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Explicit Group Methods in the Solution of the 2-D Convection-Diffusion Equations

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***Abstract*—In this paper, we present the four points Explicit Group (EG) and Explicit Decoupled Group (EDG) schemes for solving the two dimensional convection-diffusion equation with initial and Dirichlet boundary conditions. The EG method is derived from the centred difference approximation whilst EDG is derived from the rotated difference operator expressed in coordinates rotated 45^0 with respect to the standard mesh. These new formulations are shown to be unconditionally stable and the robustness of these new formulations over the existing point Crank-Nicolson scheme demonstrated through numerical experiments.**

***Index Terms*—Explicit Group (EG), Explicit Decoupled Group (EDG), Convection-Diffusion, Crank-Nicolson, Rotated Crank-Nicolson.**

I. INTRODUCTION

Consider the two dimensional convection-diffusion equation:

$$\frac{\partial U}{\partial t} = a_x \frac{\partial^2 U}{\partial x^2} + a_y \frac{\partial^2 U}{\partial y^2} - b_x \frac{\partial U}{\partial x} - b_y \frac{\partial U}{\partial y} \tag{1}$$

with initial and boundary conditions:

$$\begin{cases} u(x,y,0) = f(x,y) \\ u(0,y,t) = g_1(y,t), \quad u(X,y,t) = g_2(y,t) \\ u(x,0,t) = h_1(x,t), \quad u(x,Y,t) = h_2(x,t). \end{cases} \tag{2}$$

Here a_x, a_y, b_x, b_y are positive constants, on a rectangular grid with grid spacing Δx in x-direction and Δy in y-direction, with $x_i = x_0 + i\Delta x$, $y_j = y_0 + j\Delta y$ and $t_n = n\Delta t$ (for all $i = 0, 1, 2, \ldots, nx$, $j = 0, 1, 2 \ldots, ny$, $n = 0, 1, 2, \ldots$), $X = x_0 + nx\Delta x$, $Y = y_0 + ny\Delta y$. Equation (1) can be approximated at any point (x_i, y_j, t_n) in various ways. One commonly used integration method is the Crank-Nicolson formula:

$$\begin{aligned} \frac{u_{i,j,n+1} - u_{i,j,n}}{\Delta t} = {} & \frac{a_x}{2}\left(\frac{u_{i-1,j,n+1} - 2u_{i,j,n+1} + u_{i+1,j,n+1}}{\Delta x^2} + \frac{u_{i-1,j,n} - 2u_{i,j,n} + u_{i+1,j,n}}{\Delta x^2} \right) \\ & + \frac{a_y}{2}\left(\frac{u_{i,j-1,n+1} - 2u_{i,j,n+1} + u_{i,j+1,n+1}}{\Delta y^2} + \frac{u_{i,j-1,n} - 2u_{i,j,n} + u_{i,j+1,n}}{\Delta y^2} \right) \\ & - \frac{b_x}{2}\left(\frac{u_{i+1,j,n+1} - u_{i-1,j,n+1}}{2\Delta x} + \frac{u_{i+1,j,n} - u_{i-1,j,n}}{2\Delta x} \right) \\ & - \frac{b_y}{2}\left(\frac{u_{i,j+1,n+1} - u_{i,j-1,n+1}}{2\Delta y} + \frac{u_{i,j+1,n} - u_{i,j-1,n}}{2\Delta y} \right) \end{aligned} \tag{3}$$

Let the Courant numbers (Cx, Cy) and diffusion numbers (Sx, Sy) be defined as

$$\begin{aligned} Sx &= a_x \Delta t / \Delta x^2 \\ Sy &= a_y \Delta t / \Delta y^2 \\ Cx &= b_x \Delta t / \Delta x \\ Cy &= b_y \Delta t / \Delta y. \end{aligned} \tag{4}$$

Thus (3) can be simplified as

$$\begin{aligned} & (1 + Sx + Sy) u_{i,j,n+1} - \left(\frac{Sx}{2} + \frac{Cx}{4} \right) u_{i-1,j,n+1} - \left(\frac{Sx}{2} - \frac{Cx}{4} \right) u_{i+1,j,n+1} \\ & \quad - \left(\frac{Sy}{2} + \frac{Cy}{4} \right) u_{i,j-1,n+1} - \left(\frac{Sy}{2} - \frac{Cy}{4} \right) u_{i,j+1,n+1} \\ & = (1 - Sx - Sy) u_{i,j,n} + \left(\frac{Sx}{2} + \frac{Cx}{4} \right) u_{i-1,j,n} + \left(\frac{Sx}{2} - \frac{Cx}{4} \right) u_{i+1,j,n} \\ & \quad + \left(\frac{Sy}{2} + \frac{Cy}{4} \right) u_{i,j-1,n} + \left(\frac{Sy}{2} - \frac{Cy}{4} \right) u_{i,j+1,n} \end{aligned} \tag{5}$$

with the computational molecule as in Fig. 1.

Another integration method derived from the Crank-Nicolson formula can be obtained by rotating the x-y axis clockwise by 45°. Using Taylor series expansion, the rotated Crank-Nicolson formula for (1) can be shown to be of the following form [2]:

$$\begin{aligned} & \left(1 + \frac{Sx}{2} + \frac{Sy}{2}\right) u_{i,j,n+1} - \left(\frac{Sx}{4} + \frac{Cx}{8} - \frac{Cy}{8} \right) u_{i-1,j+1,n+1} - \left(\frac{Sx}{4} - \frac{Cx}{8} - \frac{Cy}{8} \right) u_{i+1,j+1,n+1} \\ & \quad - \left(\frac{Sy}{4} - \frac{Cx}{8} + \frac{Cy}{8} \right) u_{i+1,j-1,n+1} - \left(\frac{Sy}{4} + \frac{Cx}{8} + \frac{Cy}{8} \right) u_{i-,j-1,n+1} \\ & = \left(1 - \frac{Sx}{2} - \frac{Sy}{2}\right) u_{i,j,n} + \left(\frac{Sx}{4} + \frac{Cx}{8} - \frac{Cy}{8} \right) u_{i-1,j+1,n} + \left(\frac{Sx}{4} - \frac{Cx}{8} - \frac{Cy}{8} \right) u_{i+1,j+1,n} \\ & \quad + \left(\frac{Sy}{4} - \frac{Cx}{8} + \frac{Cy}{8} \right) u_{i+1,j-1,n} + \left(\frac{Sy}{4} + \frac{Cx}{8} + \frac{Cy}{8} \right) u_{i-,j-1,n} \end{aligned} \tag{6}$$

It is clearly seen that the application of either (3) or (6) at each time step will result in a large and sparse linear system,

$$A\, u_{n+1} = B\, u_n \tag{7}$$

where A and B are square nonsingular matrices, while u_{n+1} and u_n are specific column matrices. The solution of (7) can be obtained by direct or iterative methods. Since the equation is large and sparse, iterative method is more suitable to be used in solving this type of problem, either in point or block formulations.

