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Control and Integrability on $\mathsf{SO}(3)$

C.C. Remsing *

Abstract— **This paper considers control affine left-invariant systems evolving on matrix Lie groups. Such systems have significant applications in a variety of fields. Any left-invariant optimal control problem (with quadratic cost) can be lifted, via the celebrated Maximum Principle, to a Hamiltonian system on the dual of the Lie algebra of the underlying state space G. The (minus) Lie-Poisson structure on the dual space $\mathfrak{g}^*$ is used to describe the (normal) extremal curves. An interesting, and rather typical, single-input control system on the rotation group $\mathsf{SO}(3)$ is investigated in some detail. The reduced Hamilton equations associated with an extremal curve are derived in a simple and elegant manner. Finally, these equations are explicitly integrated by Jacobi elliptic functions.**

Keywords: ***left-invariant control system, Pontryagin maximum principle, extremal curve, Lie-Poisson structure, elliptic function***

1 Introduction

Invariant control systems on Lie groups provide a natural geometric setting for a variety of problems of mathematical physics, classical and quantum mechanics, elasticity, differential geometry and dynamical systems. Many variational problems (with constraints) can be formulated in the geometric language of modern optimal control theory. An incomplete list of such problems includes the dynamic equations of the rigid body, the ball-plate problem, various versions of the Euler and Kirchhoff elastic rod problem, the Dubins' problem as well as the (more general) sub-Riemannian geodesic problem and the motion of a particle in a magnetic or Yang-Mills field. Some of these problems (and many other) can be found, for instance, in the monographs by Jurdjevic [9], Bloch [4] or Agrachev and Sachkov [1].

In the last two decades or so, substantial work on (applied) nonlinear control has drawn attention to (left-) invariant control systems with control affine dynamics, evolving on matrix Lie groups of low dimension. These arise in problems like the airplane landing problem [23], the motion planning for wheeled robots (subject to nonholonomic constraints) [22], the control of underactuated underwater vehicles [12], the control of quantum systems [6], and the dynamic formation of DNA [7].

*Rhodes University, Dept. of Mathematics (Pure and Applied), Grahamstown 6140, South Africa. Email: c.c.remsing@ru.ac.za

A left-invariant optimal control problem consists in minimizing some (practical) cost functional over the trajectories of a given left-invariant control system, subject to appropriate boundary conditions. The application of the Maximum Principle shifts the emphasis to the language of symplectic and Poisson geometries and to the associated Hamiltonian formalism. The Maximum Principle states that the optimal solutions are projections of the extremal curves onto the base manifold. (For invariant control systems the base manifold is a Lie group G.) The extremal curves are solutions of certain Hamiltonian systems on the cotangent bundle $T^*\mathsf{G}$. The cotangent bundle $T^*\mathsf{G}$ can be realized as the direct product $\mathsf{G} \times \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual of the Lie algebra $\mathfrak{g}$ of G. As a result, each original (left-invariant) Hamiltonian induces a reduced Hamiltonian on the dual space (which comes equipped with a natural Poisson structure).

An arbitrary control affine left-invariant system on the rotation group $\mathsf{SO}(3)$ has the form

$$\dot{g} = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)\,, \quad g \in \mathsf{SO}(3),\ u \in \mathbb{R}^\ell$$

where $A, B_1, \ldots, B_\ell \in \mathfrak{so}(3)$, $1 \leq \ell \leq 3$. There are essentially four types of such systems : single-input systems with drift, underactuated (two-input) systems with or without drift, and fully actuated systems. (The single-input drift-free systems represent a degenerate case of little interest.) The (non-Euclidean) elastic problem on $\mathbb{S}^2$ is associated with control systems of the first type (see [9], [8]) whereas problems related to the attitude control of a rigid body lead to optimal control problems associated with drift-free systems, underactuated or fully actuated (see [15], [21], [20], [3]). Motion planning can be formulated as an optimal control problem associated with a control system of the third type, i.e., a two-input system with drift (see [23]).

