Curved Beams with Elastic Supports. Transfer and Stiffness Matrices

Lazaro Gimena, Pedro Gonzaga, Faustino N. Gimena and Fernando Sarria

Abstract — Authors presented in last conferences, a new numerical method (the Finite Transfer Method) to solve a linear system of ordinary differential equations, how to apply general boundary conditions in equation form, and determine in the limit the exact analytical solution as well. The method was applied to the problem of spatially curved beams. Here, the computing of elastic support conditions in twisted beams is carried out. This is a complicated problem since usual and traditional models do not contemplate the whole system with all the unknowns and all the functions. Several problems arise with the treatment of boundary conditions. The systematic model presented hereby, is complete and without holes, but is still recent and challenging. It is necessary to solve those problems, to have all data in an arranged structure that will be given in this paper. Since analytical solution is the limit of the numerical procedure proposed, exact expressions and derivations of transfer and stiffness matrices come up. For the general case, a system of twenty four algebraic equations is reached. A clear analytical example is developed to show the practice. The procedure given is general and suitable for educational purposes.

***Index Terms*— Differential system, Curved Beam, Finite Transfer Method (FTM), Transfer matrix, Stiffness matrix, boundary equations, Frenet-Serret formulas, Exact Solution.**

I. Introduction

The problem to solving a system of linear ordinary differential equations (ODE) with boundary conditions can be approached by using analytic or numerical strategies. Being normally very difficult to obtain the exact analytical solution, approximate procedures have been resorted to [1]. In last decades, several numerical methods have arisen to solve these boundary value problems; see for example, the Shooting Method [2], Finite Differences [3], Finite Element Analysis [4] and the Boundary Element [5] methods.

There exists much literature on modelling arbitrary curved beam elements [6], [7]. Traditionally, the laws governing the mechanical behavior of a curved warped beam (applying the Euler-Bernuolli and Timoshenko theories) are defined by static equilibrium and kinematics [8], [9] or dynamic motion equations [10]. Some authors present this definition by means of compact energy equations [11], [12], [13]. These interpretations have permitted to reach accurate results, for some types of beams: for example, a circular arch element loaded in plane [14], [15], [16], [17], [18] and loaded perpendicular to its plane [19], parabolic and elliptical beams loaded in plane [20], [21], [22] or a helix uniformly loaded [23].

In this paper, the Finite Transfer Method (FTM) [24] is followed and applied to a system of differential equations, obtaining an incremental equation based on the transfer matrix. Fourth order Runge-Kutta approximation is adopted.

Using the preceding finite expression, both extremes are related, reaching a system of algebraic equations with constant dimension p regardless of the number of intervals.

The establishment of the problem is completed when the p elastic supports conditions-equations are incorporated. A final algebraic system of $2p$ order is reached and solved. Once values at the initial point are known, values at any point of the domain can be obtained.

The authors apply the FTM on the arbitrary curved beam model, by means of a unique system of twelve ordinary differential equations with boundary conditions [25].

An example is given to show the procedure exposed.

II. General Beam Equation: The Differential System

A curved beam is generated by a plane cross section whose centroid sweeps through all the points of an axis curve. The vector radius $\mathbf{r} = \mathbf{r}(s)$ expresses this curved line, where s (arc length of the centroid line) is the independent variable.

The reference coordinate system used here to represent the intervening known and unknown functions of the problem is the Frenet frame P_{tnb}. Its unit vectors tangent $\mathbf{t}$, normal $\mathbf{n}$ and binormal $\mathbf{b}$ are: $\mathbf{t}=D\mathbf{r}$; $\mathbf{n}=D^2\mathbf{r}/|D^2\mathbf{r}|$; $\mathbf{b}=\mathbf{t}\wedge\mathbf{n}$; where $D=d/ds$ is the derivative with respect to the parameter s. (Please refer to. [26] for another approach expressed in the global Cartesian coordinate system).

The natural equations of the centroid line are expressed by the flexion curvature χ and the torsion curvature τ.

$$\chi(s) = \sqrt{D^2\mathbf{r} \cdot D^2\mathbf{r}}\ ;$$

$$\tau(s) = D\mathbf{r} \wedge \left(D^2\mathbf{r} \cdot D^3\mathbf{r}\right) / \left(D^2\mathbf{r} \cdot D^2\mathbf{r}\right);$$

The Frenet-Serret formulas are [27].

$$D\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix} = \begin{bmatrix} 0 & \chi & 0\\ -\chi & 0 & \tau\\ 0 & -\tau & 0\end{bmatrix}\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix} \tag{1}$$

Manuscript received April 15th , 2010.

L. Gimena is with the Public University of Navarra, Dpt of Engineering Projects, Campus Arrosadía, Pamplona, Navarra, CP 31006, Spain (✆: +34 94816 9233; fax: ext. 9644; ✉: lazaro.gimena@unavarra.es).

P. Gonzaga is with the Public University of Navarra, Dpt of Engineering Projects, Campus Arrosadía, Pamplona, Navarra, CP 31006, Spain (✉: pedro.gonzaga@unavarra.es).

F. Gimena is with the Public University of Navarra, Dpt of Engineering Projects, Campus Arrosadía, Pamplona, Navarra, CP 31006, Spain (✉: faustino@unavarra.es).

F. Sarria is with the FSestructuras company, Executive Manager, Plaza Mayor, 19-21 bajo, Sarriguren, Navarra, CP 31621, Spain (✉: fsarria@fsestructuras.es).

ISBN: 978-988-18210-8-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Assuming the habitual principles and hypotheses (Euler-Bernoulli and Timoshenko classical beam theories) and considering the stresses associated with the normal cross-section (σ, τ_n, τ_b), the geometric characteristics of the section are: area $A(s)$, shearing coefficients $\alpha_n(s)$, $\alpha_{nb}(s)$, $\alpha_{bn}(s)$, $\alpha_b(s)$, and moments of inertia $I_t(s)$, $I_n(s)$, $I_b(s)$, $I_{nb}(s)$. Longitudinal $E(s)$ and transversal $G(s)$ elasticity moduli give the elastic properties of the material. Applying equilibrium and kinematics laws to an infinitesimal element of the curve, the system of differential equations governing the structural behaviour a spatially curved beam can be obtained [25] (see Equation at the bottom below page).

The first six rows of the system (Eq. 2) represent the equilibrium equations. The functions involved in the equilibrium equation are:

Internal forces $\mathbf{V} = N\,\mathbf{t} + V_n\,\mathbf{n} + V_b\,\mathbf{b}$, given by:

$\mathbf{V} = \int_A \sigma\, dA\,\mathbf{t} + \int_A \tau_n\, dA\,\mathbf{n} + \int_A \tau_b\, dA\,\mathbf{b}$

Internal moments $\mathbf{M} = T\,\mathbf{t} + M_n\,\mathbf{n} + M_b\,\mathbf{b}$, given by:

$\mathbf{M} = \int_A (\tau_b n - \tau_n b)\, dA\,\mathbf{t} + \int_A \sigma b\, dA\,\mathbf{n} - \int_A \sigma n\, dA\,\mathbf{b}$, and

Load force $\mathbf{q_V} = q_t\mathbf{t} + q_n\mathbf{n} + q_b\mathbf{b}$

Load moment $\mathbf{q_M} = m_t\mathbf{t} + m_n\mathbf{n} + m_b\mathbf{b}$

The last six rows of the system (Eq. 2) represent the kinematics equations.