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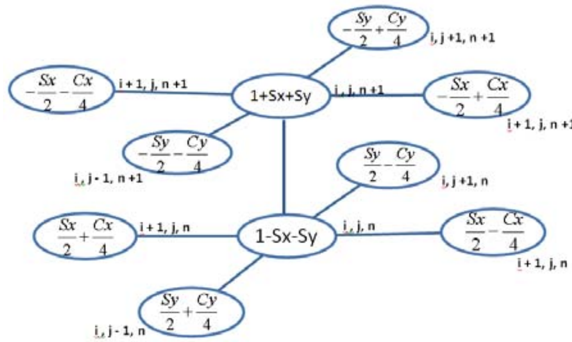

Fig. 1: The Crank-Nicolson scheme with natural ordering

The Explicit Group (EG) and Explicit Decoupled Group (EDG) schemes can be constructed based on (5) and (6) respectively. The original EG scheme was formulated by Yousif and Evans [4] in solving the two dimensional elliptic equation by constructing new grouping of the mesh points into smaller size groups of points where the gains in execution timings of the four point EG method over the 1-line smoother ranges from 25%-36%. Using the idea of smaller size groupings on rotated grids, Abdullah [1] developed the four points EDG which was shown to be more efficient computationally than the EG method. Yousif and Evans [5] later extended the method to six and nine points groupings and showed that they can be easily parallelised on an MIMD multiprocessor. Sections II and III describe the formulation the EG and EDG methods respectively, for the two dimensional convection-diffusion equation. The truncation error and consistency analysis are presented in Section IV, followed by the stability analysis in Section V. Numerical experiments and results are presented in Section VI. The concluding remark is given in Section VII.

II. Explicit Group (EG)

To formulate the EG scheme, we apply (3) to any group of four points on the solution domain at each time step. Thus, at any particular time level *(n+1),* this will result in a (4x4) system of the form:

$$\begin{bmatrix} 1+Sx+Sy & -\left(\frac{Sx}{2}-\frac{Cx}{4}\right) & 0 & -\left(\frac{Sy}{2}-\frac{Cy}{4}\right) \\ -\left(\frac{Sx}{2}+\frac{Cx}{4}\right) & 1+Sx+Sy & -\left(\frac{Sy}{2}-\frac{Cy}{4}\right) & 0 \\ 0 & -\left(\frac{Sy}{2}+\frac{Cy}{4}\right) & 1+Sx+Sy & -\left(\frac{Sx}{2}+\frac{Cx}{4}\right) \\ -\left(\frac{Sy}{2}+\frac{Cy}{4}\right) & 0 & -\left(\frac{Sx}{2}-\frac{Cx}{4}\right) & 1+Sx+Sy \end{bmatrix} \begin{bmatrix} u_{i,j,n+1} \\ u_{i+1,j,n+1} \\ u_{i+1,j+1,n+1} \\ u_{i,j+1,n+1} \end{bmatrix} = \begin{bmatrix} rh1 \\ rh2 \\ rh3 \\ rh4 \end{bmatrix} \tag{8}$$

where

$$\begin{bmatrix} rh1 \\ rh2 \\ rh3 \\ rh4 \end{bmatrix} = \begin{bmatrix} \left(\frac{Sx}{2}+\frac{Cx}{4}\right)u_{i-1,j,n+1}+\left(\frac{Sy}{2}+\frac{Cy}{4}\right)u_{i,j-1,n+1}+T_{i,j} \\ \left(\frac{Sx}{2}-\frac{Cx}{4}\right)u_{i+2,j,n+1}+\left(\frac{Sy}{2}+\frac{Cy}{4}\right)u_{i+1,j-1,n+1}+T_{i+1,j} \\ \left(\frac{Sx}{2}-\frac{Cx}{4}\right)u_{i+2,j+1,n+1}+\left(\frac{Sy}{2}-\frac{Cy}{4}\right)u_{i+1,j+2,n+1}+T_{i+1,j+1} \\ \left(\frac{Sx}{2}+\frac{Cx}{4}\right)u_{i-1,j+1,n+1}+\left(\frac{Sy}{2}-\frac{Cy}{4}\right)u_{i,j+2,n+1}+T_{i,j+1} \end{bmatrix} \tag{9}$$

$$\begin{bmatrix} T_{i,j} \\ T_{i+1,j} \\ T_{i+1,j+1} \\ T_{i,j+1} \end{bmatrix} = \begin{bmatrix} (1-Sx-Sy)u_{i,j,n}+\left(\frac{Sx}{2}+\frac{Cx}{4}\right)u_{i-1,j,n}+\left(\frac{Sx}{2}-\frac{Cx}{4}\right)u_{i+1,j,n}+\left(\frac{Sy}{2}+\frac{Cy}{4}\right)u_{i,j-1,n}+\left(\frac{Sy}{2}-\frac{Cy}{4}\right)u_{i,j+1,n} \\ (1-Sx-Sy)u_{i+1,j,n}+\left(\frac{Sx}{2}+\frac{Cx}{4}\right)u_{i,j,n}+\left(\frac{Sx}{2}-\frac{Cx}{4}\right)u_{i+2,j,n}+\left(\frac{Sy}{2}+\frac{Cy}{4}\right)u_{i+1,j-1,n}+\left(\frac{Sy}{2}-\frac{Cy}{4}\right)u_{i+1,j+1,n} \\ (1-Sx-Sy)u_{i+1,j+1,n}+\left(\frac{Sx}{2}+\frac{Cx}{4}\right)u_{i,j+1,n}+\left(\frac{Sx}{2}-\frac{Cx}{4}\right)u_{i+2,j+1,n}+\left(\frac{Sy}{2}+\frac{Cy}{4}\right)u_{i+1,j,n}+\left(\frac{Sy}{2}-\frac{Cy}{4}\right)u_{i+1,j+2,n} \\ (1-Sx-Sy)u_{i,j+1,n}+\left(\frac{Sx}{2}+\frac{Cx}{4}\right)u_{i-1,j+1,n}+\left(\frac{Sx}{2}-\frac{Cx}{4}\right)u_{i+1,j+1,n}+\left(\frac{Sy}{2}+\frac{Cy}{4}\right)u_{i,j,n}+\left(\frac{Sy}{2}-\frac{Cy}{4}\right)u_{i,j+2,n} \end{bmatrix}. \tag{10}$$

