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On the Reliability of Consecutive Systems

Spiros D. Dafnis, Frosso S. Makri and Zaharias M. Psillakis *

Abstract—**A consecutive system, the failure of which depends on the occurrence of a number of failure-success patterns, is introduced. It extends several consecutive systems studied so far in the literature. The exact system reliability is determined for systems with independently functioning components. The derivations are based on the exact distribution of properly defined random variables whose distributions are obtained by employing an appropriate Markov chain imbedding technique. The results are illustrated by numerical examples.**

Keywords: ***consecutive systems, reliability, Markov chain imbedding***

1 Introduction

A consecutive-k-out-of-n:F system, denoted as $C(k, n : F)$, consists of an ordered sequence of n components for which the existence of k (or more) consecutive failed components causes the system's failure, $1 \leq k \leq n$. $C(1, n : F)$ and $C(n, n : F)$ are series and parallel systems with n components, respectively. Since Kontoleon [12] first introduced and studied these systems in 1980, a series of articles have been published studying their reliability properties under various assumptions because of their wide applicability; e.g. they have been used to model telecommunication systems, oil pipeline systems, vacuum accelerators, etc. For extensive reviews of such systems and the used methods to evaluate their reliability under several component dependencies, we refer to [3], [13] and [19]. For recent contribution on the subject see e.g. [4], [5], [17] and the references therein.

However, a situation may occur in which the system does not weaken so as to fail because of the existence of a run of k consecutive failed components, if this failure run is followed by a run of consecutive working components with sufficiently large length. That is, we consider a system that fails if there are k_1 consecutive failed components, $k_1 \geq k$, followed by less than $k_1 + r$, $r \geq 0$, consecutive working ones. We call such a system consecutive-k, r-out-of-n:FS system and we denote it as $C(k, r, n : FS)$. Given n and k, $1 \leq k \leq n$: (a) $C(k, r, n : FS)$ reduces to $C(k, n : F)$ for $n < 2k$, $r \geq 0$ or for any $r > n - 2k \geq 0$; and (b) for $0 \leq r \leq n - 2k$, $C(k, r, n : FS)$ is a new system more reliable than $C(k, n : F)$ with the same k and n. This is so because the set of possible state configurations causing $C(k, r, n : FS)$ failure is a subset of the respective set of a $C(k, n : F)$.

To understand these systems consider one possible string $SFFFSSSSSS$ for $n = 10$, $k = 2$, $r = 2$, where the symbols S, F denote functioning, failed component states, respectively. In this case, since there are 3 (at least two) consecutive failures a $C(2, 10 : F)$ fails but, since the 3 consecutive failed components are followed by 6 (at least $3+2$) working components a $C(2, 2, 10 : FS)$ system does not fail. However, $SFFFSSFSSS$ represents a failure state of both types of systems.

A generalization of a consecutive-k-out-of-n:F system was formulated by Griffith [9], who considered a system of n ($n \geq mk$) components ordered on a line, for which m (≥ 2) non-overlapping strings of k consecutive failed components are needed for system failure. For such a system, named m-consecutive-k-out-of-n:F system and denoted as $C_m(k, n : F)$, in [16] and [1] exact formulae for its reliability, when the components of the system are iid (independent and identically distributed), have been given. In [18] the failure probability of $C_m(k, n : F)$ having independent components was obtained while in [8] the Stein-Chen method was employed to obtain Poisson approximations for the reliability.

Following the idea of this generalization Agarwal et al. [1] argued that a situation may arise in which a system fails if there are at least m non-overlapping runs of at least k consecutive failures. They called such a system m-consecutive-at least-k-out-of-n:F and they employed GERT analysis to obtain the reliability of the system when its components are iid. This system, was denoted as $C_m^+(k, n : F)$ and for $m = 1$ reduces to $C(k, n : F)$. It is mentioned that for $C_m^+(k, n : F)$ a run of failures of length rk, $r \geq 1$, is treated as one run of length at least k whereas it is treated as r runs of length k for a $C_m(k, n : F)$.

*Date of the manuscript submission: 10/01/10. Corresponding author: Spiros D. Dafnis. First author address: Department of Mathematics, University of Patras, 26500 Patras, Greece, Email:dafnisspyros@gmail.com. Second author address: Department of Mathematics, University of Patras, 26500 Patras, Greece, Email:makri@math.upatras.gr, Third author address: Department of Physics, University of Patras, 26500 Patras, Greece, Email:psillaki@physics.upatras.gr.