Rotations $\boldsymbol{\theta} = \theta_t\,\mathbf{t} + \theta_n\,\mathbf{n} + \theta_b\,\mathbf{b}$

Displacements $\mathbf{u} = u\,\mathbf{t} + v\,\mathbf{n} + w\,\mathbf{b}$

Load rotation $\mathbf{q_\theta} = \Theta_t\,\mathbf{t} + \Theta_n\,\mathbf{n} + \Theta_b\,\mathbf{b}$

Load displacement $\mathbf{q_u} = \Delta_t\,\mathbf{t} + \Delta_n\,\mathbf{n} + \Delta_b\,\mathbf{b}$

The differential system can also be expressed in the vector-matrix form as follows:

$$\frac{d\mathbf{e}(s)}{ds} = \left[\mathbf{T}(s)\right]\mathbf{e}(s) + \mathbf{q}(s) \qquad (3)$$

Where $\mathbf{e}(s)=\left\{N,\, V_n,\, V_b,\, T,\, M_n,\, M_b,\, \theta_t,\, \theta_n,\, \theta_b,\, u,\, v,\, w\right\}^T$ is the state vector $\mathbf{e}(s)$ of internal forces and deflections at a point s of the beam element, named *effect at the section*, $\mathbf{q}(s)=\left\{-q_t,\, -q_n,\, -q_b,\, -m_t,\, -m_n,\, -m_b,\, \Theta_t,\, \Theta_n,\, \Theta_b,\, \Delta_t,\, \Delta_n,\, \Delta_b\right\}^T$ is the applied load, and

$$[\mathbf{T}(s)]=\begin{bmatrix}
0 & \chi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-\chi & 0 & \tau & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -\tau & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \chi & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & -\chi & 0 & \tau & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -1 & 0 & 0 & -\tau & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \frac{1}{GI_t} & 0 & 0 & 0 & \chi & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \frac{I_b}{E\left[I_nI_b-I_{nb}^2\right]} & \frac{I_{nb}}{E\left[I_nI_b-I_{nb}^2\right]} & -\chi & 0 & \tau & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \frac{I_{nb}}{E\left[I_nI_b-I_{nb}^2\right]} & \frac{I_n}{E\left[I_nI_b-I_{nb}^2\right]} & 0 & -\tau & 0 & 0 & 0 & 0\\
\frac{1}{EA} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \chi & 0\\
0 & \frac{\alpha_n}{GA} & \frac{\alpha_{nb}}{GA} & 0 & 0 & 0 & 0 & 0 & 1 & -\chi & 0 & \tau\\
0 & \frac{\alpha_{bn}}{GA} & \frac{\alpha_b}{GA} & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -\tau & 0
\end{bmatrix}$$

is the *Derivative Infinitesimal Transfer Matrix*.

III. Exact Analytical Solution

The exact analytical solution is given by [28]:

$$\mathbf{e}(s) = e^{\int_{s_\mathbf{I}}^{s}\left[\mathbf{T}(s)\right]ds}\left[\mathbf{e}(s_\mathbf{I}) + \int_{s_\mathbf{I}}^{s}\mathbf{q}(s)\, e^{-\int_{s_\mathbf{I}}^{s}\left[\mathbf{T}(s)\right]ds}\, ds\right] \qquad (4)$$

Or in compact form:

$$\mathbf{e}(s) = \left[\mathbf{T}\left(s_\mathbf{I}, s\right)\right]\mathbf{e}(s_\mathbf{I}) + \mathbf{q}\left(s_\mathbf{I}, s\right) \qquad (5)$$

Where, $\left[\mathbf{T}\left(s_\mathbf{I}, s\right)\right] = e^{\int_{s_\mathbf{I}}^{s}\left[\mathbf{T}(s)\right]ds}$ is the Transfer matrix from a general point s to the initial $\mathbf{I}$.

$\mathbf{q}\left(s_\mathbf{I}, s\right) = e^{\int_{s_\mathbf{I}}^{s}\left[\mathbf{T}(s)\right]ds}\int_{s_\mathbf{I}}^{s}\mathbf{q}(s)\, e^{-\int_{s_\mathbf{I}}^{s}\left[\mathbf{T}(s)\right]ds}\, ds$ is the load transmitted from initial $\mathbf{I}$ to a general point s.

$$\begin{array}{l}
DN - \chi V_n + q_t = 0\\
\chi N + DV_n - \tau V_b + q_n = 0\\
\tau V_n + DV_b + q_b = 0\\
DT - \chi M_n + m_t = 0\\
-V_b + \chi T + DM_n - \tau M_b + m_n = 0\\
V_n + \tau M_n + DM_b + m_b = 0\\
-\frac{T}{GI_t} + D\theta_t - \chi\theta_n - \Theta_t = 0\\
-\frac{M_nI_b}{E\left[I_nI_b-I_{nb}^2\right]} - \frac{M_bI_{nb}}{E\left[I_nI_b-I_{nb}^2\right]} + \chi\theta_t + D\theta_n - \tau\theta_b - \Theta_n = 0\\
-\frac{M_nI_{nb}}{E\left[I_nI_b-I_{nb}^2\right]} - \frac{M_bI_n}{E\left[I_nI_b-I_{nb}^2\right]} + \tau\theta_n + D\theta_b - \Theta_b = 0\\
-\frac{N}{EA} + Du - \chi v - \Delta_t = 0\\
-\frac{\alpha_nV_n}{GA} - \frac{\alpha_{nb}V_b}{GA} - \theta_b + \chi u + Dv - \tau w - \Delta_n = 0\\
-\frac{\alpha_{bn}V_n}{GA} - \frac{\alpha_bV_b}{GA} + \theta_n + \tau v + Dw - \Delta_b = 0
\end{array} \qquad (2)$$

Equation 2. Differential System for Spatially Curved Beams.

IV. EXACT ANALYTICAL TRANSFER MATRIX

Previous solution particularized for both extremes **I** and **II** of the curved beam, gives the next relation:

$$\mathbf{e}(s_{\mathbf{II}}) = e^{\int_{s_{\mathbf{I}}}^{s_{\mathbf{II}}}[\mathbf{T}(s)]ds}\left[\mathbf{e}(s_{\mathbf{I}}) + \int_{s_{\mathbf{I}}}^{s_{\mathbf{II}}} \mathbf{q}(s)\, e^{-\int_{s_{\mathbf{I}}}^{s_{\mathbf{II}}}[\mathbf{T}(s)]ds} ds\right] \quad (6)$$

or in compact form:

$$\mathbf{e}(s_{\mathbf{II}}) = [\mathbf{T}]\,\mathbf{e}(s_{\mathbf{I}}) + \mathbf{q} \quad (7)$$

where,

$$[\mathbf{T}] = \begin{bmatrix} [\mathbf{T}_{\mathbf{V_I}}^{\mathbf{V_{II}}}] & [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{T}_{\mathbf{V_I}}^{\mathbf{M_{II}}}] & [\mathbf{T}_{\mathbf{M_I}}^{\mathbf{M_{II}}}] & [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{T}_{\mathbf{V_I}}^{\mathbf{\theta_{II}}}] & [\mathbf{T}_{\mathbf{M_I}}^{\mathbf{\theta_{II}}}] & [\mathbf{T}_{\mathbf{\theta_I}}^{\mathbf{\theta_{II}}}] & [\mathbf{0}] \\ [\mathbf{T}_{\mathbf{V_I}}^{\mathbf{u_{II}}}] & [\mathbf{T}_{\mathbf{M_I}}^{\mathbf{u_{II}}}] & [\mathbf{T}_{\mathbf{\theta_I}}^{\mathbf{u_{II}}}] & [\mathbf{T}_{\mathbf{u_I}}^{\mathbf{u_{II}}}] \end{bmatrix}$$ is the Exact Analytical Transfer Matrix, and $\mathbf{q} = \left\{\mathbf{q}_{\mathbf{V_{I,II}}}{}^{T}, \mathbf{q}_{\mathbf{M_{I,II}}}{}^{T}, \mathbf{q}_{\mathbf{\theta_{I,II}}}{}^{T}, \mathbf{q}_{\mathbf{u_{I,II}}}{}^{T}\right\}^{T}$ the load transfer vector transferred.