Equation (8) can be inverted to obtain the four-point EG equation:

$$\begin{bmatrix} u_{i,j,n+1} \\ u_{i+1,j,n+1} \\ u_{i+1,j+1,n+1} \\ u_{i,j+1,n+1} \end{bmatrix} = \frac{1}{const} \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \\ q_5 & q_1 & q_4 & q_6 \\ q_7 & q_8 & q_1 & q_5 \\ q_8 & q_9 & q_2 & q_1 \end{bmatrix} \begin{bmatrix} rh1 \\ rh2 \\ rh3 \\ rh4 \end{bmatrix} \tag{11}$$

$$const=(1+Sx+Sy)^4+\left(\frac{Sx}{2}+\frac{Cx}{4}\right)^2\left(\frac{Sx}{2}-\frac{Cx}{4}\right)^2+\left(\frac{Sy}{2}+\frac{Cy}{4}\right)^2\left(\frac{Sy}{2}-\frac{Cy}{4}\right)^2$$

$$q_1=(1+Sx+Sy)^3-(1+Sx+Sy)\left(\frac{Sx}{2}+\frac{Cx}{4}\right)\left(\frac{Sx}{2}-\frac{Cx}{4}\right)-(1+Sx+Sy)\left(\frac{Sy}{2}+\frac{Cy}{4}\right)\left(\frac{Sy}{2}-\frac{Cy}{4}\right)$$

$$q_2=(1+Sx+Sy)^2\left(\frac{Sx}{2}-\frac{Cx}{4}\right)-\left(\frac{Sx}{2}+\frac{Cx}{4}\right)\left(\frac{Sx}{2}-\frac{Cx}{4}\right)^2+\left(\frac{Sx}{2}-\frac{Cx}{4}\right)\left(\frac{Sy}{2}+\frac{Cy}{4}\right)\left(\frac{Sy}{2}-\frac{Cy}{4}\right)$$

$$q_3=2(1+Sx+Sy)\left(\frac{Sx}{2}-\frac{Cx}{4}\right)\left(\frac{Sy}{2}-\frac{Cy}{4}\right)$$

$$q_4=(1+Sx+Sy)^2\left(\frac{Sy}{2}-\frac{Cy}{4}\right)+\left(\frac{Sx}{2}+\frac{Cx}{4}\right)\left(\frac{Sx}{2}-\frac{Cx}{4}\right)\left(\frac{Sy}{2}-\frac{Cy}{4}\right)-\left(\frac{Sy}{2}+\frac{Cy}{4}\right)\left(\frac{Sy}{2}-\frac{Cy}{4}\right)^2$$

$$q_5=(1+Sx+Sy)^2\left(\frac{Sx}{2}+\frac{Cx}{4}\right)-\left(\frac{Sx}{2}+\frac{Cx}{4}\right)^2\left(\frac{Sx}{2}-\frac{Cx}{4}\right)+\left(\frac{Sx}{2}+\frac{Cx}{4}\right)\left(\frac{Sy}{2}+\frac{Cy}{4}\right)\left(\frac{Sy}{2}-\frac{Cy}{4}\right)$$

$$q_6=2(1+Sx+Sy)\left(\frac{Sx}{2}+\frac{Cx}{4}\right)\left(\frac{Sy}{2}-\frac{Cy}{4}\right)$$

$$q_7=2(1+Sx+Sy)\left(\frac{Sx}{2}+\frac{Cx}{4}\right)\left(\frac{Sy}{2}+\frac{Cy}{4}\right)$$

$$q_8=(1+Sx+Sy)^2\left(\frac{Sy}{2}+\frac{Cy}{4}\right)+\left(\frac{Sx}{2}+\frac{Cx}{4}\right)\left(\frac{Sx}{2}-\frac{Cx}{4}\right)\left(\frac{Sy}{2}+\frac{Cy}{4}\right)-\left(\frac{Sy}{2}+\frac{Cy}{4}\right)^2\left(\frac{Sy}{2}-\frac{Cy}{4}\right)$$

$$q_9=2(1+Sx+Sy)\left(\frac{Sx}{2}-\frac{Cx}{4}\right)\left(\frac{Sy}{2}+\frac{Cy}{4}\right)$$

The solutions may be obtained by imposing the Gauss-Seidel iterative scheme to the four-point EG formula (11) at each time level. Iterations are generated in groups of four points over the entire spatial domain until the convergence test is satisfied. The converged solutions are then taken as initial guesses for the iterations at the next time level.