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Extending the above generalization we consider a system of n components ordered on a line that fails if there are at least m ($m \geq 1$) runs of failed components of lengths k_i ($k_i \geq k$), $i = 1, \ldots, m, \ldots$, such that the i-th failure run (of length k_i) is followed by less than $k_i + r$ working components, $r \geq 0$, $1 \leq k \leq n$. We call such a system m-consecutive-k,r-out-of-n:FS system and we denote it as $C_m(k, r, n : FS)$. This system reduces to an m-consecutive-at least-k-out-of-n:F system for any $r > n - 2k \geq 0$ and obviously for $n < 2k$, $r \geq 0$. Readily, for a fixed $m \geq 1$, $C_m(k, r, n : FS)$ with $0 \leq r \leq n-2k$, is more reliable than $C_m^+(k, n : F)$ which in turns is more reliable than $C_m(k, n : F)$ with the same parameters k and n. These are true, because the set of possible state configurations causing $C_m(k, r, n : FS)$ failure is a subset of the set of possible state configurations causing $C_m^+(k, n : F)$ failure which in turns is a subset of the respective set of a $C_m(k, n : F)$. For the following sequence of $n = 15$ trials $FFSSSSSFFFSSSSS$ a $C_2^+(2, 15 : F)$ fails whereas a $C_2(2, 2, 15 : FS)$ functions.

In this paper we employ an appropriate Markov chain imbedding technique to obtain the reliabilities of an m-consecutive-at least-k-out-of-n:F ($m \geq 1$), a consecutive-k, r-out-of-n:FS system and the generalized m-consecutive-k, r-out-of-n:FS ($m \geq 2$), when the system components are independently functioning with not necessarily equal reliabilities, via the determination of the exact distribution of properly defined random variables (RVs). The theoretical results are clarified further by numerical examples. Specifically, the study is organized as follows.

In Section 2.1 we establish the reliability of a general class of consecutive systems along with a brief discussion of the Markov chain imbedding method for enumerating RVs. In Section 2.2 we derived the reliability of the under study systems which are presented in Theorems 1 and 2. Finally, in Section 3 we highlight a potential use of such systems in applied research.

2 Reliability of consecutive systems

The reliability of any consecutive system mentioned in Section 1 can be formulated using the following general setup.

2.1 Preliminaries and general results

Let a system consist of an ordered (linear) sequence of n ($n > 0$) components. Each component and the system itself is either good (working or functioning or in state 1) or not-good (failed or no-functioning or in state 0). Let the indicator RVs $Z_1, Z_2, \ldots, Z_n$ represent the states of the system components, i.e. $Z_i = 1$ if component i works; 0, if component i fails. We say that the system fails or it is in state 0 if there are at least m ($m > 0$) occurrences of a pattern $\mathcal{E}$. The pattern $\mathcal{E}$ may be an un-interrupted sequence of 0s or a pre-specified composition of 0s and 1s. If $X_n(\mathcal{E})$ is an enumerating (non-negative) RV denoting the number of occurrences of $\mathcal{E}$ in the sequence $Z_1, Z_2, \ldots, Z_n$ and $\Gamma_n = \{Z_1 = Z_2 = \ldots = Z_n = 0\}$, then the system failure probability is

$$Q_{n,m}(\mathcal{E}) = P(X_n(\mathcal{E}) \geq m), \text{ if } \Gamma_n \in (X_n(\mathcal{E}) \geq m) \quad (1)$$

and

$$Q_{n,m}(\mathcal{E}) = P(X_n(\mathcal{E}) \geq m) + P(\prod_{i=1}^{n}(1 - Z_i) = 1), \quad \text{if} \quad \Gamma_n \notin (X_n(\mathcal{E}) \geq m); \quad (2)$$

whereas the system working (functioning) probability, i.e. the system reliability is

$$R_{n,m}(\mathcal{E}) = 1 - Q_{n,m}(\mathcal{E}). \quad (3)$$

In many cases the exact distribution of $X_n(\mathcal{E})$, therefore the system reliability $R_{n,m}(\mathcal{E})$, may be captured by employing a Markov chain imbedding technique (MCIT) that projects the random variable $X_n(\mathcal{E})$ to appropriate subspaces of the state space of a properly defined Markov chain. Usually, a typical element of the state space is represented by a 2-tuple (x, j). The first component x stands for the number of occurrences of the pattern $\mathcal{E}$ whereas the second component j provides information about the stage of the formation of the next pattern.