V. NUMERICAL SOLUTION. FINITE TRANSFER METHOD *RK*4

The approximation of the differential system (Eq. 2) is given by:

$$\frac{d\mathbf{e}(s)}{dt} \cong \frac{\Delta\tilde{\mathbf{e}}(s_i)}{\Delta s} = \frac{\tilde{\mathbf{e}}(s_{i+1}) - \tilde{\mathbf{e}}(s_i)}{\Delta s} = \frac{\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4}{6}$$

being,

$$\mathbf{k}_1 = [\mathbf{T}_i]\tilde{\mathbf{e}}(s_i) + \mathbf{q}_i$$

$$\mathbf{k}_2 = [\mathbf{T}_{i+1/2}][\tilde{\mathbf{e}}(s_i) + \mathbf{k}_1 \Delta s/2] + \mathbf{q}_{i+1/2}$$

$$\mathbf{k}_3 = [\mathbf{T}_{i+1/2}][\tilde{\mathbf{e}}(s_i) + \mathbf{k}_2 \Delta s/2] + \mathbf{q}_{i+1/2}$$

$$\mathbf{k}_4 = [\mathbf{T}_{i+1}][\tilde{\mathbf{e}}(s_i) + \mathbf{k}_3 \Delta s] + \mathbf{q}_{i+1}$$

Assuming that approximated functions are:
$\mathbf{e}(s_{i+1}) \cong \tilde{\mathbf{e}}(s_{i+1})$; $\mathbf{e}(s_i) \cong \tilde{\mathbf{e}}(s_i)$

Thus, the *Finite Transfer Equation* of four order is:

$$\begin{aligned} \tilde{\mathbf{e}}(s_{i+1}) = & \left[[\mathbf{I}] + \left[[\mathbf{T}_{i+1}] + 4[\mathbf{T}_{i+1/2}] + [\mathbf{T}_i]\right]\Delta s/6 + \right. \\ & + \left[[\mathbf{T}_{i+1}][\mathbf{T}_{i+1/2}] + [\mathbf{T}_{i+1/2}]^2 + [\mathbf{T}_{i+1/2}][\mathbf{T}_i]\right]\Delta s^2/6 + \\ & + \left[[\mathbf{T}_{i+1}][\mathbf{T}_{i+1/2}]^2 + [\mathbf{T}_{i+1/2}]^2[\mathbf{T}_i]\right]\Delta s^3/12 + \\ & \left. + [\mathbf{T}_{i+1}][\mathbf{T}_{i+1/2}]^2[\mathbf{T}_i]\Delta s^4/24\right]\tilde{\mathbf{e}}(s_i) + \\ & + \left(\mathbf{q}_{i+1} + 4\mathbf{q}_{i+1/2} + \mathbf{q}_i\right)\Delta s/6 + \\ & + \left([\mathbf{T}_{i+1}]\mathbf{q}_{i+1/2} + [\mathbf{T}_{i+1/2}]\mathbf{q}_{i+1/2} + [\mathbf{T}_{i+1/2}]\mathbf{q}_i\right)\Delta s^2/6 + \\ & + \left([\mathbf{T}_{i+1}][\mathbf{T}_{i+1/2}]\mathbf{q}_{i+1/2} + [\mathbf{T}_{i+1/2}]^2\mathbf{q}_i\right)\Delta s^3/12 + \\ & + [\mathbf{T}_{i+1}][\mathbf{T}_{i+1/2}]^2\mathbf{q}_i\,\Delta s^4/24 = \\ = & [\mathbf{T_{RK4}}(s_i)]\tilde{\mathbf{e}}(s_i) + \mathbf{q_{RK4}}(s_i) \end{aligned}$$

Finally the general numerical solution is written:

$$\begin{aligned} \tilde{\mathbf{e}}(s_{i+1}) = & \left[\prod_{j=0}^{j=i}[\mathbf{T_{RK4}}(s_j)]\right]\tilde{\mathbf{e}}(s_{\mathbf{I}}) + \sum_{j=0}^{j=i}\left[\prod_{k=j+1}^{k=i}[\mathbf{T_{RK4}}(s_k)]\right]\mathbf{q_{RK4}}(s_j) = \\ = & \left[\mathbf{T_{RK4}}(s_{\mathbf{I}}, s_{i+1})\right]\tilde{\mathbf{e}}(s_{\mathbf{I}}) + \mathbf{q_{RK4}}(s_{\mathbf{I}}, s_{i+1}) \end{aligned}$$

with $\mathbf{e}(s_{\mathbf{I}}) \cong \tilde{\mathbf{e}}(s_{\mathbf{I}})$.

VI. NUMERICAL TRANSFER MATRIX

Establishing n intervals, both end points **I** and **II** of the curved line can be related:

$$\tilde{\mathbf{e}}(s_{\mathbf{II}}) = \left[\prod_{j=0}^{j=n-1}[\mathbf{T_{RK4}}(s_j)]\right]\tilde{\mathbf{e}}(s_{\mathbf{I}}) + \sum_{j=0}^{j=n-1}\left[\prod_{k=j+1}^{k=n-1}[\mathbf{T_{RK4}}(s_k)]\right]\mathbf{q_{RK4}}(s_j) \quad (8)$$

In compact form,

$$\tilde{\mathbf{e}}(s_{\mathbf{II}}) = [\mathbf{T_{RK4}}]\tilde{\mathbf{e}}(s_{\mathbf{I}}) + \mathbf{q_{RK4}}$$

Where $[\mathbf{T_{RK4}}]$ is the RK-4th order-Numerical Transfer Matrix and $\mathbf{q_{RK4}}$ the load transfer vector.

The Numerical Solution converges to the Analytical as was demonstrated [28]:

$\mathbf{e}(s_{\mathbf{II}}) \cong \tilde{\mathbf{e}}(s_{\mathbf{II}})$, $[\mathbf{T_{RK4}}] \cong [\mathbf{T}]$ and $\mathbf{q_{RK4}} \cong \mathbf{q}$.

VII. STIFFNESS MATRIX

The former terms of the above equation (Eq. 6) o (Eq. 8) are subsequently arrayed [29], yielding:

$$\begin{bmatrix} [\mathbf{T}_{\mathbf{V_I}}^{\mathbf{V_{II}}}] & [\mathbf{0}] & [\mathbf{I}] & [\mathbf{0}] \\ [\mathbf{T}_{\mathbf{V_I}}^{\mathbf{M_{II}}}] & [\mathbf{T}_{\mathbf{M_I}}^{\mathbf{M_{II}}}] & [\mathbf{0}] & [\mathbf{I}] \\ [\mathbf{T}_{\mathbf{V_I}}^{\mathbf{\theta_{II}}}] & [\mathbf{T}_{\mathbf{M_I}}^{\mathbf{\theta_{II}}}] & [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{T}_{\mathbf{V_I}}^{\mathbf{u_{II}}}] & [\mathbf{T}_{\mathbf{M_I}}^{\mathbf{u_{II}}}] & [\mathbf{0}] & [\mathbf{0}] \end{bmatrix} \begin{bmatrix} -\mathbf{V_I} \\ -\mathbf{M_I} \\ \mathbf{V_{II}} \\ \mathbf{M_{II}} \end{bmatrix} = \begin{bmatrix} [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{T}_{\mathbf{\theta_I}}^{\mathbf{\theta_{II}}}] & [\mathbf{0}] & -[\mathbf{I}] \\ [\mathbf{T}_{\mathbf{u_I}}^{\mathbf{u_{II}}}] & [\mathbf{T}_{\mathbf{\theta_I}}^{\mathbf{u_{II}}}] & -[\mathbf{I}] & [\mathbf{0}] \end{bmatrix} \begin{bmatrix} \mathbf{u_I} \\ \mathbf{\theta_I} \\ \mathbf{u_{II}} \\ \mathbf{\theta_{II}} \end{bmatrix} + \begin{bmatrix} \mathbf{q}_{\mathbf{V_{I,II}}} \\ \mathbf{q}_{\mathbf{M_{I,II}}} \\ \mathbf{q}_{\mathbf{\theta_{I,II}}} \\ \mathbf{q}_{\mathbf{u_{I,II}}} \end{bmatrix} \quad (9)$$