In this paper, we consider an optimal control problem associated with a single-input control-affine system on the rotation group $\mathsf{SO}(3)$, known as a stiff Serret-Frenet control system (see [9]). The problem is lifted, via the Pontryagin Maximum Principle, to a Hamiltonian system on the dual of the Lie algebra $\mathfrak{so}(3)$. Now, the (minus) Lie-Poisson structure on $\mathfrak{so}(3)^*$ (identified here with $\mathbb{R}^3_\wedge$) can be used to derive, in a general and elegant manner, the equations for extrema (cf. [9], [1], [11], [19], [17], [18]). Jacobi elliptic functions are used to derive *explicit* expressions for the extremal curves (cf. [15], [16]).

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The paper is organized as follows. Section 2 contains mathematical preliminaries including invariant control systems, elements of Hamilton-Poisson formalism as well as a (coordinate-free) statement of the Maximum Principle. In section 3, a class of optimal control problems is identified and a particular result due to P.S. Krishnaprasad [11] is recalled. Sections 4 and 5 deal with a particular case of a single-input optimal control on the rotation group $\mathsf{SO}(3)$. The later section contains the explicit equations of extrema. Finally, section 6 contains the integration procedures which lead to explicit expressions (in terms of Jacobi elliptic functions) of the extremal curves.

2 Preliminaries

2.1 Left-Invariant Control Systems

Invariant control systems on Lie groups were first considered in 1972 by Brockett [5] and by Jurdjevic and Sussmann [10]. A *left-invariant control system* is a (smooth) control system evolving on some (real) Lie group, whose dynamics is invariant under left translations. For the sake of convenience, we shall assume that the state space of the system is a matrix Lie group and that there are no constraints on the controls. Such a control system (evolving on G) is described as follows (cf. [9], [17], [18])

$$\dot{g} = g\,\Xi(\mathbf{1}, g), \quad g \in \mathsf{G},\ u \in \mathbb{R}^\ell \tag{1}$$

where the parametrisation map $\Xi(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{g}$ is a (smooth) embedding. (Here $\mathbf{1} \in \mathsf{G}$ denotes the identity matrix and $\mathfrak{g}$ denotes the Lie algebra associated with G.) An admissible control is a map $u(\cdot) : [0, T] \to \mathbf{R}^\ell$ that is bounded and measurable. ("Measurable" means "almost everywhere limit of piecewise constant maps".) A *trajectory* for an admissible control $u(\cdot) : [0, T] \to \mathbf{R}^\ell$ is an absolutely continuous curve $g(\cdot) : [0, T] \to \mathsf{G}$ such that $\dot{g}(t) = g(t)\,\Xi(1, u(t))$ for almost every $t \in [0, T]$. The Carathéodory existence and uniqueness theorem of ordinary differential equations implies the local existence and global uniqueness of trajectories. A controlled trajectory is a pair $(g(\cdot), u(\cdot))$, where $u(\cdot)$ is an admissible control and $g(\cdot)$ is the trajectory corresponding to $u(\cdot)$.

The attainable set from $g \in \mathsf{G}$ is the set $\mathcal{A}(g)$ of all terminal points $g(T)$ of all trajectories $g(\cdot) : [0, T] \to \mathsf{G}$ starting at g. It follows that $\mathcal{A}(g) = g\,\mathcal{A}(\mathbf{1})$. Thus, $\mathcal{A}(g) = \mathsf{G}$ if and only if $\mathcal{A}(\mathbf{1}) = \mathsf{G}$. Control systems which satisfy $\mathcal{A}(\mathbf{1}) = \mathsf{G}$ are called *controllable.* Let $\Gamma \subseteq \mathfrak{g}$ be the image of the parametrisation map $\Xi(\mathbf{1}, \cdot)$, and let $\mathsf{Lie}(\Gamma)$ denote the Lie subalgebra of $\mathfrak{g}$ generated by Γ. It is well known that a necessary condition for the control system (1) to be controllable is that G be connected and that $\mathsf{Lie}(\Gamma) = \mathfrak{g}$. If the group G is compact, then the condition is also sufficient.

For many practical control applications, (left-invariant) control systems contain a drift term and are affine in controls, i.e., are of the form

$$\dot{g} = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)\,, \quad g \in \mathsf{G},\ u \in \mathbb{R}^\ell \tag{2}$$

where $A, B_1, \ldots, B_\ell \in \mathfrak{g}$. Usually the elements (matrices) $B_1, \ldots, B_\ell$ are assumed to be linearly independent.