It was the novel paper of Fu and Koutras [6] that established the concept of a Markov chain imbeddable RV (MV) and it popularized MCIT. After that, Koutras and Alexandrou [15] refined the method by providing a general recursive scheme for the probability distribution of a Markov chain imbeddable RV of binomial type (MVB). The concept of MVB was extended later by Han and Aki [10] who introduced a Markov chain imbeddable RV of returnable type (MVR) and also gave a general recursive scheme for its probability distribution. The papers [2], [11], [14] as well as the treatise [7] and the references therein present many aspects and applications of MCIT and its versions. In the sequel, a brief -but sufficient for our study, description of MCIT is given. It presents only a summary of the involved concepts and the necessary notation in order to make the article self-contained.

Definition 1. A random variable X_n (n a non-negative integer) with support $\{0, 1, \ldots, \ell_n\}$, $\ell_n = \max\{x; P(X_n = x) > 0\}$, will be called Markov chain imbeddable variable if
(a) there exists a Markov chain $\{Y_t; t \geq 0\}$ defined on a state space Ω
(b) there exists a partition $\{C_x, x = 0, 1, \ldots\}$ on Ω, $C_x = \{c_{x,0}, c_{x,1}, \ldots, c_{x,s-1}\}$, $s =| C_x |$ and
(c) for every $x = 0, 1, \ldots, \ell_n$

$$P(X_n = x) = P(Y_n \in C_x).$$

Definition 2. A Markov chain imbeddable random variable will be called of
(a) Binomial type (MVB) if $P(Y_t = c_{y,j} \mid Y_{t-1} = c_{x,i}) = 0$, for all $y \neq x, x+1$, $t \geq 1$, or equivalently $P(Y_t \in C_y \mid Y_{t-1} \in C_x) = 0$, for all $y \neq x, x+1$,
(b) Returnable type (MVR) if $P(Y_t = c_{y,j} \mid Y_{t-1} = c_{x,i}) = 0$, for all $y \neq x-1, x, x+1$, $t \geq 1$, or equivalently $P(Y_t \in C_y \mid Y_{t-1} \in C_x) = 0$, for all $y \neq x-1, x, x+1$.

It is noted that for an MVB there are transitions within the same sub-state set C_x and transitions from set C_x to C_{x+1} while in the case of an MVR there are, in addition, transitions from set C_x to C_{x-1}, i.e. the process can also move backwards.

Next, let the one-step $s \times s$ transition matrices

$$A_t(x) = (\alpha_{ij}^{(t)}(x)),\ B_t(x) = (\beta_{ij}^{(t)}(x)),\ D_t(x) = (d_{ij}^{(t)}(x))$$

with

$$\alpha_{ij}^{(t)}(x) = P(Y_t = c_{x,j} \mid Y_{t-1} = c_{x,i}),$$

$$\beta_{ij}^{(t)}(x) = P(Y_t = c_{x+1,j} \mid Y_{t-1} = c_{x,i}),$$

$$d_{ij}^{(t)}(x) = P(Y_t = c_{x-1,j} \mid Y_{t-1} = c_{x,i})$$

and $\mathbf{f}_t(x)$ the probability vector associated with time t and sub-state set C_x, i.e., for $0 \leq t \leq n$,

$$\mathbf{f}_t(x) = (P(Y_t = c_{x,0}), P(Y_t = c_{x,1}), \ldots, P(Y_t = c_{x,s-1})).$$

Then, readily

$$P(X_n = x) = \mathbf{f}_n(x)\mathbf{1}^{'}, \quad x = 0, 1, \ldots, \ell_n \tag{4}$$

with $\mathbf{1} = (1, 1, \ldots, 1) \in R^s$. Also, the convention $P(X_0 = 0) = 1$ implies that if π_x is the (row) vector of initial probabilities of the Markov chain, i.e. $\pi_x = \mathbf{f}_0(x)$ then $\pi_0 \mathbf{1}^{'} = 1$ and $\pi_x \mathbf{1}^{'} = 0$, $x > 0$.

The following Lemmas 1 and 2 provide recursive relations for the probability vectors $\mathbf{f}_t(x)$.