The stiffness matrix is determined, with this simple operation:

$$\begin{bmatrix} -\mathbf{V_I} \\ -\mathbf{M_I} \\ \mathbf{V_{II}} \\ \mathbf{M_{II}} \end{bmatrix} = \begin{bmatrix} [\mathbf{T}_{\mathbf{V_I}}^{\mathbf{V_{II}}}] & [\mathbf{0}] & [\mathbf{I}] & [\mathbf{0}] \\ [\mathbf{T}_{\mathbf{V_I}}^{\mathbf{M_{II}}}] & [\mathbf{T}_{\mathbf{M_I}}^{\mathbf{M_{II}}}] & [\mathbf{0}] & [\mathbf{I}] \\ [\mathbf{T}_{\mathbf{V_I}}^{\mathbf{\theta_{II}}}] & [\mathbf{T}_{\mathbf{M_I}}^{\mathbf{\theta_{II}}}] & [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{T}_{\mathbf{V_I}}^{\mathbf{u_{II}}}] & [\mathbf{T}_{\mathbf{M_I}}^{\mathbf{u_{II}}}] & [\mathbf{0}] & [\mathbf{0}] \end{bmatrix}^{-1} \left[\begin{bmatrix} [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{T}_{\mathbf{\theta_I}}^{\mathbf{\theta_{II}}}] & [\mathbf{0}] & -[\mathbf{I}] \\ [\mathbf{T}_{\mathbf{u_I}}^{\mathbf{u_{II}}}] & [\mathbf{T}_{\mathbf{\theta_I}}^{\mathbf{u_{II}}}] & -[\mathbf{I}] & [\mathbf{0}] \end{bmatrix} \begin{bmatrix} \mathbf{u_I} \\ \mathbf{\theta_I} \\ \mathbf{u_{II}} \\ \mathbf{\theta_{II}} \end{bmatrix} + \begin{bmatrix} \mathbf{q}_{\mathbf{V_{I,II}}} \\ \mathbf{q}_{\mathbf{M_{I,II}}} \\ \mathbf{q}_{\mathbf{\theta_{I,II}}} \\ \mathbf{q}_{\mathbf{u_{I,II}}} \end{bmatrix}\right] \quad (10)$$

Expressed in a compact form it can be written as follows:

$$\mathbf{f} = [\mathbf{K}]\boldsymbol{\delta} + \mathbf{q_K} \quad (11)$$

Being,

$\mathbf{f}$ and $\boldsymbol{\delta}$ vectors of reactions and displacements unknowns at both ends,

$$[\mathbf{K}] = \begin{bmatrix} [\mathbf{K}_{\mathbf{V_I}}^{\mathbf{u_I}}] & [\mathbf{K}_{\mathbf{V_I}}^{\mathbf{\theta_I}}] & [\mathbf{K}_{\mathbf{V_I}}^{\mathbf{u_{II}}}] & [\mathbf{K}_{\mathbf{V_I}}^{\mathbf{\theta_{II}}}] \\ [\mathbf{K}_{\mathbf{M_I}}^{\mathbf{u_I}}] & [\mathbf{K}_{\mathbf{M_I}}^{\mathbf{\theta_I}}] & [\mathbf{K}_{\mathbf{M_I}}^{\mathbf{u_{II}}}] & [\mathbf{K}_{\mathbf{M_I}}^{\mathbf{\theta_{II}}}] \\ [\mathbf{K}_{\mathbf{V_{II}}}^{\mathbf{u_I}}] & [\mathbf{K}_{\mathbf{V_{II}}}^{\mathbf{\theta_I}}] & [\mathbf{K}_{\mathbf{V_{II}}}^{\mathbf{u_{II}}}] & [\mathbf{K}_{\mathbf{V_{II}}}^{\mathbf{\theta_{II}}}] \\ [\mathbf{K}_{\mathbf{M_{II}}}^{\mathbf{u_I}}] & [\mathbf{K}_{\mathbf{M_{II}}}^{\mathbf{\theta_I}}] & [\mathbf{K}_{\mathbf{M_{II}}}^{\mathbf{u_{II}}}] & [\mathbf{K}_{\mathbf{M_{II}}}^{\mathbf{\theta_{II}}}] \end{bmatrix}$$ the stiffness matrix and

$\mathbf{q_K} = \left\{\mathbf{q}_{\mathbf{V_{I,II}}}^{\mathbf{K}}{}^{T}, \mathbf{q}_{\mathbf{M_{I,II}}}^{\mathbf{K}}{}^{T}, \mathbf{q}_{\mathbf{\theta_{I,II}}}^{\mathbf{K}}{}^{T}, \mathbf{q}_{\mathbf{u_{I,II}}}^{\mathbf{K}}{}^{T}\right\}^{T}$ the equivalent load vector.

Note, that in general, the isolated beam has not been yet supported, so there are twenty four unknowns, twelve of forces and moments and other twelve in rotations and displacements:

$$\left[\begin{array}{c|c} [\mathbf{I}] & -[\mathbf{K}] \\ \hline [\mathbf{0}] & [\mathbf{0}] \end{array}\right] \begin{bmatrix} \mathbf{f} \\ \boldsymbol{\delta} \end{bmatrix} = \begin{bmatrix} \mathbf{q_K} \\ \mathbf{0} \end{bmatrix} \quad (12)$$

It is very interesting to write the whole algebraic system with all the intervening unknowns in a single vector as this form, because it will be better to implement the elastic conditions, as it is shown later.

ISBN: 978-988-18210-8-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

VIII. Elastic Support Equations to Assemble

A. Initial support

Equilibrium and compatibility relations applied in forces and displacements at the elastic initial support extreme of the beam, yields:

$$-\left[[\mathbf{I}]-\left[\boldsymbol{\alpha}_{\mathbf{V}_\mathbf{I}}^{\mathbf{u}_\mathbf{I}}\right]\right](\mathbf{V}_\mathbf{I}+\mathbf{Q}_\mathbf{I})=[\mathbf{I}]\left[\mathbf{K}_{\mathbf{V}_\mathbf{I}}^{\mathbf{u}_\mathbf{I}}\right]\left[\boldsymbol{\alpha}_{\mathbf{V}_\mathbf{I}}^{\mathbf{u}_\mathbf{I}}\right](\mathbf{u}_\mathbf{I}-\boldsymbol{\Lambda}_\mathbf{I}) \quad (13)$$

where, $\left[\boldsymbol{\alpha}_{\mathbf{V}_\mathbf{I}}^{\mathbf{u}_\mathbf{I}}\right]=\begin{bmatrix}\alpha_{N_\mathrm{I}}^{u_\mathrm{I}} & 0 & 0\\ 0 & \alpha_{V_{n\mathrm{I}}}^{v_\mathrm{I}} & 0\\ 0 & 0 & \alpha_{V_{b\mathrm{I}}}^{w_\mathrm{I}}\end{bmatrix}$ is the matrix of the displacements factors of restraints at the elastic support in the initial extreme, $\mathbf{Q}_\mathbf{I}=\left\{Q_t^\mathrm{I},\ Q_n^\mathrm{I},\ Q_b^\mathrm{I}\right\}^T$ is the punctual forces loads applied and $\boldsymbol{\Lambda}_\mathbf{I}=\left\{\Lambda_t^\mathrm{I},\ \Lambda_n^\mathrm{I},\ \Lambda_b^\mathrm{I}\right\}^T$ is the imposed (if applied) punctual displacements at the initial point.