2.2 Optimal Control Problems

Consider a left-invariant control system (1) evolving on some matrix Lie group $\mathsf{G} \leq \mathsf{GL}(n, \mathbb{R})$ of dimension m. In addition, it is assumed that there is a prescribed (smooth) *cost function* $L : \mathbb{R}^\ell \to \mathbb{R}_{>0}$ (which is also called a Lagrangian). Let g_0 and g_1 be arbitrary but fixed points of G. We shall be interested in finding a controlled trajectory $(g(\cdot), u(\cdot))$ which satisfies

$$g(0) = g_0, \quad g(T) = g_1 \tag{3}$$

and which in addition *minimizes* the total cost functional $\mathcal{J} = \int_0^T L(u(t))\,dt$ among all trajectories of (1) which satisfy the same boundary conditions (3). The terminal time $T > 0$ can be either fixed or it can be free.

2.3 Symplectic and Poisson Structures

The cotangent bundle $T^*\mathsf{G}$ can be trivialized (from the left) such that $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual space of the Lie algebra $\mathfrak{g}$. Explicitly, $\xi \in T_g^*\mathsf{G}$ is identified with $(g, p) \in \mathsf{G} \times \mathfrak{g}^*$ via $p = dL_g^*(\xi)$. (Here, dL_g^* denotes the dual of the tangent map $dL_g = (L_g)_{*,\mathbf{1}} : \mathfrak{g} \to T_g\mathsf{G}$.) That is, $\xi(gA) = p(A)$ for $g \in \mathsf{G}$, $A \in \mathfrak{g}$. Each element (matrix) $A \in \mathfrak{g}$ defines a (smooth) function H_A on the cotangent bundle $T^*\mathsf{G}$ defined by $H_A(\xi) = \xi\,(gA)$ for $\xi \in T_g^*\mathsf{G}$. Viewed as a function on $\mathsf{G} \times \mathfrak{g}^*$, H_A is left-invariant, which is equivalent to saying that H_A is a function on $\mathfrak{g}^*$.

The canonical symplectic form ω on $T^*\mathsf{G}$ sets up a correspondence between (smooth) functions H on $T^*\mathsf{G}$ and vector fields $\vec{H}$ on $T^*\mathsf{G}$ given by $\omega_\xi\left(\vec{H}(\xi), V\right) = dH(\xi) \cdot V$ for $V \in T_\xi(T^*\mathsf{G})$. The Poisson bracket of two functions F, G on $T^*\mathsf{G}$ is defined by $\{F, G\}(\xi) = \omega_\xi\left(\vec{F}(\xi), \vec{G}(\xi)\right)$ for $\xi \in T^*\mathsf{G}$. If (ϕ_t) is the flow of the Hamiltonian vector field $\vec{H}$, then $H \circ \phi_t = H$ (conservation of energy) and $\frac{d}{dt}(F \circ \phi_t) = \{F, H\} \circ \phi_t = \{F \circ \phi_t, H\}$. For short, for any $F \in C^\infty(T^*\mathsf{G})$,

$$\dot{F} = \{F, H\} \tag{4}$$

(the equation of motion in Poisson bracket form).

The dual space $\mathfrak{g}^*$ has a natural *Poisson structure*, called the "minus Lie-Poisson structure" and given by

$$\{F, G\}_-(p) = -p\,([dF(p), dG(p)])$$

for $p \in \mathfrak{g}^*$ and $F, G \in C^\infty(\mathfrak{g}^*)$. (Note that $dF(p)$ is a linear function on $\mathfrak{g}^*$ and hence is an element of $\mathfrak{g}$.)

