} \text{If } n \text{ is odd:} & \text{all } n\text{-tuples of weights } 2, 3, \ldots, \frac{n-1}{2}. \\ \text{If } n \text{ is even:} & \begin{cases} \text{all } n\text{-tuples of weights } 2, 3, \ldots, \frac{n}{2}-1, \\ \text{those } n\text{-tuples of weight } \frac{n}{2} \text{ s.t. } u_n = 1. \end{cases} \end{cases} \quad (4.2)$$

The construction of such chains must be performed as follows. For each fixed admissible Hamming weight m, consider all the admissible n-tuples with weight m, and sort them by lexicographic order of the vectors of positions of their 1-bits. Then, beginning with the first vector $[1, 2, \ldots, m]_n$ of this sorted list and ending with its last vector $[n-m+1, n-m+2, \ldots, n]_n$, we proceed as follows. Two any consecutive elements $u = [i_1, i_2, \ldots, i_m]_n$ and $v = [j_1, j_2, \ldots, j_m]_n$ of this list (i.e., u immediately precedes v in the list) will be consecutive n-tuples in the same saturated chain (i.e., $u \rhd v$) iff their vectors of positions of 1s only differ in one component (position). Otherwise, v will be the first element of a new chain, and we proceed exactly in the same way with v and its immediate successor w in the sorted list, and so on.

Indeed, note that, for every fixed weight m, if the vectors $u = [i_1, i_2, \ldots, i_m]_n$ and $v = [j_1, j_2, \ldots, j_m]_n$ only differ in one component, say $i_p \neq j_p$, then $j_p = i_p + 1$. This is because v immediately succeeds to u in the list of the vectors of positions of 1s arranged by lexicographic order. But to say that the vectors of u and v only differ in the p-th component with $j_p = i_p + 1$ is equivalent to saying that matrix M_v^u has the pattern (2.4), and thus $u \rhd v$.

(iii) Complementary chains of those obtained in (i) & (ii):

Call S the set of all binary n-tuples included in all chains generated in steps (i) & (ii). Clearly, there are $\binom{n}{m}$ binary n-tuples with weight m. Hence, from (4.1) and (4.2), we have that the cardinality of S is given by

$$\begin{cases} \text{If } n \text{ is odd:} & \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{\frac{n-3}{2}} + \binom{n}{\frac{n-1}{2}} = 2^{n-1}, \\ \text{If } n \text{ is even:} & \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{\frac{n}{2}-1} + \frac{1}{2}\binom{n}{\frac{n}{2}} = 2^{n-1}, \end{cases}$$

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so that, in both cases (n odd, n even), we have used in steps (i) & (ii) exactly half of the set $\{0,1\}^n$ of binary n-tuples (whose cardinality is 2^n).

Now, taking into account again (4.1) and (4.2), we have that all the n-tuples of S have weight $m < \frac{n}{2}$ (with the only exception of those n-tuples of weight $m = \frac{n}{2}$ s.t. $u_n = 1$, if n is even). Note that the complementary n-tuple of a binary n-tuple of weight m obviously has weight $n-m$. Then all the complementary n-tuples of the n-tuples of S have weight $n-m > n-\frac{n}{2} = \frac{n}{2}$ (with the only exception of those n-tuples of weight $n-m = n-\frac{n}{2} = \frac{n}{2}$ s.t. $u_n = 0$, if n is even). This implies that

$$S \cap S^c = \emptyset$$

and this empty intersection means that, among the 2^{n-1} nodes of S (i.e., the ones used in steps (i) & (ii)), there is no pair of complementary n-tuples. Consequently, using Lemma 4.1, we conclude that it suffices to take the complementary chains of those covering the half S of I_n (steps (i) & (ii)), to cover the other half S^c of I_n (step (iii)). Finally, since all the chains generated in this way are, by construction, saturated, pairwise disjoint, and they completely cover our poset I_n, the proof is concluded. □

Example 4.1 The "diagonal" chain cover of I_4 ($n = 4$, even) has the following 4 chains, displayed from top to bottom (and also from right to left) in I_4 (see the right-most graph in Fig. 1, and use (2.1) for step(iii)):

$$\begin{aligned}
\text{(i) } C_1 &= \left\{0 \rhd [1]_4 \rhd [2]_4 \rhd [3]_4 \rhd [4]_4\right\} = \{0 \rhd 1 \rhd 2 \rhd 4 \rhd 8\}, \\
\text{(ii) } C_2 &= \left\{[1,2]_4 \rhd [1,3]_4 \rhd [1,4]_4\right\} = \{3 \rhd 5 \rhd 9\}, \\
\text{(iii) } C_2^c &= \{6 \rhd 10 \rhd 12\}, \\
C_1^c &= \{7 \rhd 11 \rhd 13 \rhd 14 \rhd 15\}.
\end{aligned}$$

Example 4.2 The "diagonal" chain cover of I_5 ($n = 5$, odd) has the following 8 chains, displayed from top to bottom (and also from right to left) in I_5 (see Fig. 2, and use (2.1) for step(iii)):

$$\begin{aligned}
\text{(i) } C_1 &= \left\{0 \rhd [1]_5 \rhd [2]_5 \rhd [3]_5 \rhd [4]_5 \rhd [5]_5\right\} \\
&= \{0 \rhd 1 \rhd 2 \rhd 4 \rhd 8 \rhd 16\}, \\
\text{(ii) } C_2 &= \left\{[1,2]_5 \rhd [1,3]_5 \rhd [1,4]_5 \rhd [1,5]_5\right\} = \{3 \rhd 5 \rhd 9 \rhd 17\}, \\
C_3 &= \left\{[2,3]_5 \rhd [2,4]_5 \rhd [2,5]_5\right\} = \{6 \rhd 10 \rhd 18\}, \\
C_4 &= \left\{[3,4]_5 \rhd [3,5]_5 \rhd [4,5]_5\right\} = \{12 \rhd 20 \rhd 24\}, \\
\text{(iii) } C_4^c &= \{7 \rhd 11 \rhd 19\}, \\
C_3^c &= \{13 \rhd 21 \rhd 25\}, \\
C_2^c &= \{14 \rhd 22 \rhd 26 \rhd 28\}, \\
C_1^c &= \{15 \rhd 23 \rhd 27 \rhd 29 \rhd 30 \rhd 31\}.
\end{aligned}$$

5 Concluding Remarks

We have partitioned the IOG for CSBSs in two different ways. The first one consists of successively bisecting I_n into smaller and smaller subgraphs. The main interest of this iterative bisection process is the following. The (ordering between the) occurrence probabilities of the 2^k equal-sized subgraphs –obtained after k successive bisections of I_n– exactly coincides with the (ordering between the) occurrence probabilities of the corresponding 2^k binary k-tuples whose components are the k left-most bits of all nodes lying on those subgraphs. Hence, k successive bisections of I_n lead to I_k: a useful, nice, fractal property for the analysis of CSBSs! The second one consists of covering I_n with the "diagonal" chains displayed from top to bottom in its IOG. The main interest of this chain cover is the following. One of the main problems in Reliability Engineering and Risk Analysis is to determine the system elementary states with the largest occurrence probabilities [1]. Such elementary states are usually those having the smallest Hamming weights, and with their 1-bits placed among the right-most positions. But, precisely, the up-most "diagonal" chains of our chain partition contains the binary n-tuples satisfying both conditions. This can be also applied, in Fault Tree Analysis, for evaluating the system unavailability.

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