III. Explicit Decoupled Group (EDG)

Similar to the EG method, we apply (6) to any group of four points in the solution domain at each time step to obtain the following (4x4) system of equations:

$$\begin{bmatrix} 1+\frac{Sx}{2}+\frac{Sy}{2} & -\left(\frac{Sx}{4}-\frac{Cx}{8}-\frac{Cy}{8}\right) & 0 & 0 \\ -\left(\frac{Sy}{4}+\frac{Cx}{8}+\frac{Cy}{8}\right) & 1+\frac{Sx}{2}+\frac{Sy}{2} & 0 & 0 \\ 0 & 0 & 1+\frac{Sx}{2}+\frac{Sy}{2} & -\left(\frac{Sx}{4}+\frac{Cx}{8}-\frac{Cy}{8}\right) \\ 0 & 0 & -\left(\frac{Sy}{4}-\frac{Cx}{8}+\frac{Cy}{8}\right) & 1+\frac{Sx}{2}+\frac{Sy}{2} \end{bmatrix} \begin{bmatrix} u_{i,j,n+1} \\ u_{i+1,j+1,n+1} \\ u_{i+1,j,n+1} \\ u_{i,j+1,n+1} \end{bmatrix} = \begin{bmatrix} rh1 \\ rh2 \\ rh3 \\ rh4 \end{bmatrix} \tag{12}$$

with

$$\begin{bmatrix} rh1 \\ rh2 \\ rh3 \\ rh4 \end{bmatrix} = \begin{bmatrix} bu_{i-1,j+1,n+1}+du_{i+1,j-1,n+1}+eu_{i-1,j-1,n+1}+T_{i,j} \\ bu_{i,j+2,n+1}+cu_{i+2,j+2,n+1}+du_{i+2,j,n+1}+T_{i+1,j+1} \\ cu_{i+2,j+1,n+1}+du_{i+2,j-1,n+1}+eu_{i,j-1,n+1}+T_{i+1,j} \\ bu_{i-1,j+2,n+1}+cu_{i+1,j+2,n+1}+eu_{i-1,j,n+1}+T_{i,j+1} \end{bmatrix} \tag{13}$$

$$\begin{bmatrix} T_{i,j} \\ T_{i+1,j+1} \\ T_{i+1,j} \\ T_{i,j+1} \end{bmatrix} = \begin{bmatrix} au_{i,j,n} + bu_{i-1,j+1,n} + cu_{i+1,j+1,n} + du_{i+1,j-1,n} + eu_{i-1,j-1,n} \\ au_{i+1,j+1,n} + bu_{i,j+2,n} + cu_{i+2,j+2,n} + du_{i+2,j,n} + eu_{i,j,n} \\ au_{i+1,j,n} + bu_{i,j+1,n} + cu_{i+2,j+1,n} + du_{i+2,j-1,n} + eu_{i,j-1,n} \\ au_{i,j+1,n} + bu_{i-1,j+2,n} + cu_{i+1,j+2,n} + du_{i+1,j,n} + eu_{i-1,j,n} \end{bmatrix} \tag{14}$$

The system (12) leads to a decoupled system of 2x2 equations in explicit form:

$$\begin{bmatrix} 1+\frac{Sx}{2}+\frac{Sy}{2} & -\left(\frac{Sx}{4}-\frac{Cx}{8}-\frac{Cy}{8}\right) \\ -\left(\frac{Sy}{4}+\frac{Cx}{8}+\frac{Cy}{8}\right) & 1+\frac{Sx}{2}+\frac{Sy}{2} \end{bmatrix} \begin{bmatrix} u_{i,j,n+1} \\ u_{i+1,j+1,n+1} \end{bmatrix} = \begin{bmatrix} rh1 \\ rh2 \end{bmatrix} \tag{15}$$

and

$$\begin{bmatrix} 1+\frac{Sx}{2}+\frac{Sy}{2} & -\left(\frac{Sx}{4}+\frac{Cx}{8}-\frac{Cy}{8}\right) \\ -\left(\frac{Sy}{4}-\frac{Cx}{8}+\frac{Cy}{8}\right) & 1+\frac{Sx}{2}+\frac{Sy}{2} \end{bmatrix} \begin{bmatrix} u_{i+1,j,n+1} \\ u_{i,j+1,n+1} \end{bmatrix} = \begin{bmatrix} rh3 \\ rh4 \end{bmatrix} \tag{16}$$

Referring to Fig. 2(a), it is observed that the iterative evaluation of (15) at any time level involves points of type ● only, while the evaluation of (16) involves points of type ○ only (see Fig. 2(b)). Thus, the iterations may be chosen to involve only one type of points. Suppose we choose to iterate on points of type ●. Hence, the EDG scheme corresponds to generation of iterations on these points using the group formula (15) until a convergence test is satisfied. After convergence is achieved, the solutions at the points of type ○ are evaluated directly once using the Crank-Nicolson formula (5) before proceeding to the next time level.

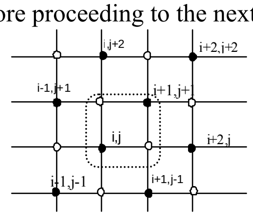

Fig. 2 (a) Computational Molecule of (15)

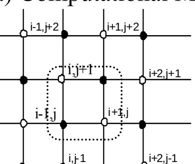

Fig. 2 (b) Computational Molecule of (16)

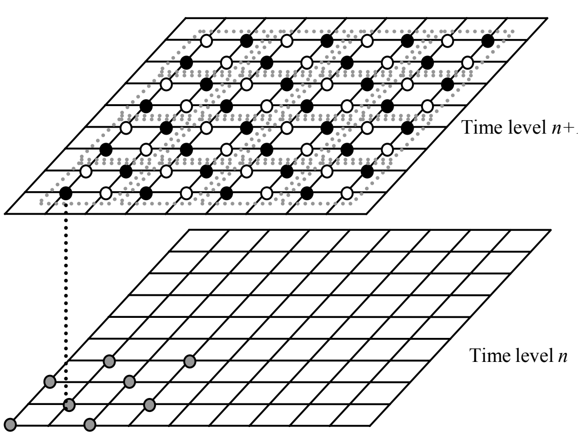

Fig. 3 Grid generation at time level $n+1$ and n (mesh size N=9)