If the support has a rigidity that restrains the longitudinal displacement K_{Nu}^I it can be determined its factor by the next expression:

$$\alpha_{N_\mathrm{I}}^{u_\mathrm{I}}=\frac{K_{Nu}^\mathrm{I}}{K_{Nu}^\mathrm{I}-K_{N_\mathrm{I}}^{u_\mathrm{I}}} \quad (14)$$

Rest of α factors can be obtained in the same manner.

For the rotation the elastic support relation at the initial point, it can be written as:

$$-\left[[\mathbf{I}]-\left[\boldsymbol{\alpha}_{\mathbf{M}_\mathbf{I}}^{\boldsymbol{\theta}_\mathbf{I}}\right]\right](\mathbf{M}_\mathbf{I}+\boldsymbol{\Pi}_\mathbf{I})=[\mathbf{I}]\left[\mathbf{K}_{\mathbf{M}_\mathbf{I}}^{\boldsymbol{\theta}_\mathbf{I}}\right]\left[\boldsymbol{\alpha}_{\mathbf{M}_\mathbf{I}}^{\boldsymbol{\theta}_\mathbf{I}}\right](\boldsymbol{\theta}_\mathbf{I}-\mathbf{O}_\mathbf{I}) \quad (15)$$

being, $\left[\boldsymbol{\alpha}_{\mathbf{M}_\mathbf{I}}^{\boldsymbol{\theta}_\mathbf{I}}\right]=\begin{bmatrix}\alpha_{T_\mathrm{I}}^{\theta_{t\mathrm{I}}} & 0 & 0\\ 0 & \alpha_{M_{n\mathrm{I}}}^{\theta_{n\mathrm{I}}} & 0\\ 0 & 0 & \alpha_{M_{b\mathrm{I}}}^{\theta_{b\mathrm{I}}}\end{bmatrix}$ is the matrix of the rotation factors of restraints at the elastic support in the initial extreme, $\boldsymbol{\Pi}_\mathbf{I}=\left\{\Pi_t^\mathrm{I},\ \Pi_n^\mathrm{I},\ \Pi_b^\mathrm{I}\right\}^T$ is the punctual moment actions exerted and $\mathbf{O}_\mathbf{I}=\left\{O_t^\mathrm{I},\ O_n^\mathrm{I},\ O_b^\mathrm{I}\right\}^T$ is the imposed (if applied) punctual displacements at the initial point.

B. Final support

In a similar way, extreme elastic conditions can be derivate for the final end of the spatially curved beam. First, on forces and displacements:

$$\left[[\mathbf{I}]-\left[\boldsymbol{\alpha}_{\mathbf{V}_\mathbf{II}}^{\mathbf{u}_\mathbf{II}}\right]\right](\mathbf{V}_\mathbf{II}-\mathbf{Q}_\mathbf{II})=[\mathbf{I}]\left[\mathbf{K}_{\mathbf{V}_\mathbf{II}}^{\mathbf{u}_\mathbf{II}}\right]\left[\boldsymbol{\alpha}_{\mathbf{V}_\mathbf{II}}^{\mathbf{u}_\mathbf{II}}\right](\mathbf{u}_\mathbf{II}-\boldsymbol{\Lambda}_\mathbf{II}) \quad (16)$$

where, $\left[\boldsymbol{\alpha}_{\mathbf{V}_\mathbf{II}}^{\mathbf{u}_\mathbf{II}}\right]=\begin{bmatrix}\alpha_{N_\mathrm{II}}^{u_\mathrm{II}} & 0 & 0\\ 0 & \alpha_{V_{n\mathrm{II}}}^{v_\mathrm{II}} & 0\\ 0 & 0 & \alpha_{V_{b\mathrm{II}}}^{w_\mathrm{II}}\end{bmatrix}$ is the matrix of the displacements factors of restraints at the elastic support in the final extreme, $\mathbf{Q}_\mathbf{II}=\left\{Q_t^\mathrm{II},\ Q_n^\mathrm{II},\ Q_b^\mathrm{II}\right\}^T$ is the punctual forces loads applied and $\boldsymbol{\Lambda}_\mathbf{II}=\left\{\Lambda_t^\mathrm{II},\ \Lambda_n^\mathrm{II},\ \Lambda_b^\mathrm{II}\right\}^T$ is the imposed (if applied) punctual displacements at the final point.

Secondly, with respect the rotations, it is obtained:

$$\left[[\mathbf{I}]-\left[\boldsymbol{\alpha}_{\mathbf{M}_\mathbf{II}}^{\boldsymbol{\theta}_\mathbf{II}}\right]\right](\mathbf{M}_\mathbf{II}-\boldsymbol{\Pi}_\mathbf{II})=[\mathbf{I}]\left[\mathbf{K}_{\mathbf{M}_\mathbf{II}}^{\boldsymbol{\theta}_\mathbf{II}}\right]\left[\boldsymbol{\alpha}_{\mathbf{M}_\mathbf{II}}^{\boldsymbol{\theta}_\mathbf{II}}\right](\boldsymbol{\theta}_\mathbf{II}-\mathbf{O}_\mathbf{II}) \quad (17)$$

where, $\left[\boldsymbol{\alpha}_{\mathbf{M}_\mathbf{II}}^{\boldsymbol{\theta}_\mathbf{II}}\right]=\begin{bmatrix}\alpha_{T_\mathrm{II}}^{\theta_{t\mathrm{II}}} & 0 & 0\\ 0 & \alpha_{M_{n\mathrm{II}}}^{\theta_{n\mathrm{II}}} & 0\\ 0 & 0 & \alpha_{M_{b\mathrm{II}}}^{\theta_{b\mathrm{II}}}\end{bmatrix}$ is the matrix of the rotation factors of restraints at the elastic support in the final extreme, $\boldsymbol{\Pi}_\mathbf{II}=\left\{\Pi_t^\mathrm{II},\ \Pi_n^\mathrm{II},\ \Pi_b^\mathrm{II}\right\}^T$ is the punctual forces loads applied and $\mathbf{O}_\mathbf{II}=\left\{O_t^\mathrm{II},\ O_n^\mathrm{II},\ O_b^\mathrm{II}\right\}^T$ is the imposed (if applied) punctual rotations at the final point.

If we joint previous equations (Eq. 13), (Eq. 15), (Eq. 16) and (Eq. 17) in a single matrix equation, we reach the following expression of elastic stiffness conditions:

$$[[\mathbf{I}]-[\boldsymbol{\alpha}]](\mathbf{f}-\mathbf{Q})=[\boldsymbol{\alpha}](\boldsymbol{\delta}-\boldsymbol{\Lambda}) \quad (18)$$

Where, $[\boldsymbol{\alpha}]=\begin{bmatrix}\left[\boldsymbol{\alpha}_{\mathbf{Vu}}^\mathbf{I}\right] & [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}]\\ [\mathbf{0}] & \left[\boldsymbol{\alpha}_{\mathbf{M}\boldsymbol{\theta}}^\mathbf{I}\right] & [\mathbf{0}] & [\mathbf{0}]\\ [\mathbf{0}] & [\mathbf{0}] & \left[\boldsymbol{\alpha}_{\mathbf{Vu}}^\mathbf{II}\right] & [\mathbf{0}]\\ [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] & \left[\boldsymbol{\alpha}_{\mathbf{M}\boldsymbol{\theta}}^\mathbf{II}\right]\end{bmatrix}$ is the restraints support matrix, $\mathbf{Q}=\left\{\mathbf{Q}_\mathbf{I}{}^T,\boldsymbol{\Pi}_\mathbf{I}{}^T,\mathbf{Q}_\mathbf{II}{}^T,\boldsymbol{\Pi}_\mathbf{II}{}^T\right\}^T$ is the complete vector of forces and moments $\boldsymbol{\Lambda}=\left\{\boldsymbol{\Lambda}_\mathbf{I}{}^T,\mathbf{O}_\mathbf{I}{}^T,\boldsymbol{\Lambda}_\mathbf{II}{}^T,\mathbf{O}_\mathbf{II}{}^T\right\}^T$ and is the punctual loads of rotations and displacements in both ends.