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The (minus) Lie-Poisson bracket can be derived from the canonical Poisson structure on the cotangent bundle $T^*\mathsf{G}$ by a process called Poisson reduction (cf. [13], [11]). The Poisson manifold $(\mathfrak{g}, \{\cdot,\cdot\})$ is denoted by $\mathfrak{g}^*_-$. Each left-invariant Hamiltonian on the cotangent bundle $T^*\mathsf{G}$ is identified with its reduction on the dual space $\mathfrak{g}^*_-$. In the left-invariant realization of $T^*\mathsf{G}$, the equations of motion for the left-invariant Hamiltonian H are

$$\begin{aligned} \dot{g} &= g\, dH(p) \\ \dot{p} &= \mathrm{ad}^*_{dH(p)} p \end{aligned}$$

where ad^* denotes the coadjoint representation of $\mathfrak{g}$ (cf. [13], [9]). Note that for non-commutative Lie groups, the representation $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$ invariably leads to non-canonical coordinates.

If $(E_k)_{1\leq k\leq m}$ is a basis for the Lie algebra $\mathfrak{g}$, the structure constants $\left(c^k_{ij}\right)$ are defined by $[E_i, E_j] = \sum_{k=1}^m c^k_{ij} E_k$. Any element $p \in \mathfrak{g}^*$ can be expressed uniquely as $p = \sum_{k=1}^m p_k E^*_k$, where $(E^*_k)_{1\leq k\leq m}$ is the basis of $\mathfrak{g}^*$ dual to $(E_k)_{1\leq k\leq m}$. Then the (minus) Lie-Poisson bracket becomes

$$\{F, G\}_-(p) = -\sum_{i,j,k=1}^m c^k_{ij} p_k \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial p_j}.$$

A *Casimir function* of (the Poisson structure of) $\mathfrak{g}^*_-$ is a (smooth) function C on $\mathfrak{g}^*$ such that $\{C, F\}_- = 0$ for all $F \in C^\infty(\mathfrak{g}^*)$. The Casimir functions have the remarkable property that they are integrals of motion for any Hamiltonian system (i.e., they are constant along the flow of any Hamiltonian vector field) on $\mathfrak{g}^*_-$.

2.4 The Maximum Principle

The *Pontryagin Maximum Principle* is a necessary condition for optimality expressed most naturally in the language of the geometry of the cotangent bundle $T^*\mathsf{G}$ of G (cf. [1], [9]). To an optimal control problem (with fixed terminal time)

$$\int_0^T L(u(t))\, dt \to \min \tag{5}$$

subject to (1) and (3), we associate, for each real number λ and each control parameter $u \in \mathbb{R}^\ell$, a Hamiltonian function on $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$:

$$\begin{aligned} H^\lambda_u(\xi) &= \lambda\, L(u) + \xi\left(g\, \Xi(\mathbf{1}, u)\right) \\ &= \lambda\, L(u) + p\left(\Xi(\mathbf{1}, u)\right), \quad \xi = (g, p) \in T^*\mathsf{G}. \end{aligned}$$

The Maximum Principle can be stated, in terms of the above Hamiltonians, as follows :

THE MAXIMUM PRINCIPLE. *Suppose the controlled trajectory $(\bar{g}(\cdot), \bar{u}(\cdot))$ defined over the interval $[0, T]$ is a solution for the optimal control problem (1)-(3)-(5). Then, there exists a curve $\xi(\cdot) : [0, T] \to T^*\mathsf{G}$ with $\xi(t) \in T^*_{\bar{g}(t)}\mathsf{G}$, $t \in [0, T]$, and a real number $\lambda \leq 0$, such that the following conditions hold for almost every $t \in [0, T]$:*

$$(\lambda, \xi(t)) \not\equiv (0, 0) \tag{6}$$

$$\dot{\xi}(t) = \vec{H}^\lambda_{\bar{u}(t)}(\xi(t)) \tag{7}$$

$$H^\lambda_{\bar{u}(t)} = \max_u H^\lambda_u(\xi(t)) = \textit{constant}. \tag{8}$$

An optimal trajectory $\bar{g}(\cdot) : [0, T] \to \mathsf{G}$ is the projection of an integral curve $\xi(\cdot)$ of the (time-varying) Hamiltonian vector field $\vec{H}^\lambda_{\bar{u}(t)}$ defined for all $t \in [0, T]$. A trajectory-control pair $(\xi(\cdot), u(\cdot))$ defined on $[0, T]$ is said to be an *extremal pair* if $\xi(\cdot)$ is such that the conditions (6), (7) and (8) of the Maximum Principle hold. The projection $\xi(\cdot)$ of an extremal pair is called an extremal. An extremal curve is called normal if $\lambda = -1$ and abnormal if $\lambda = 0$. In this paper, we shall be concerned only with normal extremals.