Lemma 1. ([15]). For an MVB X_n the sequence $\mathbf{f}_t(x)$, $t = 1, 2, \ldots, n$ satisfies the recurrence relations

$$\mathbf{f}_t(0) = \mathbf{f}_{t-1}(0) A_t(0)$$

$$\mathbf{f}_t(x) = \mathbf{f}_{t-1}(x) A_t(x) + \mathbf{f}_{t-1}(x-1) B_t(x-1), \quad 1 \leq x \leq \ell_n.$$

Lemma 2. ([10]). For an MVR X_n the sequence $\mathbf{f}_t(x)$, $t = 1, 2, \ldots, n$ satisfies the recursive relations

$$\mathbf{f}_t(x) = \mathbf{0}, \quad x < 0, \quad \text{or} \quad x > \ell_t$$

$$\begin{aligned} \mathbf{f}_t(x) \quad &= \quad \mathbf{f}_{t-1}(x) A_t(x) + \mathbf{f}_{t-1}(x-1) B_t(x-1) \\ &\quad + \mathbf{f}_{t-1}(x+1) D_t(x+1), \quad 0 \leq x \leq \ell_t. \end{aligned}$$

As we can see the method requires, the proper state space Ω, its partition $\{C_x, 0 \leq x \leq \ell_n\}$ and the transition matrices, A, B and D, to be determined. Then, relation (4) along with Lemmas 1 or 2, provide the probability distribution of the under study RV X_n. Therefore, within the framework of MCIT, what remains for the computation of the reliabilities of the under study consecutive systems, in particular $C_m^+(k, n : F)$ and $C_m(k, r, n : FS)$, is: First, the definition of the failure patterns $\mathcal{E}$ which correspond to the respective systems (i.e. the associated RVs) and second, the determination of the proper state spaces and transition probability matrices. The first task is given next whereas the second is presented in Section 2.2.

Proposition 1. The correspondences among consecutive systems, patterns $\mathcal{E}$ causing their failures and the enumerating RVs used to evaluate the systems reliabilities are:

(I) $C_m^+(k, n : F)$: Let $\mathcal{E} \equiv \mathcal{E}_k = \underbrace{FF \ldots F}_{k_1 \geq k}$, then $X_n(\mathcal{E}) \equiv G_{n,k}$ denoting the number of failure runs of length at least k in a sequence of n binary (success-failure) trials ordered on a line. Its support is $S_{G_{n,k}} = \{0, 1, \ldots, \ell_n = [\frac{n+1}{k+1}]\}$, where by $[x]$ we denote the greatest integer less than or equal to x.

(II) $C_m(k, r, n : FS)$: Let $\mathcal{E} \equiv \mathcal{E}_{k,r} = \underbrace{FF \ldots F}_{k_1 \geq k} \underbrace{SS \ldots S}_{k_2 < k_1 + r}$, then $X_n(\mathcal{E}) \equiv N_{n,k,r}$ denoting the number of occurrences of the pattern $\mathcal{E}_{k,r}$ in a sequence of n binary (success-failure) trials ordered on a line. Its support is $S_{N_{n,k,r}} = \{0, 1, \ldots, \ell_n\}$, with $\ell_n = [\frac{n+2}{3}]$, if $k = 1$, $r = 0$; $[\frac{n+1}{k+1}]$, otherwise.

We note that for $n < 2k$, $r \geq 0$ or for $r > n - 2k \geq 0$, $N_{n,k,r}$ counts as $G_{n,k}$ does since the numbers of the patterns $\mathcal{E}_k$ and $\mathcal{E}_{k,r}$ coincide, hence $C_m(k, r, n : FS)$ reduces to $C_m^+(k, n : F)$, for $1 \leq m \leq \ell_n$. In general, $N_{n,k,r} \leq G_{n,k}$ for $r \geq 0$. Specifically, it holds: $N_{n,k,r} \leq G_{n,k}$ for $0 \leq r \leq n - 2k$ and $N_{n,k,r} = G_{n,k}$ for $r > n - 2k \geq 0$ or for $n < 2k$, $r \geq 0$. As an example we consider again the binary sequence: $FFSSSSSFFFSSSSS$ of $n = 15$ trials. Then, we have: $G_{n,2} = 2$, $N_{n,2,2} = 0$ and $N_{n,2,r} = 2$ for $r > 11$.