Former equation Eq.18, can be written as:

$$\left[\begin{array}{cc}[\mathbf{0}] & [\mathbf{0}]\\ \hline [[\mathbf{I}]-[\boldsymbol{\alpha}]] & -[\boldsymbol{\alpha}]\end{array}\right]\left[\begin{array}{c}\mathbf{f}\\ \hline \boldsymbol{\delta}\end{array}\right]=\left[\begin{array}{c}\mathbf{0}\\ \hline [[\mathbf{I}]-[\boldsymbol{\alpha}]]\mathbf{Q}-[\boldsymbol{\alpha}]\boldsymbol{\Lambda}\end{array}\right] \quad (19)$$

As we comment in section VII where the stiffness matrix was develop and demonstrated in a particular approach, it is important to write the whole algebraic system with all the intervening unknowns in a single vector, because it will be solved at a single step without adding other sophisticated and artificial procedure. This method, as it can be rapidly gotten, is general, easier and flexible to implement the elastic conditions and present several advantages from other used.

IX. Full System of Equations of the Model

Stiffness equation (Eq. 11) contains twelve algebraic equations, which relates reactions and displacements in both ends of the beam. The other twelve equations necessaries to solve the problem are given by the elastic condition of the support in Eq. 18. As we mentioned formerly, now we can directly sum the expressions derived (Eq. 12) and (Eq. 19) to obtain the most general system, complete with twenty four equations, which are written down here:

$$\left[\begin{array}{cc}[\mathbf{I}] & -[\mathbf{K}]\\ \hline [[\mathbf{I}]-[\boldsymbol{\alpha}]] & -[\boldsymbol{\alpha}]\end{array}\right]\left[\begin{array}{c}\mathbf{f}\\ \hline \boldsymbol{\delta}\end{array}\right]=\left[\begin{array}{c}\mathbf{q}_\mathbf{K}\\ \hline [[\mathbf{I}]-[\boldsymbol{\alpha}]]\mathbf{Q}-[\boldsymbol{\alpha}]\boldsymbol{\Lambda}\end{array}\right] \quad (20)$$

The algebraic system can be solved directly by obtaining the inverse of the this matrix, given the vector of reactions and displacements straightforward by:

$$\left[\begin{array}{c}\mathbf{f}\\ \hline \boldsymbol{\delta}\end{array}\right]=\left[\begin{array}{cc}[\mathbf{I}] & -[\mathbf{K}]\\ \hline [[\mathbf{I}]-[\boldsymbol{\alpha}]] & -[\boldsymbol{\alpha}]\end{array}\right]^{-1}\left[\begin{array}{c}\mathbf{q}_\mathbf{K}\\ \hline [[\mathbf{I}]-[\boldsymbol{\alpha}]]\mathbf{Q}-[\boldsymbol{\alpha}]\boldsymbol{\Lambda}\end{array}\right] \quad (21)$$

In spite of the singularity of stiffness matrix, thus with no inverse, if the structure is stable, always a solution is provided because of the compatibility determinate system.

ISBN: 978-988-18210-8-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

X. EXAMPLE

A. Bending in a beam. General solution.

For simplicity, the example to be considered here is the particular of a bar under bending effect, but it is suitable to show the procedure to solve the whole problem exposed formerly.

Let's consider a straight beam with uniform force load and elastic support as show in next figure:

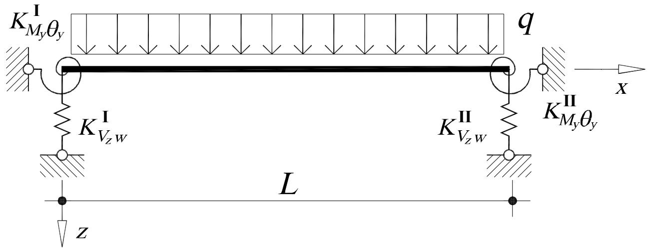

Figure 1. Bending with elastic support conditions.

The differential system in this case will be a particular case of the general equation (Eq.2) given:

$$
\begin{aligned}
&\frac{dV_z}{dx} && + q_z = 0\\
&-V_z + \frac{dM_y}{dx} && + m_y = 0\\
&-\frac{M_y}{EI_y} + \frac{d\theta_y}{dx} && -\Theta_y = 0\\
&\theta_y + \frac{dw}{dx} && -\Delta_z = 0
\end{aligned}
$$

Integrating with only force load applied, it is obtained:

$$
\begin{aligned}
V_z(x) &= V_{z0} && -qx\\
M_y(x) &= V_{z0}x + M_{y0} && -\frac{qx^2}{2}\\
\theta_y(x) &= V_{z0}\frac{x^2}{2EI_y} + M_{y0}\frac{x}{EI_y} + \theta_{y0} && -\frac{qx^3}{6EI_y}\\
w(x) &= -V_{z0}\frac{x^3}{6EI_y} - M_{y0}\frac{x^2}{2EI_y} - \theta_{y0}x + w_0 + \frac{qx^4}{24EI_y}
\end{aligned}
$$

In matricial form we get,

$$
\begin{bmatrix} V_{zL}\\ M_{yL}\\ \theta_{yL}\\ w_L \end{bmatrix}
=
\begin{bmatrix}
1 & 0 & 0 & 0\\
L & 1 & 0 & 0\\
\frac{L^2}{2EI_y} & \frac{L}{EI_y} & 1 & 0\\
-\frac{L^3}{6EI_y} & -\frac{L^2}{2EI_y} & -L & 1
\end{bmatrix}
\begin{bmatrix} V_{z0}\\ M_{y0}\\ \theta_{y0}\\ w_0 \end{bmatrix}
+
\begin{bmatrix} -qL\\ -\frac{qL^2}{2}\\ -\frac{qL^3}{6EI_y}\\ \frac{qL^4}{24EI_y} \end{bmatrix}
$$

which is the expression in transference as mentioned in Eq. 6 if analytically or Eq. 8 if numerically.

Reordering reactions and displacements to each member as was shown in Eq. 9, yields:

$$
\begin{bmatrix}
1 & 0 & 1 & 0\\
L & 1 & 0 & 1\\
\frac{L^2}{2EI_y} & \frac{L}{EI_y} & 0 & 0\\
\frac{L^3}{6EI_y} & \frac{L^2}{2EI_y} & 0 & 0
\end{bmatrix}
\begin{bmatrix} -V_{z0}\\ -M_{y0}\\ V_{zL}\\ M_{yL} \end{bmatrix}
=
\begin{bmatrix}
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0\\
0 & 1 & 0 & -1\\
1 & -L & -1 & 0
\end{bmatrix}
\begin{bmatrix} w_0\\ \theta_{y0}\\ w_L\\ \theta_{yL} \end{bmatrix}
+
\begin{bmatrix} -qL\\ -\frac{qL^2}{2}\\ -\frac{qL^3}{6EI_y}\\ \frac{qL^4}{24EI_y} \end{bmatrix}
$$

After inverting the former matrix and multiplying, we put all the unknowns in a single vector as said in Eq. 12:

$$
\left[\begin{array}{cccc:cccc}
1 & 0 & 0 & 0 & -\frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} & \frac{12EI_y}{L^3} & \frac{6EI_y}{L^2}\\
0 & 1 & 0 & 0 & \frac{6EI_y}{L^2} & -\frac{4EI_y}{L} & -\frac{6EI_y}{L^2} & -\frac{2EI_y}{L}\\
0 & 0 & 1 & 0 & \frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2} & -\frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2}\\
0 & 0 & 0 & 1 & \frac{6EI_y}{L^2} & -\frac{2EI_y}{L} & -\frac{6EI_y}{L^2} & -\frac{4EI_y}{L}\\
\hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\begin{bmatrix} -V_{z0}\\ -M_{y0}\\ V_{zL}\\ M_{yL}\\ w_0\\ \theta_{y0}\\ w_L\\ \theta_{yL} \end{bmatrix}
=
\left[\begin{array}{c}
-\frac{qL}{2}\\ \frac{qL^2}{12}\\ -\frac{qL}{2}\\ -\frac{qL^2}{12}\\ \hdashline 0\\ 0\\ 0\\ 0
\end{array}\right]
$$