If the maximum condition (8) eliminates the parameter u from the family of Hamiltonians (H_u), and as a result of this elimination, we obtain a smooth function H (without parameters) on $T^*\mathsf{G}$ (in fact, on $\mathfrak{g}^*_-$), then the whole (left-invariant) optimal control problem reduces to the study of trajectories of a fixed Hamiltonian vector field $\vec{H}$.

3 A Class of Optimal Control Problems

Consider now a left-invariant optimal control problem (2)-(3)-(5) with quadratic cost of the form

$$L(u_1, \ldots, u_\ell) = \frac{1}{2}\left(c_1 u_1^2 + \cdots + c_\ell u_\ell^2\right)$$

where $c_1, \ldots, c_\ell$ are (positive) constants. The terminal time $T > 0$ is fixed in advance. The maximum condition (8) of the Maximum Principle implies that (for $\lambda = -1$) the optimal controls $\bar{u}(\cdot)$ satisfy

$$-\frac{\partial L}{\partial u_i} + \frac{\partial}{\partial u_i}\left(p\left(A + u_1 B_1 + \cdots + u_\ell B_\ell\right)\right) = 0$$

or

$$-c_i u_i + p(B_i) = 0, \quad i = 1, \ldots, \ell.$$

The following result holds (see [11]) :

Proposition 1 (Krishnaprasad, 1993) *For the optimal control problem (2)-(3)-(5), every normal extremal is given by*

$$\bar{u}_i(t) = \frac{1}{c_i}\, p(t)(B_i), \quad i = 1, \ldots, \ell$$

where $p(\cdot) : [0, T] \to \mathfrak{g}^*$ is an integral curve of the vector field $\vec{H}$ on $\mathfrak{g}^*_-$ corresponding to the reduced Hamiltonian

$$H(p) = p(A) + \frac{1}{2}\left(\frac{1}{c_1} p(B_1)^2 + \cdots + \frac{1}{c_\ell} p(B_\ell)^2\right).$$

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Furthermore, in coordinates on $\mathfrak{g}_-^*$*, the (components of the) integral curves satisfy*

$$\dot{p}_i = -\sum_{j,k=1}^{m} c_{ij}^k p_k \frac{\partial H}{\partial p_j}, \quad i = 1, \ldots, m. \tag{9}$$

4 A Left-Invariant Control Problem on the Rotation Group $\mathsf{SO}(3)$

The rotation group

$$\mathsf{SO}(3) = \left\{ a \in \mathsf{GL}(3, \mathbb{R}) \, : \, a^\top a = \mathbf{1}, \, \det a = 1 \right\}$$

is a three-dimensional compact and connected matrix Lie group. The associated Lie algebra is given by

$$\mathfrak{so}(3) = \left\{ \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \, : \, a_1, a_2, a_3 \in \mathbb{R} \right\}.$$

Let

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

be the standard basis of $\mathfrak{so}(3)$ with the following table for the bracket operation

$[\cdot,\cdot]$	E_1	E_2	E_3
E_1	0	E_3	$-E_2$
E_2	$-E_3$	0	E_1
E_3	E_2	$-E_1$	0

The linear map $\widehat{\cdot} : \mathfrak{so}(3) \to \mathbb{R}^3$ defined by

$$A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \mapsto \widehat{A} = (a_1, a_2, a_3)$$

is a Lie algebra isomorphism. Hence, we identify $\mathfrak{so}(3)$ with (the cross-product Lie algebra) $\mathbb{R}^3_\wedge$. We consider the following optimal control problem

$$\dot{g} = g\,(E_3 + uE_1)\,, \quad g \in \mathsf{SO}(3), \; u \in \mathbb{R} \tag{10}$$

$$g(0) = g_0, \quad g(T) = g_1 \quad (g_0, g_1 \in \mathsf{SO}(3)) \tag{11}$$

$$\mathcal{J} = \frac{1}{2}\int_0^T u^2(t)\, dt \to \min. \tag{12}$$

This problem models a variation of the classical elastic problem of Euler and Kirchhoff (cf. [9], [1], [8]). Note that the underlying control system is controllable.