2.2 Systems with independent components

In the sequel we determine the exact reliabilities of $C_m^+(k, n : F)$ and $C_m(k, r, n : FS)$ under the assumptions that the components of the system fail independently of each other and their reliabilities $p_i = P(Z_i = 1) = 1 - P(Z_i = 0) = 1 - q_i$ are known for $i = 1, 2, \ldots, n$. The p_i's are not necessarily the same; if this happens we have the particular case of systems with iid components, that is $p_i = p$, $q_i = q = 1 - p$ for $i = 1, 2, \ldots, n$.

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Let $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. Since the system reliability is a function of $\mathbf{p}$, we denote by $R_m^+(k, n; \mathbf{p})$ and $R_m(k, r, n; \mathbf{p})$ the reliability functions of $C_m^+(k, n : F)$ and $C_m(k, r, n : FS)$ which are given in Theorems 1 and 2, respectively. The proofs of the theorems are provided in the Appendix A.

2.2.1 Reliability of an m-consecutive-at least-k-out-of-n:F system, $n \geq 2$

In [15] (see also [14]) it was proved that $G_{n,k}$ is a MVB with

$$C_x = \{(x, j); j = -1, 0, 1, \ldots, k-1\}$$

and

$$A_t(x) = A_t = \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \cdot & k-1 & -1 \end{array} \\ \begin{pmatrix} p_t & q_t & 0 & \cdot & 0 & 0 \\ p_t & 0 & q_t & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ p_t & 0 & 0 & \cdot & q_t & 0 \\ p_t & 0 & 0 & \cdot & 0 & 0 \\ p_t & 0 & 0 & \cdot & 0 & q_t \end{pmatrix} \end{array},$$

$$B_t(x) = B_t = \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \cdot & k-1 & -1 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & q_t \\ 0 & 0 & 0 & \cdot & 0 & 0 \end{pmatrix} \end{array}.$$

Theorem 1. For positive integers k, m and n the reliability $R_m^+(k, n; \mathbf{p})$ of a $C_m^+(k, n : F)$ with independent components and reliability of the i-th component p_i, $i = 1, 2, \ldots, n$, is given by

$$R_m^+(k, n; \mathbf{p}) = 1 - \prod_{i=1}^{n}(1 - p_i), \quad \text{if} \quad n < mk + m - 1$$

and

$$R_m^+(k, n; \mathbf{p}) = \sum_{x=0}^{m-2} \mathbf{f}_{n-1}(x)(A_n + B_n)\mathbf{1}' + \mathbf{f}_{n-1}(m-1)A_n\mathbf{1}' - \zeta(1, m)\prod_{i=1}^{n}(1 - p_i), \quad \text{if} \quad n \geq mk + m - 1,$$

with A_n and B_n being the matrices given previously, $\mathbf{f}$'s are evaluated via the recursive scheme of Lemma 1 and $\zeta(1, m) = 1$ if $m > 1$; 0, otherwise.

Remark 1. For $m = 1$, $C_m^+(k, n : F)$ reduces to $C(k, n : F)$. Hence, its reliability $R(k, n; \mathbf{p})$ can be computed via the relationship

$$R(k, n; \mathbf{p}) = R_1^+(k, n; \mathbf{p}) = \mathbf{f}_{n-1}(0)A_n\mathbf{1}', \quad n \geq k.$$

2.2.2 Reliability of an m-consecutive-k, r-out-of-n:FS system, $n \geq 2$

The random variable $N_{n,k,r}$ is an MVR (see, Appendix A.2) with

$$C_x = \begin{cases} \{(x, j); j = 0, 1, 2\}, & \text{if} \quad k = 1, \quad r = 0, \quad n = 2, 3 \\ \{(x, j); j = -[\frac{n-r}{2}] - r + 1, \ldots, -1, 0, 1, \ldots, [\frac{n-r}{2}], [\frac{n-r}{2}] + 1\}, & \text{otherwise} \end{cases}$$

and transition matrices A, B, D given as follows.