It is easy then, to write the general elastic supports conditions:

$$
\left[\begin{array}{cccc:cccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\hdashline
1-\alpha^{w_{\mathrm{I}}}_{V_{z\mathrm{I}}} & 0 & 0 & 0 & -\alpha^{w_{\mathrm{I}}}_{V_{z\mathrm{I}}} & 0 & 0 & 0\\
0 & 1-\alpha^{\theta_{y\mathrm{I}}}_{M_{y\mathrm{I}}} & 0 & 0 & 0 & -\alpha^{\theta_{y\mathrm{I}}}_{M_{y\mathrm{I}}} & 0 & 0\\
0 & 0 & 1-\alpha^{w_{\mathrm{II}}}_{V_{z\mathrm{II}}} & 0 & 0 & 0 & -\alpha^{w_{\mathrm{II}}}_{V_{z\mathrm{II}}} & 0\\
0 & 0 & 0 & 1-\alpha^{\theta_{y\mathrm{II}}}_{M_{y\mathrm{II}}} & 0 & 0 & 0 & -\alpha^{\theta_{y\mathrm{II}}}_{M_{y\mathrm{II}}}
\end{array}\right]
\begin{bmatrix} -V_{z0}\\ -M_{y0}\\ V_{zL}\\ M_{yL}\\ w_0\\ \theta_{y0}\\ w_L\\ \theta_{yL} \end{bmatrix}
=
\left[\begin{array}{c}
0\\ 0\\ 0\\ 0\\ \hline 0\\ 0\\ 0\\ 0
\end{array}\right]
$$

The two above algebraic systems can be directly summed and yields the system eight by eight, to be solved:

$$
\left[\begin{array}{cccc:cccc}
1 & 0 & 0 & 0 & -\frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} & \frac{12EI_y}{L^3} & \frac{6EI_y}{L^2}\\
0 & 1 & 0 & 0 & \frac{6EI_y}{L^2} & -\frac{4EI_y}{L} & -\frac{6EI_y}{L^2} & -\frac{2EI_y}{L}\\
0 & 0 & 1 & 0 & \frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2} & -\frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2}\\
0 & 0 & 0 & 1 & \frac{6EI_y}{L^2} & -\frac{2EI_y}{L} & -\frac{6EI_y}{L^2} & -\frac{4EI_y}{L}\\
\hdashline
1-\alpha^{w_{\mathrm{I}}}_{V_{z\mathrm{I}}} & 0 & 0 & 0 & -\alpha^{w_{\mathrm{I}}}_{V_{z\mathrm{I}}} & 0 & 0 & 0\\
0 & 1-\alpha^{\theta_{y\mathrm{I}}}_{M_{y\mathrm{I}}} & 0 & 0 & 0 & -\alpha^{\theta_{y\mathrm{I}}}_{M_{y\mathrm{I}}} & 0 & 0\\
0 & 0 & 1-\alpha^{w_{\mathrm{II}}}_{V_{z\mathrm{II}}} & 0 & 0 & 0 & -\alpha^{w_{\mathrm{II}}}_{V_{z\mathrm{II}}} & 0\\
0 & 0 & 0 & 1-\alpha^{\theta_{y\mathrm{II}}}_{M_{y\mathrm{II}}} & 0 & 0 & 0 & -\alpha^{\theta_{y\mathrm{II}}}_{M_{y\mathrm{II}}}
\end{array}\right]
\begin{bmatrix} -V_{z0}\\ -M_{y0}\\ V_{zL}\\ M_{yL}\\ w_0\\ \theta_{y0}\\ w_L\\ \theta_{yL} \end{bmatrix}
=
\left[\begin{array}{c}
-\frac{qL}{2}\\ \frac{qL^2}{12}\\ -\frac{qL}{2}\\ -\frac{qL^2}{12}\\ \hdashline 0\\ 0\\ 0\\ 0
\end{array}\right]
$$

The solution is written directly as:

$$
\begin{bmatrix} -V_{z0}\\ -M_{y0}\\ V_{zL}\\ M_{yL}\\ w_0\\ \theta_{y0}\\ w_L\\ \theta_{yL} \end{bmatrix}
=
\left[\begin{array}{cccc:cccc}
1 & 0 & 0 & 0 & -\frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} & \frac{12EI_y}{L^3} & \frac{6EI_y}{L^2}\\
0 & 1 & 0 & 0 & \frac{6EI_y}{L^2} & -\frac{4EI_y}{L} & -\frac{6EI_y}{L^2} & -\frac{2EI_y}{L}\\
0 & 0 & 1 & 0 & \frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2} & -\frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2}\\
0 & 0 & 0 & 1 & \frac{6EI_y}{L^2} & -\frac{2EI_y}{L} & -\frac{6EI_y}{L^2} & -\frac{4EI_y}{L}\\
\hdashline
1-\alpha^{w_{\mathrm{I}}}_{V_{z\mathrm{I}}} & 0 & 0 & 0 & -\alpha^{w_{\mathrm{I}}}_{V_{z\mathrm{I}}} & 0 & 0 & 0\\
0 & 1-\alpha^{\theta_{y\mathrm{I}}}_{M_{y\mathrm{I}}} & 0 & 0 & 0 & -\alpha^{\theta_{y\mathrm{I}}}_{M_{y\mathrm{I}}} & 0 & 0\\
0 & 0 & 1-\alpha^{w_{\mathrm{II}}}_{V_{z\mathrm{II}}} & 0 & 0 & 0 & -\alpha^{w_{\mathrm{II}}}_{V_{z\mathrm{II}}} & 0\\
0 & 0 & 0 & 1-\alpha^{\theta_{y\mathrm{II}}}_{M_{y\mathrm{II}}} & 0 & 0 & 0 & -\alpha^{\theta_{y\mathrm{II}}}_{M_{y\mathrm{II}}}
\end{array}\right]^{-1}
\left[\begin{array}{c}
-\frac{qL}{2}\\ \frac{qL^2}{12}\\ -\frac{qL}{2}\\ -\frac{qL^2}{12}\\ \hdashline 0\\ 0\\ 0\\ 0
\end{array}\right]
$$

Note that all possible elastic supports can be considered in

ISBN: 978-988-18210-8-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

this manner. Also, if complete restraints are imposed, as for example, fixed-fixed, is as simple as substituting the actual values of the factors α's, and get the next expression:

$$\begin{bmatrix} -V_{z0} \\ -M_{y0} \\ V_{zL} \\ M_{yL} \\ w_0 \\ \theta_{y0} \\ w_L \\ \theta_{yL} \end{bmatrix} = \left[\begin{array}{cccc|cccc} 1&0&0&0 & -\frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} & \frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} \\ 0&1&0&0 & \frac{6EI_y}{L^2} & -\frac{4EI_y}{L} & -\frac{6EI_y}{L^2} & -\frac{2EI_y}{L} \\ 0&0&1&0 & \frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2} & -\frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2} \\ 0&0&0&1 & \frac{6EI_y}{L^2} & -\frac{2EI_y}{L} & -\frac{6EI_y}{L^2} & -\frac{4EI_y}{L} \\ \hline 0&0&0&0 & -1 & 0 & 0 & 0 \\ 0&0&0&0 & 0 & -1 & 0 & 0 \\ 0&0&0&0 & 0 & 0 & -1 & 0 \\ 0&0&0&0 & 0 & 0 & 0 & -1 \end{array}\right]^{-1} \left[\begin{array}{c} -\frac{qL}{2} \\ \frac{qL^2}{12} \\ -\frac{qL}{2} \\ -\frac{qL^2}{12} \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