5 Extremal Curves in $\mathfrak{so}(3)^*$

We will identify $\mathfrak{so}(3)^*$ with $\mathfrak{so}(3)$ via the pairing

$$\left\langle \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix} \right\rangle = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Then each extremal curve $p(\cdot)$ is identified with a curve $P(\cdot)$ in $\mathfrak{so}(3)$ via the formula $\langle P(t), A\rangle = p(t)(A)$ for all $A \in \mathfrak{so}(3)$. Thus

$$P(t) = \begin{bmatrix} 0 & -P_3(t) & P_2(t) \\ P_3(t) & 0 & -P_1(t) \\ -P_2(t) & P_1(t) & 0 \end{bmatrix} \tag{13}$$

where $P_i(t) = \langle P(t), E_i \rangle = p(t)(E_i), \quad i = 1, 2, 3.$

The (minus) Lie-Poisson bracket on $\mathfrak{so}(3)^*$ is given by

$$\begin{aligned} \{F, G\}_-(p) &= -\sum_{i,j,k=1}^{3} c_{ij}^k p_k \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial p_j} \\ &= -\begin{vmatrix} p_1 & p_2 & p_3 \\ \frac{\partial F}{\partial p_1} & \frac{\partial F}{\partial p_2} & \frac{\partial F}{\partial p_3} \\ \frac{\partial G}{\partial p_1} & \frac{\partial G}{\partial p_2} & \frac{\partial G}{\partial p_3} \end{vmatrix} \\ &= -\widehat{P} \bullet (\nabla F \times \nabla g)\,. \end{aligned}$$

Here, $\mathfrak{so}(3)^*$ is identified with $\mathbb{R}^3_\wedge$. Explicitly, the covector $p = p_1 E_1^* + p_2 E_2^* + p_3 E_3^*$ is identified with the vector $\widehat{P} = (P_1, P_2, P_3)$. The equation of motion (4) becomes

$$\begin{aligned} \dot{F} &= \{F, H\}_- \\ &= -\widehat{P} \bullet (\nabla F \times \nabla H) \\ &= \nabla F \bullet \left(\widehat{P} \times \nabla H\right) \end{aligned}$$

and so

$$\begin{aligned} \begin{bmatrix} \dot{P}_1 \\ \dot{P}_2 \\ \dot{P}_3 \end{bmatrix} &= \widehat{P} \times \nabla H \\ &= P \cdot \nabla H \\ &= \begin{bmatrix} 0 & -P_3 & P_2 \\ P_3 & 0 & -P_1 \\ -P_2 & P_1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \\ \frac{\partial H}{\partial p_3} \end{bmatrix}. \end{aligned}$$

Hence, we get the following (scalar) equations of motion

$$\dot{P}_1 = \frac{\partial H}{\partial p_3} P_2 - \frac{\partial H}{\partial p_2} P_3 \tag{14}$$

$$\dot{P}_2 = \frac{\partial H}{\partial p_1} P_3 - \frac{\partial H}{\partial p_3} P_1 \tag{15}$$

$$\dot{P}_3 = \frac{\partial H}{\partial p_2} P_1 - \frac{\partial H}{\partial p_1} P_2. \tag{16}$$

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The function

$$C = P_1^2 + P_2^2 + P_3^2 \tag{17}$$

is a Casimir function.

Proposition 2 *Given the left-invariant optimal control problem (10)-(11)-(12), the extremal control is*

$$\bar{u} = P_1$$

where $P_1 : [0, T] \to \mathbb{R}$ *(together with* P_2 *and* P_3*) is a solution of the system of differential equations*

$$\dot{P}_1 = P_2 \tag{18}$$
$$\dot{P}_2 = P_1 P_3 - P_1 \tag{19}$$
$$\dot{P}_3 = -P_1 P_2. \tag{20}$$

Proof : The reduced Hamiltonian (on $\mathfrak{so}(3)^* = \mathbb{R}^3_\wedge$) is

$$H = \frac{1}{2} P_1^2 + P_3. \tag{21}$$

The desired result now follows from **Proposition 1** and (14)-(15)-(16).