(a) If $k > 1$ then

$A_t(x) = A_t$

$$= \begin{array}{c} \begin{array}{ccccccccccccccccc} 0 & 1 & 2 & \cdot & k-1 & k & k+1 & \cdot & [\frac{n-r}{2}] & [\frac{n-r}{2}]+1 & -1 & -2 & \cdot & -k-r+1 & \cdot & -[\frac{n-r}{2}]-r+2 & -[\frac{n-r}{2}]-r+1 \end{array} \\ \begin{pmatrix} p_t & q_t & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & 0 & \cdot & 0 & 0 \\ p_t & 0 & q_t & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & 0 & \cdot & 0 & 0 \\ p_t & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ p_t & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & q_t & \cdot & 0 & 0 & 0 & 0 & \cdot & p_t & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & q_t & 0 & 0 & \cdot & 0 & \cdot & 0 & p_t \\ p_t & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & q_t & 0 & 0 & \cdot & 0 & \cdot & 0 & 0 \\ 0 & q_t & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & 0 & \cdot & 0 & 0 \\ 0 & q_t & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & p_t & 0 & \cdot & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & q_t & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & q_t & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & 0 & \cdot & 0 & 0 \\ 0 & q_t & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & 0 & \cdot & p_t & 0 \end{pmatrix} \end{array}$$

$$B_t(x) = B_t = (\beta_{ij}^{(t)})_{(2[\frac{n-r}{2}]+r+1)\times(2[\frac{n-r}{2}]+r+1)} \text{ with } \beta_{k-1,k}^{(t)} = q_t \text{ and } \beta_{ij}^{(t)} = 0 \text{ for } (i, j) \neq (k-1, k)$$

and

$$D_t(x) = D_t = (d_{ij}^{(t)})_{(2[\frac{n-r}{2}]+r+1)\times(2[\frac{n-r}{2}]+r+1)} \text{ with } d_{-1,0}^{(t)} = p_t \text{ and } d_{ij}^{(t)} = 0 \text{ for } (i, j) \neq (-1, 0).$$

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(b) If $k=1$, $r>0$, then

$$
\begin{aligned}
A_t(x) &= A_t \\
&= \left(\begin{array}{cccccccccccccc}
0 & 1 & 2 & \cdot & [\frac{n-r}{2}] & [\frac{n-r}{2}]+1 & -1 & -2 & \cdot & -r & -r-1 & \cdot & -[\frac{n-r}{2}]-r+2 & -[\frac{n-r}{2}]-r+1 \\
\hline
p_t & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & \cdot & 0 & 0 \\
0 & 0 & q_t & \cdot & 0 & 0 & 0 & 0 & \cdot & p_t & 0 & \cdot & 0 & 0 \\
0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & 0 & p_t & \cdot & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdot & 0 & q_t & 0 & 0 & \cdot & 0 & 0 & \cdot & 0 & p_t \\
p_t & 0 & 0 & \cdot & 0 & q_t & 0 & 0 & \cdot & 0 & 0 & \cdot & 0 & 0 \\
0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & \cdot & 0 & 0 \\
0 & 0 & 0 & \cdot & 0 & 0 & p_t & 0 & \cdot & 0 & 0 & \cdot & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & \cdot & 0 & 0 \\
0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & p_t & 0 & \cdot & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & \cdot & 0 & 0 \\
0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & \cdot & p_t & 0
\end{array}\right)
\end{aligned}
$$

$B_t(x) = B_t = (\beta_{ij}^{(t)})_{(2[\frac{n-r}{2}]+r+1)\times(2[\frac{n-r}{2}]+r+1)}$ with $\beta_{i1}^{(t)} = q_t$, $i = 0, -1, -2, \ldots, -[\frac{n-r}{2}]-r+1$, $\beta_{ij}^{(t)} = 0$ for all the other entries and

$D_t(x) = D_t = (d_{ij}^{(t)})_{(2[\frac{n-r}{2}]+r+1)\times(2[\frac{n-r}{2}]+r+1)}$ with $d_{-1,0}^{(t)} = p_t$ and $d_{ij}^{(t)} = 0$ for $(i,j) \neq (-1,0)$.

(c) If $k=1$, $r=0$, $n>3$ then

$$
\begin{aligned}
A_t(x) &= A_t \\
&= \left(\begin{array}{ccccccccccc}
0 & 1 & 2 & \cdot & [\frac{n}{2}] & [\frac{n}{2}]+1 & -1 & -2 & \cdot & -[\frac{n}{2}]+2 & -[\frac{n}{2}]+1 \\
\hline
p_t & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\
0 & 0 & q_t & \cdot & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\
0 & 0 & 0 & \cdot & 0 & 0 & p_t & 0 & \cdot & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdot & 0 & q_t & 0 & 0 & \cdot & 0 & p_t \\
p_t & 0 & 0 & \cdot & 0 & q_t & 0 & 0 & \cdot & 0 & 0 \\
0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\
0 & 0 & 0 & \cdot & 0 & 0 & p_t & 0 & \cdot & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\
0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & p_t & 0
\end{array}\right)
\end{aligned}
$$