It results that, the inverse matrix is the same, so reactions and displacements unknowns are directly obtained:

$$\begin{bmatrix} -V_{z0} \\ -M_{y0} \\ V_{zL} \\ M_{yL} \\ w_0 \\ \theta_{y0} \\ w_L \\ \theta_{yL} \end{bmatrix} = \left[\begin{array}{cccc|cccc} 1&0&0&0 & -\frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} & \frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} \\ 0&1&0&0 & \frac{6EI_y}{L^2} & -\frac{4EI_y}{L} & -\frac{6EI_y}{L^2} & -\frac{2EI_y}{L} \\ 0&0&1&0 & \frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2} & -\frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2} \\ 0&0&0&1 & \frac{6EI_y}{L^2} & -\frac{2EI_y}{L} & -\frac{6EI_y}{L^2} & -\frac{4EI_y}{L} \\ \hline 0&0&0&0 & -1 & 0 & 0 & 0 \\ 0&0&0&0 & 0 & -1 & 0 & 0 \\ 0&0&0&0 & 0 & 0 & -1 & 0 \\ 0&0&0&0 & 0 & 0 & 0 & -1 \end{array}\right] \left[\begin{array}{c} -\frac{qL}{2} \\ \frac{qL^2}{12} \\ -\frac{qL}{2} \\ -\frac{qL^2}{12} \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array}\right] = \left[\begin{array}{c} -\frac{qL}{2} \\ \frac{qL^2}{12} \\ -\frac{qL}{2} \\ -\frac{qL^2}{12} \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

Other type of combinations of different support can easily be implemented and the system solved in the same way.

XI. Conclusions

It is presented the general system of differential equations that governs the behaviour of a spatially curved beam (Eq.2). This system can be solved either by an analytical or numerical method. Authors presented in last conferences, a new numerical method, the Finite Transfer Method (FTM) to solve a linear system of ordinary differential equations, how to apply general boundary conditions in equation form, and determine in the limit the exact analytical solution as well. The Finite Transfer Method exposed seems to be the most suitable to reach the desire result. The Transfer Matrix (exact or numerical) is directly reached. Rearranging it in a new expression, Stiffness Matrix expression is derived consequently. Since analytical solution is the limit of the numerical procedure proposed, exact expressions and derivations of transfer and stiffness matrices come up.

It is important to note that the algebraic system is extended to twenty four equations to let space for next equations of support. That permits the generality of the statement of the problem. This treatment of the problem is new and intriguing.

Establishing the elastic conditions has been a complicated classical problem since usual and traditional models do not contemplate the whole system with all the unknowns and all the functions.

An analytical example of a particular case of a straight beam is developed to show the practice. The procedure given is general and suitable for educational purposes.

References

[1] A.P. Boresi, K.P. Chong, S. Sagal, *Aproximate solution methods in engineering mechanics*. New Jersey: John Wiley & Sons; 2003.
[2] R. Burden, D. Faires, *Numerical Analysis*. Boston: PWS, 1985.
[3] M. Rahman, *Applied Differential Equations for Scientists and Engineers: Ordinary Differential Equations*. Glasgow: Computational Mechanics Publications; 1991.
[4] K.J. Bathe, *Finite Element Procedures*. New Jersey: Prentice Hall; 1996.
[5] E. Alarcon, C. Brebbia, J. Dominguez, "The boundary element method in elasticity". *Eng Anal Bound Elem*, 1978;20(9):625-639.
[6] J.I. Parcel, Moorman RB, *Analysis of statically indeterminate structures*. New York: John Wiley; 1955.
[7] K. Washizu, "Some considerations on a naturally curved and twisted slender beam". *J Appl Math Phys*, 1964;43(2):111-116.
[8] A.E.H. Love, *A Treatise on the mathematical theory of elasticity*. New York: Dover; 1944.
[9] S. Timoshenko, *Strength of materials*. New York: D. Van Nostrand Company; 1957.
[10] H. Weiss, "Dynamics of Geometrically Nonlinear Rods: I. Mechanical Models and Equations of Motion". *Nonlinear Dynamics*, 2002;30:357-381.
[11] D.L. Moris, "Curved beam stiffness coefficients". *J Struct Div* 1968;94:1165-1178.
[12] V. Leontovich, *Frames and arches; condensed solutions for structural analysis*. New York: McGraw-Hill; 1959.
[13] H. Kardestuncer, *Elementary matrix analysis of structures*. New York: McGraw-Hill; 1974.
[14] Y. Yamada, Y. Ezawa, "On curved finite elements for the analysis of curved beams". *Int J Numer Meth Engng* 1977;11:1635-1651.
[15] D.J. Just, "Circularly curved beams under plane loads". *J Struct Div* 1982;108:1858-1873.
[16] A.F. Saleeb, T.Y. Chang, "On the hybrid-mixed formulation C0 curved beam elements". *Comput Meth Appl Mech Engng* 1987;60;95-121.
[17] G. Shi, G.Z. Voyiadjis, "Simple and efficient shear flexible two-node arch/beam and four-node cylindrical shell/plate finite elements". *Int J Numer Meth Engng* 1991;31:759-776.
[18] L. Molari, F. Ubertini, "A flexibility-based model for linear analysis of arbitrarily curved arches". *Int J Numer Meth Engng* 2006;65:1333-1353.
[19] H.P. Lee, "Generalized stiffness matrix of a curved-beam element". *AIAA J* 1969;7:2043-2045.
[20] T.M. Wang, T.F. Merrill, "Stiffness coefficients of noncircular curved beams". *J Struct Engng* 1988;114:1689-1699.
[21] A. Benedetti, A. Tralli, "A new hybrid f.e. model for arbitrarily curved beam-I. Linear analysis". *Comput Struct* 1989;33:1437-1449.
[22] J.P. Marquis, T.M. Wang, "Stiffness matrix of parabolic beam element". *Comput Struct* 1989;6:863–870.
[23] A.C. Scordelis, "Internal forces in uniformly loaded helicoidal girders". *J Amer Conc Inst* 1960;31:1013-1026.
[24] Gimena, L., Gimena, F.N., Gonzaga, P. "Finite Transfer Method for the problem of spatially curved beams" 2008; *World Congress on Engineering 2008*, London, Vol. II, 868-873.
[25] F.N. Gimena, P. Gonzaga, L. Gimena, "3D-curved beam element with varying cross-sectional area under generalized loads". *Eng Struct* 2008;30:404-411.
[26] L. Gimena, F.N. Gimena, P. Gonzaga, "Structural analysis of a curved beam element defined in global coordinates".*Eng Struct* 2008;30: 3355-3364.
[27] I.S. Sokolnikoff, R.M. Redeffer, *Mathematics of Physics and Modern Engineering*. Tokyo: McGraw-Hill; 1958.
[28] F.N. Gimena, P. Gonzaga, L. Gimena, "Numerical transfer-method with boundary conditions for arbitrary curved beam elements". *Eng Analysis with Boundary Elements* 2009;33:249-257.
[29] F.N. Gimena, P. Gonzaga, L. Gimena, "New procedure to obtain the stiffness and transfer matrices of an elastic linear element". 2003; *16th Engineering Mechanics Conference*: ASCE.
[30] Gimena, L., Gonzaga, P., Gimena, F.N. "Implementation of Boundary Equations in the Finite Transfer Method" 2009; *World Congress on Engineering 2009*, London, Vol. II, 963-968.

ISBN: 978-988-18210-8-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)