IV. Truncation Error and consistency

The local truncation for the Crank-Nicolson scheme may be obtained by using the Taylor series expansion about the point (x_i, y_j, $t_{n+1/2}$):

$$\begin{aligned} T_{CN} = & -\frac{\Delta t^2}{24}\frac{\partial^3 u}{\partial t^3} + \frac{\Delta t^2}{8}\left(a_x \frac{\partial^2}{\partial t^2}\frac{\partial^2 u}{\partial x^2}\bigg|_{i,j,n+0.5} + a_y \frac{\partial^2}{\partial t^2}\frac{\partial^2 u}{\partial x^2}\bigg|_{i,j,n+0.5} \right. \\ & \left. -b_x \frac{\partial^2}{\partial t^2}\frac{\partial u}{\partial x}\bigg|_{i,j,n+0.5} - b_y \frac{\partial^2}{\partial t^2}\frac{\partial u}{\partial y}\bigg|_{i,j,n+0.5} + \ldots \right) + \Delta x^2 \left(\frac{a_x}{12}\frac{\partial^4 u}{\partial x^4}\bigg|_{i,j,n+0.5} - \frac{b_x}{6}\frac{\partial^3 u}{\partial x^3}\bigg|_{i,j,n+0.5} \right) \\ & + \Delta y^2 \left(\frac{a_y}{12}\frac{\partial^4 u}{\partial y^4}\bigg|_{i,j,n+0.5} - \frac{b_y}{6}\frac{\partial^3 u}{\partial y^3}\bigg|_{i,j,n+0.5} \right) \\ & + \frac{\Delta x^2 \Delta t^2}{48}\left(\frac{a_x}{2}\frac{\partial^2}{\partial t^2}\frac{\partial^4 u}{\partial x^4}\bigg|_{i,j,n+0.5} - b_x \frac{\partial^2}{\partial t^2}\frac{\partial^3 u}{\partial x^3}\bigg|_{i,j,n+0.5} \right) \\ & + \frac{\Delta y^2 \Delta t^2}{48}\left(\frac{a_y}{2}\frac{\partial^2}{\partial t^2}\frac{\partial^4 u}{\partial y^4}\bigg|_{i,j,n+0.5} - b_y \frac{\partial^2}{\partial t^2}\frac{\partial^3 u}{\partial y^3}\bigg|_{i,j,n+0.5} \right) + \ldots \end{aligned}$$

i.e. $T_{CN} = O(\Delta t^2) + O(\Delta x^2) + O(\Delta y^2)$ (17)

Let $h = \Delta x = \Delta y$, $k = \Delta t$, the local truncation error for this scheme is then

$$\begin{aligned} T_{CN} = & -\frac{k^2}{24}\frac{\partial^3 u}{\partial t^3} \\ & + \frac{k^2}{8}\left(a_x \frac{\partial^2}{\partial t^2}\frac{\partial^2 u}{\partial x^2}\bigg|_{i,j,n+0.5} + a_y \frac{\partial^2}{\partial t^2}\frac{\partial^2 u}{\partial x^2}\bigg|_{i,j,n+0.5} - b_x \frac{\partial^2}{\partial t^2}\frac{\partial u}{\partial x}\bigg|_{i,j,n+0.5} - b_y \frac{\partial^2}{\partial t^2}\frac{\partial u}{\partial y}\bigg|_{i,j,n+0.5} + \ldots \right) \\ & + h^2 \left(\frac{a_x}{12}\frac{\partial^4 u}{\partial x^4}\bigg|_{i,j,n+0.5} + \frac{a_y}{12}\frac{\partial^4 u}{\partial y^4}\bigg|_{i,j,n+0.5} - \frac{b_x}{6}\frac{\partial^3 u}{\partial x^3}\bigg|_{i,j,n+0.5} - \frac{b_y}{6}\frac{\partial^3 u}{\partial y^3}\bigg|_{i,j,n+0.5} \right) \\ & + \frac{h^2 k^2}{48}\left(\frac{a_x}{2}\frac{\partial^2}{\partial t^2}\frac{\partial^4 u}{\partial x^4}\bigg|_{i,j,n+0.5} + \frac{a_y}{2}\frac{\partial^2}{\partial t^2}\frac{\partial^4 u}{\partial y^4}\bigg|_{i,j,n+0.5} - b_x \frac{\partial^2}{\partial t^2}\frac{\partial^3 u}{\partial x^3}\bigg|_{i,j,n+0.5} - b_y \frac{\partial^2}{\partial t^2}\frac{\partial^3 u}{\partial y^3}\bigg|_{i,j,n+0.5} \right) + \ldots \end{aligned}$$

i.e. $T_{CN} = O(k^2) + O(h^2)$. (18)

As $\Delta x, \Delta y, \Delta t \to 0$, the truncation error T_{CN} tends to zero. Hence, as the grid spacings Δx, Δy, $\Delta t \to 0$ in the limit sense, the Crank Nicolson formula (5) is equivalent to the convection-diffusion equation and thus is consistent. EG is also consistent and its truncation error is similar with the Crank-Nicolson scheme since it is derived from the same formula.

Assuming that $a = a_x = a_y$, the truncation error for the rotated Crank-Nicolson scheme becomes:

$$\begin{aligned} T_{R-CN} = & -\frac{k^2}{24}\frac{\partial^3 u}{\partial t^3} \\ & + \frac{k^2}{8}\left(a \frac{\partial^2}{\partial t^2}\frac{\partial^2 u}{\partial x^2}\bigg|_{i,j,n+0.5} + a \frac{\partial^2}{\partial t^2}\frac{\partial^2 u}{\partial x^2}\bigg|_{i,j,n+0.5} - b_x \frac{\partial^2}{\partial t^2}\frac{\partial u}{\partial x}\bigg|_{i,j,n+0.5} - b_y \frac{\partial^2}{\partial t^2}\frac{\partial u}{\partial y}\bigg|_{i,j,n+0.5} + \ldots \right) \\ & + \frac{h^2}{2}\left(\frac{a}{12}\frac{\partial^4 u}{\partial x^4}\bigg|_{i,j,n+0.5} + \frac{a}{2}\frac{\partial^4 u}{\partial x^2 \partial y^2}\bigg|_{i,j,n+0.5} + \frac{a}{12}\frac{\partial^4 u}{\partial y^4}\bigg|_{i,j,n+0.5} \right. \\ & \left. - \frac{b_x}{6}\frac{\partial^3 u}{\partial x^3}\bigg|_{i,j,n+0.5} - \frac{b_x}{2}\frac{\partial^3 u}{\partial x \partial y^2}\bigg|_{i,j,n+0.5} - \frac{b_y}{6}\frac{\partial^3 u}{\partial y^3}\bigg|_{i,j,n+0.5} - \frac{b_y}{2}\frac{\partial^3 u}{\partial x^2 \partial y}\bigg|_{i,j,n+0.5} \right) \\ & + \frac{h^2 k^2}{48}\left(\frac{a}{2}\frac{\partial^2}{\partial t^2}\frac{\partial^4 u}{\partial x^4}\bigg|_{i,j,n+0.5} + 4a \frac{\partial^2}{\partial t^2}\frac{\partial^4 u}{\partial x^2 \partial y^2}\bigg|_{i,j,n+0.5} + \frac{a}{2}\frac{\partial^2}{\partial t^2}\frac{\partial^4 u}{\partial y^4}\bigg|_{i,j,n+0.5} \right. \\ & \left. - b_x \frac{\partial^2}{\partial t^2}\frac{\partial^3 u}{\partial x^3}\bigg|_{i,j,n+0.5} - 4b_x \frac{\partial^2}{\partial t^2}\frac{\partial^3 u}{\partial x \partial y^2}\bigg|_{i,j,n+0.5} - b_y \frac{\partial^2}{\partial t^2}\frac{\partial^3 u}{\partial y^3}\bigg|_{i,j,n+0.5} - 4b_y \frac{\partial^2}{\partial t^2}\frac{\partial^3 u}{\partial x^2 \partial y}\bigg|_{i,j,n+0.5} \right) + \ldots \end{aligned}$$