$B_t(x) = B_t = (\beta_{ij}^{(t)})_{(2[\frac{n}{2}]+1)\times(2[\frac{n}{2}]+1)}$ with $\beta_{i1}^{(t)} = q_t$, $i = 0, -1, -2, \ldots, -[\frac{n}{2}]+1$, $\beta_{ij}^{(t)} = 0$, for all the other entries and

$D_t(x) = D_t = (d_{ij}^{(t)})_{(2[\frac{n}{2}]+1)\times(2[\frac{n}{2}]+1)}$ with $d_{-1,0}^{(t)} = p_t$, $d_{10}^{(t)} = p_t$ and $d_{ij}^{(t)} = 0$, for all the other entries.

(d) If $k=1$, $r=0$, $n=2,3$, then

$A_t(x) = A_t = (a_{ij}^{(t)})_{3\times3}$ with $a_{00}^{(t)} = a_{20}^{(t)} = p_t$, $a_{12}^{(t)} = a_{22}^{(t)} = q_t$ and $a_{ij}^{(t)} = 0$, for all the other entries,

$B_t(x) = B_t = (\beta_{ij}^{(t)})_{3\times3}$ with $\beta_{01}^{(t)} = q_t$; $\beta_{ij}^{(t)} = 0$, for $(i,j) \neq (0,1)$ and

$D_t(x) = D_t = (d_{ij}^{(t)})_{3\times3}$ with $d_{10}^{(t)} = p_t$, $d_{ij}^{(t)} = 0$, $(i,j) \neq (1,0)$.

In cases (a)-(d) the states in matrices B and D are labelled as in the respective matrices A.

Theorem 2. The reliability $R_m(k,r,n;\mathbf{p})$ of a $C_m(k,r,n:FS)$ for $0 \le r \le n-2k$, with independent components and reliability of the i-th component p_i, $i = 1,2,\ldots,n$, is given by:

(a) $\quad R_1(k,r,n;\mathbf{p}) = \mathbf{f}_{n-1}(0)A_n\mathbf{1}' + \mathbf{f}_{n-1}(1)D_n\mathbf{1}'$;

(b) $\quad R_2(k,r,n;\mathbf{p}) = \mathbf{f}_{n-1}(0)(A_n+B_n)\mathbf{1}' + \mathbf{f}_{n-1}(1)(A_n+D_n)\mathbf{1}' + \mathbf{f}_{n-1}(2)D_n\mathbf{1}' - \prod_{i=1}^{n}(1-p_i)$;

and for $m \ge 3$,

(c) $\quad R_m(k,r,n;\mathbf{p}) = \sum_{x=1}^{m-2}\mathbf{f}_{n-1}(x)(A_n+B_n+D_n)\mathbf{1}' + \mathbf{f}_{n-1}(0)(A_n+B_n)\mathbf{1}' + \mathbf{f}_{n-1}(m-1)(A_n+D_n)\mathbf{1}' + \mathbf{f}_{n-1}(m)D_n\mathbf{1}' - \prod_{i=1}^{n}(1-p_i)$,

where A_n, B_n, D_n are the matrices given above and $\mathbf{f}$'s are evaluated via the recursive scheme of Lemma 2.

Remark 2. Since $C_m(k,r,n:FS)$ reduces to $C_m^+(k,n:F)$ for $r > n-2k \ge 0$, $R_m(k,r,n;\mathbf{p})$ becomes $R_m^+(k,n;\mathbf{p})$ which is provided by Theorem 1.

3 A note for application

In applied reliability studies we need to work with specific systems. To this end, we consider possible examples of $C_m^+(k, n : F)$ and $C_m(k, r, n : FS)$.

Example Let an alarm system of an accelerator consist of n (≥ 2) detectors (feelers) posed along the surface of an accelerator. The feelers (i.e. the system components) might measure the temperature or the radioactivity level of the accelerator. Their failures are likely to occur independently. The reliabilities of the detectors may be different because of the holding conditions and the operational procedures among the individual feelers or they may be identical due to economic reasons and maintenance policy. We consider that such a system fails if there are at least m (≥ 1) clusters of feelers that either: (I) each has at least k consecutive feelers failed or (II) the i-th cluster consists of k_i $(\geq k)$ failed feelers and is followed by less than k_i+r working ones, $r \geq 0$ (i.e. a situation which implies a malfunction of the system which is not possible to be compensated). Readily, cases (I) and (II) correspond to $C_m^+(k, n : F)$ and $C_m(k, r, n : FS)$, respectively.