It follows that the extremal trajectories (i.e., the solution curves of the reduced Hamilton equations) are the intersections of the parabolic cylinders $P_1^2 + 2P_3 = 2H$ and the spheres $P_1^2 + P_2^2 + P_3^2 = 2C$.

6 Integration by Jacobi Elliptic Functions

The Jacobi elliptic functions are inverses of elliptic integrals. Given a number $k \in [0, 1]$, the function $F(\varphi, k) = \int_0^\varphi \frac{dt}{\sqrt{1-k^2 \sin^2 t}}$ is called an (incomplete) *elliptic integral* of the first kind. The parameter k is known as the modulus. The inverse function $\mathrm{am}(\cdot, k) = F(\cdot, k)^{-1}$ is called the amplitude, from which the basic *Jacobi elliptic functions* are derived :

$$\begin{aligned} \mathrm{sn}(x,k) &= \sin \mathrm{am}(x,k) \quad \text{(sine amplitude)} \\ \mathrm{cn}(x,k) &= \cos \mathrm{am}(x,k) \quad \text{(cosine amplitude)} \\ \mathrm{dn}(x,k) &= \sqrt{1 - k^2 \sin^2 \mathrm{am}(x,k)} \quad \text{(delta amplitude).} \end{aligned}$$

(For the degenerate cases $k = 0$ and $k = 1$, we recover the circular functions and the hyperbolic functions, respectively.) Alternatively, the Jacobi elliptic functions $\mathrm{sn}(\cdot, k), \mathrm{cn}(\cdot, k)$ and $\mathrm{dn}(\cdot, k)$ can be defined as the solutions of the system of differential equations

$$\begin{aligned} \dot{x} &= yz \\ \dot{y} &= -zx \\ \dot{z} &= -k^2 xy \end{aligned}$$

that satisfy the initial conditions (see [14])

$$x(0) = 0, \quad y(0) = 1, \quad z(0) = 1.$$

Furthermore, these functions are solutions to certain nonlinear differential equations. For instance, the Jacobi elliptic function $x(\cdot) = \mathrm{sn}(\cdot, k)$ solves the differential equation $\dot{x}^2 = (1-x^2)(1-k^2 x^2)$. Nine other elliptic functions are defined by taking reciprocals and quotients; in particular, we get $\mathrm{ns}(\cdot, \mathrm{k}) = \frac{1}{\mathrm{sn}(\cdot,k)}$ and $\mathrm{dc}(\cdot, k) = \frac{\mathrm{dn}(\cdot,k)}{\mathrm{cn}(\cdot,k)}$.

An integral of the type $\int R(x, y)\, dx$, where y^2 is a cubic or quartic polynomial in x and $R(\cdot, \cdot)$ denotes a rational function, is called an elliptic integral. General elliptic integrals may be expressed as a finite sum of elementary integrals and the three types of integral given by the Legendre normal forms (of the first, second and third kinds). Simple elliptic integrals can be expressed in terms of the appropriate inverse functions. Specifically, the following two formulas hold true for $b < a \leq x$ (see e.g. [2]) :

$$\int_a^x \frac{dt}{\sqrt{(t^2 - a^2)(t^2 - b^2)}} = \frac{1}{a} \mathrm{dc}^{-1} \left(\frac{x}{a}, \frac{b}{a} \right) \tag{22}$$

$$\int_x^\infty \frac{dt}{\sqrt{(t^2 - a^2)(t^2 - b^2)}} = \frac{1}{a} \mathrm{ns}^{-1} \left(\frac{x}{a}, \frac{b}{a} \right). \tag{23}$$