i.e. $T_{R-CN} = O(k^2) + O(h^2)$. (19)

Similarly, the rotated Crank-Nicolson equation (6) is consistent and the consistency of EDG is also maintained since it is based on the same formula.

V. Stability analysis

Explicit Group (EG)

Equation (8) can be written explicitly in difference form as $u_{n+1} = T\,u_n$ where $T = A^{-1}B$. Here,

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$$A=\begin{bmatrix} R_1 & R_2 & & & \\ R_3 & R_1 & R_2 & & \\ & \ddots & \ddots & \ddots & \\ & & R_3 & R_1 & R_2 \\ & & & R_3 & R_1 \end{bmatrix},$$

$$R_1=\begin{bmatrix} G_1 & G_3 & & & \\ G_2 & G_1 & G_3 & & \\ & \ddots & \ddots & \ddots & \\ & & G_2 & G_1 & G_3 \\ & & & G_2 & G_1 \end{bmatrix}, R_2=\begin{bmatrix} G_5 & & \\ & \ddots & \\ & & G_5 \end{bmatrix},$$

$$R_3=\begin{bmatrix} G_4 & & \\ & \ddots & \\ & & G_4 \end{bmatrix}, G_1=\begin{bmatrix} a & -c & -e & 0 \\ -b & a & 0 & -e \\ -d & 0 & a & -c \\ 0 & -d & -b & a \end{bmatrix},$$

$$G_2=\begin{bmatrix} 0 & -b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b \\ 0 & 0 & 0 & 0 \end{bmatrix}, G_3=\begin{bmatrix} 0 & 0 & 0 & 0 \\ -c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -c & 0 \end{bmatrix},$$

$$G_4=\begin{bmatrix} 0 & 0 & -d & 0 \\ 0 & 0 & 0 & -d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, G_5=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -e & 0 & 0 & 0 \\ 0 & -e & 0 & 0 \end{bmatrix}$$

where

$a=1+Sx+Sy$, $b=\frac{Sx}{2}+\frac{Cx}{4}$, $c=\frac{Sx}{2}-\frac{Cx}{4}$, $d=\frac{Sy}{2}+\frac{Cy}{4}$, $e=\frac{Sy}{2}-\frac{Cy}{4}$, $f=1-Sx-Sy.$

$$\therefore \|A\|_\infty = (|1+Sx+Sy|+|0.5Sx-0.25Cx|+|0.5Sx+0.25Cx| + |0.5Sy+0.25Cy|+|0.5Sy-0.25Cy|).$$

$$B=\begin{bmatrix} S_1 & S_2 & & & \\ S_3 & S_1 & S_2 & & \\ & \ddots & \ddots & \ddots & \\ & & S_3 & S_1 & S_2 \\ & & & S_3 & S_1 \end{bmatrix},$$

where

$$S_1=\begin{bmatrix} H_1 & H_3 & & & \\ H_2 & H_1 & H_3 & & \\ & \ddots & \ddots & \ddots & \\ & & H_2 & H_1 & H_3 \\ & & & H_2 & H_1 \end{bmatrix}, S_2=\begin{bmatrix} H_5 & & \\ & \ddots & \\ & & H_5 \end{bmatrix},$$

$$S_3=\begin{bmatrix} H_4 & & \\ & \ddots & \\ & & H_4 \end{bmatrix}, H_1=\begin{bmatrix} f & c & e & 0 \\ b & f & 0 & e \\ d & 0 & f & c \\ 0 & d & b & f \end{bmatrix},$$

$$H_2=\begin{bmatrix} 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{bmatrix}, H_3=\begin{bmatrix} 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 \end{bmatrix},$$

$$H_4=\begin{bmatrix} 0 & 0 & d & 0 \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, H_5=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e & 0 & 0 & 0 \\ 0 & e & 0 & 0 \end{bmatrix}$$

$$\therefore \|B\|_\infty = (|1-Sx-Sy|+|0.5Sx-0.25Cx|+|0.5Sx+0.25Cx| + |0.5Sy+0.25Cy|+|0.5Sy-0.25Cy|)$$

$$\therefore \|T\|_\infty = \|A^{-1}B\|_\infty \le \|A^{-1}\|_\infty \|B\|_\infty \le \frac{1}{\|A\|_\infty}\|B\|_\infty \le 1$$

for all $Cx, Cy, Sx, Sy \ge 0$. Therefore the EG iterative method is unconditionally stable.

Explicit Decoupled Group (EDG)

Equation (12) may also be expressed explicitly as $u_{n+1} = T u_n$ where $T = A^{-1}B$. The matrix A is of the form:

$$A=\begin{bmatrix} R_1 & R_2 & & & \\ R_3 & R_1 & R_2 & & \\ & \ddots & \ddots & \ddots & \\ & & R_3 & R_1 & R_2 \\ & & & R_3 & R_1 \end{bmatrix}$$

where

$$R_1=\begin{bmatrix} G_1 & G_2 & & & \\ G_3^T & G_1 & G_2 & & \\ & \ddots & \ddots & \ddots & \\ & & G_3^T & G_1 & G_2 \end{bmatrix}, R_2=\begin{bmatrix} G_3 & G_4 & & & \\ & G_3 & G_4 & & \\ & & \ddots & \ddots & \\ & & & G_3 & G_4 \\ & & & & G_3 \end{bmatrix},$$

$$R_3=\begin{bmatrix} G_2^T & & & \\ G_5 & G_2^T & & \\ & \ddots & \ddots & \\ & & G_5 & G_2^T \\ & & & G_5 & G_2^T \end{bmatrix}, G_1=\begin{bmatrix} a & -e \\ -d & a \end{bmatrix}$$

$$G_2=\begin{bmatrix} 0 & 0 \\ -c & 0 \end{bmatrix} G_3=\begin{bmatrix} 0 & 0 \\ -b & 0 \end{bmatrix} G_4=\begin{bmatrix} 0 & 0 \\ -e & 0 \end{bmatrix} G_5=\begin{bmatrix} 0 & -d \\ 0 & 0 \end{bmatrix}$$

with $a=1+\frac{Sx}{2}+\frac{Sy}{2}$, $b=\frac{Sx}{4}+\frac{Cx}{8}-\frac{Cy}{8}$, $c=\frac{Sx}{4}-\frac{Cx}{8}+\frac{Cy}{8}$, $d=\frac{Sy}{4}+\frac{Cx}{8}+\frac{Cy}{8}$, $e=\frac{Sy}{4}-\frac{Cx}{8}-\frac{Cy}{8}$, $f=1-\frac{Sx}{2}-\frac{Sy}{2}.$

$$\therefore \|A\|_\infty = \left(\left|1+\frac{Sx}{2}+\frac{Sy}{2}\right|+\left|\frac{Sx}{4}+\frac{Cx}{8}-\frac{Cy}{8}\right|+\left|\frac{Sx}{4}-\frac{Cx}{8}+\frac{Cy}{8}\right| + \left|\frac{Sx}{4}+\frac{Cx}{8}+\frac{Cy}{8}\right|+\left|\frac{Sx}{4}-\frac{Cx}{8}-\frac{Cy}{8}\right|\right).$$

$$B=\begin{bmatrix} S_1 & S_2 & & & \\ S_3 & S_1 & S_2 & & \\ & \ddots & \ddots & \ddots & \\ & & S_3 & S_1 & S_2 \\ & & & S_3 & S_1 \end{bmatrix},$$

where

$$S_1=\begin{bmatrix} H_1 & H_2 & & & \\ H_3^T & H_1 & H_2 & & \\ & \ddots & \ddots & \ddots & \\ & & H_3^T & H_1 & H_2 \end{bmatrix}, S_2=\begin{bmatrix} H_3 & H_4 & & & \\ & H_3 & H_4 & & \\ & & \ddots & \ddots & \\ & & & H_3 & H_4 \\ & & & & H_3 \end{bmatrix},$$

$$S_3=\begin{bmatrix} H_2^T & & & \\ H_5 & H_2^T & & \\ & \ddots & \ddots & \\ & & H_5 & H_2^T \\ & & & H_5 & H_2^T \end{bmatrix},$$

$$H_1=\begin{bmatrix} f & e \\ d & f \end{bmatrix}, H_2=\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}, H_3=\begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix},$$

$$H_4=\begin{bmatrix} 0 & 0 \\ e & 0 \end{bmatrix}, H_5=\begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix}$$

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$$\therefore \|B\|_\infty = \left(\left|1 - \frac{Sx}{2} - \frac{Sy}{2}\right| + \left|\frac{Sx}{4} + \frac{Cx}{8} - \frac{Cy}{8}\right| + \left|\frac{Sx}{4} - \frac{Cx}{8} + \frac{Cy}{8}\right| + \left|\frac{Sx}{4} + \frac{Cx}{8} + \frac{Cy}{8}\right| + \left|\frac{Sx}{4} - \frac{Cx}{8} - \frac{Cy}{8}\right| \right)$$

Since the amplification matrix $T=A^{-1}B$,

$$\therefore \|T\|_\infty = \|A^{-1}B\|_\infty \le \|A^{-1}\|_\infty \|B\|_\infty \le \frac{1}{\|A\|_\infty}\|B\|_\infty \le 1$$

for all $Cx, Cy, Sx, Sy \ge 0$. Therefore, the EDG iterative scheme is unconditionally stable.

VI. Numerical Experiments

The experiments were carried out on a PC with Intel (R) Corel(TM)2 Duo CPU E7400 @ 2.80 GHz, 1.98 GB of RAM running Windows XP Pro using C compiler in Cygwin. Throughout the whole experiments, the absolute error test was used with tolerance equals to 10^{-10}. One average error was obtained at each time step. The Average Error depicted in Tables I-III denotes the maximum of all the average errors for the particular mesh size. Tables I, II and III present the numerical results of the four methods, the classical Crank-Nicolson, rotated point Crank-Nicolson, EG and EDG, in solving *Examples 1, 2* and *3* respectively, for the number of time step NT = 100 and Δt = 0.01.

Example 1(Diffusion problem)

We consider the following example ($a_x=a_y=1,\ b_x=b_y=0$):

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}, \quad 0 \le x \le 1, 0 \le y \le 1, 0 \le t \le T.$$

The initial and boundary conditions are defined so that they satisfy the exact solution [3]:

$$U(x,y,t) = \frac{1}{4t+1}\exp\left\{\frac{-(x-0.5)^2}{4t+1} - \frac{(y-0.5)^2}{4t+1}\right\}, t > 0. \quad (20)$$

EG reduces the execution times up to 50% of the classical Crank-Nicolson while maintaining the same degree of accuracies. The execution timings of EDG are nearly 65% of the rotated Crank-Nicolson scheme. The latter was also observed to require lesser computing timings than the original Crank-Nicolson scheme.

Example 2

Consider the following example ($ax = ay = b_x = b_y = 1$):

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - \frac{\partial U}{\partial x} - \frac{\partial U}{\partial y}, \quad 0 \le x \le 1, 0 \le y \le 1, 0 \le t \le T$$

The exact solution of the problem above is as follows [3]:

$$U(x,y,t) = \frac{1}{4t+1}\exp\left\{\frac{-(x-t-0.5)^2}{4t+1} - \frac{(y-t-0.5)^2}{4t+1}\right\}, t > 0. \quad (21)$$

Similar with *Example 1*, EG is faster than the Crank-Nicolson scheme, while EDG is faster than the rotated Crank-Nicolson and the EG schemes.