Proposition 3 *The reduced Hamilton equations (18)-(19)-(20) can be explicitly integrated by Jacobi elliptic functions. More precisely, we have*

$$\begin{aligned} P_1 &= \pm \sqrt{2(H - P_3)} \\ P_2 &= \pm \sqrt{C - 2(H - 2P_3) - P_3^2} \\ P_3 &= \frac{\alpha - \beta\, \delta\, \Phi\left((\alpha - \beta) M \delta\, t, \frac{\epsilon}{\delta}\right)}{1 - \delta\, \Phi\left((\alpha - \beta) M \delta\, t, \frac{\epsilon}{\delta}\right)} \end{aligned}$$

whenever $H^2 - C > 0$. *(Here* $\alpha = H + \sqrt{H^2 - C}$, $\beta = H - \sqrt{H^2 - C}$, $M = \frac{H - \sqrt{H^2 - C} - 1}{4(H^2 - C)}$, $\delta^2 = \frac{1 - \sqrt{H^2 - C} - H}{1 + \sqrt{H^2 - C} - H}$, $\epsilon^2 = 1$, *and* $\Phi(\cdot, k)$ *denotes one of the Jacobi elliptic functions* $\mathrm{dc}(\cdot, k)$ *or* $\mathrm{ns}(\cdot, k)$.*)*

Proof : The reduced Hamiltonian (21) and the Casimir function (17) are constants of motion. We get $P_1^2 = 2(H - P_3)$ and $P_2^2 = C - 2(H - P_3) - P_3^2$. Hence,

$$\dot{P}_3^2 = 2(H - P_3)\left(C - 2(H - P_3) - P_3^2\right). \tag{24}$$

The right-hand side of this equation can be written as

$$\left(\mu_1 (P_3 - \alpha)^2 + \nu_1 (P_3 - \beta)^2\right)\left(\mu_2 (P_3 - \alpha)^2 + \nu_2 (P_3 - \beta)^2\right)$$

where

$$\mu_1 = \frac{H - \sqrt{H^2 - C} - 1}{2\sqrt{H^2 - C}} \qquad \mu_2 = \frac{1}{2\sqrt{H^2 - C}}$$

$$\nu_1 = \frac{1 - \sqrt{H^2 - C} - H}{2\sqrt{H^2 - C}} \qquad \nu_2 = -\frac{1}{2\sqrt{H^2 - C}}$$

$$\alpha = H + \sqrt{H^2 - C} \qquad \beta = H - \sqrt{H^2 - C}.$$

Notice that

$$\frac{1 - \sqrt{H^2 - C} - H}{1 + \sqrt{H^2 - C} - H} \quad \text{and} \quad \frac{H - \sqrt{H^2 - C} - 1}{4(H^2 - C)}$$

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are both positive (whenever $H^2 - C$ is positive). Denote $\sqrt{\mu_1 \mu_2}$ by M and let

$$\delta^2 = \frac{1 - \sqrt{H^2 - K} - H}{1 + \sqrt{H^2 - C} - H} \quad \text{and} \quad \epsilon = \pm 1.$$

Now, straightforward algebraic manipulation as well as simple integration and appropriate change of variables yield explicit expressions (in terms of Jacobi elliptic functions) for the solutions of the (first-order) ordinary differential equation (24). We get

$$P_3(t) = \frac{\alpha - \beta\,\delta\,\mathrm{dc}\left((\alpha-\beta)M\delta\,t, \frac{\epsilon}{\delta}\right)}{1 - \delta\,\mathrm{dc}\left((\alpha-\beta)M\delta\,t, \frac{\epsilon}{\delta}\right)}$$

(corresponding to the integral (22)) or

$$P_3(t) = \frac{\alpha - \beta\,\delta\,\mathrm{ns}\left((\alpha-\beta)M\delta\,t, \frac{\epsilon}{\delta}\right)}{1 - \delta\,\mathrm{ns}\left((\alpha-\beta)M\delta\,t, \frac{\epsilon}{\delta}\right)}$$

(corresponding to the integral (23)).

7 Final Remark

Invariant optimal control problems on matrix Lie groups other than the rotation group $\mathsf{SO}(3)$ (like the Euclidean groups $\mathsf{SE}(2)$ and $\mathsf{SE}(3)$, the Lorentz groups $\mathsf{SO}(1,2)$ and $\mathsf{SO}(1,3)$, or the Heisenberg group) can also be considered. It is to be expected that explicit integration of the reduced Hamilton equations will be possible in all these cases. Further work is in progress.

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