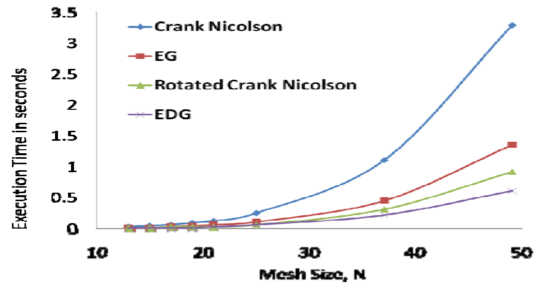

Fig. 4: Experimental Results of *Example 1*

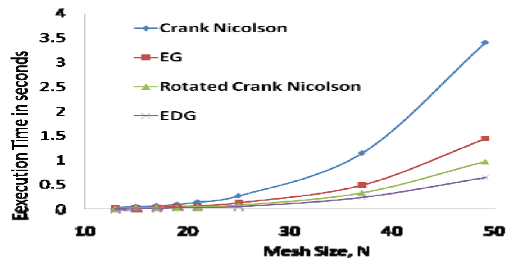

Fig. 5: Experimental Results of *Example 2*

Example 3

We will consider a convection dominant problem. Let $a_x = a_y = 0.1,\ b_x = b_y = 1.0,$ then the exact solution of the problem above is denoted as below [3]:

$$U(x,y,t) = \frac{1}{4t+1}\exp\left\{-10\frac{(x-t-0.5)^2}{(4t+1)} - 10\frac{(y-t-0.5)^2}{(4t+1)}\right\}, t > 0. \quad (22)$$

As shown in Table III and Fig. 6, EDG scheme requires the least execution timings compared to the other three methods. In all of the examples, the EG method produces almost the same accuracies as the classical Crank-Nicolson, while the EDG method is almost as accurate as the rotated Crank-Nicolson.

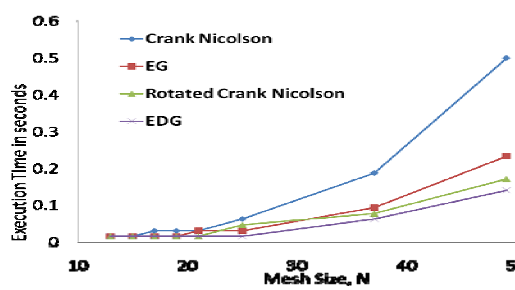

Fig. 6: Experimental results of *Example 3*

VII. Conclusions

In this paper, we have presented effective unconditionally stable group explicit iterative algorithms in solving the two dimensional convection-diffusion problem. The methods serve as viable alternative solvers to the problem with the group scheme derived from the rotated finite difference approximation requiring the least computing efforts among the schemes tested. The parallel implementation of these group methods are still under investigation and will be reported soon.

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Table I: Experimental results of *Example 1* for each scheme with natural ordering (NT= 100, Δt = 0.01)

Mesh Size, N	Classical C-N			Explicit Group (EG)		
	Total iteration Number	Average Error	Execution Time (s)	Total iteration Number	Average Error	Execution Time (s)
13	3400	2.63323E-04	0.031	2015	2.63323E-04	0.016
21	7343	5.71861E-05	0.125	4086	5.71848E-05	0.063
37	19826	3.42176E-05	1.109	10622	3.42213E-05	0.453
49	32993	5.19494E-05	3.297	17528	5.19566E-05	1.360
Mesh Size, N	Rotated C-N			Explicit Decoupled Group (EDG)		
	Total iteration Number	Average Error	Execution Time (s)	Total iteration Number	Average Error	Execution Time (s)
13	2069	6.04169E-04	0.015	1585	6.04169E-04	0.015
21	4163	1.87250E-04	0.031	3173	1.87250E-04	0.031
37	10717	1.55451E-05	0.313	8171	1.55443E-05	0.219
49	17623	2.88300E-05	0.922	13448	2.88311E-05	0.610

Table II: Experimental results of *Example 2* for each scheme with natural ordering (NT= 100, Δt = 0.01)

Mesh Size, N	Classical C-N			Explicit Group (EG)		
	Total iteration Number	Average Error	Execution Time (s)	Total iteration Number	Average Error	Execution Time (s)
13	3358	3.09292E-04	0.031	2000	3.09292E-04	0.016
21	7294	8.19951E-05	0.141	4062	8.19938E-05	0.063
37	19815	2.50416E-05	1.141	10615	2.50448E-05	0.485
49	33046	4.21901E-05	3.406	17551	4.21967E-05	1.438
Mesh Size, N	Rotated C-N			Explicit Decoupled Group (EDG)		
	Total iteration Number	Average Error	Execution Time (s)	Total iteration Number	Average Error	Execution Time (s)
13	2056	6.87329E-04	0.016	1583	6.87329E-04	0.000
21	4144	2.24525E-04	0.047	3169	2.24525E-04	0.031
37	10710	3.32556E-05	0.328	8185	3.32550E-05	0.235
49	17647	1.99782E-05	0.969	13497	1.99791E-05	0.641

Table III: Experimental results of *Example 3* for each scheme with natural ordering (NT= 100, Δt = 0.01)

Mesh Size, N	Classical C-N			Explicit Group (EG)		
	Total iteration Number	Average Error	Execution Time (s)	Total iteration Number	Average Error	Execution Time (s)
13	905	0.010664914	0.016	675	0.010664914	0.000
21	1431	0.004067973	0.031	966	0.004067973	0.031
37	2954	0.001327565	0.188	1789	0.001327564	0.094
49	4551	0.000783234	0.500	2631	0.000783233	0.234
Mesh Size, N	Rotated C-N			Explicit Decoupled Group (EDG)		
	Total iteration Number	Average Error	Execution Time (s)	Total iteration Number	Average Error	Execution Time (s)
13	742	0.020605961	0.016	588	0.020605961	0.016
21	1064	0.007780534	0.016	819	0.007780534	0.016
37	1918	0.002522236	0.078	1465	0.002522363	0.063
49	2769	0.001458329	0.172	2122	0.001458329	0.141