Next, in order to evaluate the reliability of the alarm systems discussed we consider some specific configurations of them. The system reliabilities are computed via Theorems 1 or 2 and they present results helpful to a practically minded reader.

Table 1 Exact reliabilities of $C_m^+(k, 15 : F)$ and $C_m(k, r, 15 : FS)$ with independent components, for (A) $p_i = 0.85$ if i is odd; 0.95, if i is even (B) $p_i = p = 0.90$, $i = 1, 2, \ldots, n$

k	r	m	System type	Reliability	
				(A)	(B)
1		1	series	0.190290	0.205891
1	2	1	consecutive:FS	0.580285	0.618671
1	0	1	consecutive:FS	0.836277	0.880764
3		1	consecutive:F	0.990575	0.988237
3		2	at least consecutive:F	0.999975	0.999963
3	2	1	consecutive:FS	0.994128	0.992845
3	2	2	consecutive:FS	0.999984	0.999976
3	0	1	consecutive:FS	0.996079	0.995358
3	0	2	consecutive:FS	0.999993	0.999990
15		1	parallel	1.000000	1.000000

Let a system consist of $n = 15$ independently functioning feelers. Table 1 depicts the exact system reliabilities $R_m^+(k, n; \mathbf{p})$ and $R_m(k, r, n; \mathbf{p})$ for various values of k, r, m and for $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ such that $p_i = 0.85$, if i is odd; 0.95, if i is even (Part A, non-iid case) and $p_i = p = 0.90$ (Part B, iid case). The entries of Table 1 show how the reliabilities of the several presented systems vary depending on their type as well as on their internal structure. Further, they confirm that $R_m(k, r_1, n; \mathbf{p}) \geq R_m(k, r_2, n; \mathbf{p}) \geq R_m^+(k, n; \mathbf{p})$, $0 \leq r_1 \leq r_2 \leq n - 2k$.

4 Summary and discussion

In this article, we studied the reliability of two generalizations of the classical consecutive system. The system components were assumed to function independently of each other. The results were derived via Markov chain imbedding.

A potential application concerning an alarm system was discussed to justify the usefulness of such systems. It was illustrated further by indicative numerical results.

Closing this section we mention that the approach used for the study of the new RV $N_{n,k,r}$ can be modified to capture also its behavior under a Markovian setup on a sequence of failure-success trials. This study might be connected with forecasting in financial markets.

Appendix A

A.1 Proof of Theorem 1

Let $Z_1, Z_2, \ldots, Z_n$ be a sequence of n independent binary $(0-1)$ random variables with $P(Z_i = 1) = p_i$, $i = 1, 2, \ldots, n$ and Γ_n as in Section 2.1. For $n < mk + m - 1$,

$$R_m^+(k, n; \mathbf{p}) = 1 - P(\Gamma_n) = 1 - \prod_{i=1}^{n}(1 - p_i),$$

because of the independence of the components. For $n \geq mk + m - 1$,

$$R_1(n, k; \mathbf{p}) = P(G_{n,k} < 1)$$

and for $m \geq 2$,

$$\begin{aligned} R_m^+(k, n; \mathbf{p}) &= P((G_{n,k} < m) \cap \Gamma_n^c) \\ &= \sum_{x=0}^{m-1} \mathbf{f}_n(x)\mathbf{1}^{'} - \prod_{i=0}^{n}(1 - p_i). \end{aligned}$$

Next, noting that $G_{n,k}$ is an MVB we get the result using the recursive scheme of Lemma 1.

A.2 Proof of Theorem 2

Let C_x be as in Section 2.2.2 and $\Omega = \cup_{x \geq 0} C_x$. To introduce a proper Markov chain $\{Y_t; t \geq 0\}$ on Ω for the RV $N_{n,k,r}$, for $k > 1$, we define $Y_t = (x, j)$ as follows:

$Y_t = (x, j)$, if at the first t outcomes a pattern $\mathcal{E}_{k,r}$ has occurred x times and

- $j = 0$, if the t-th outcome is a success and (a) no failures are preceded (the success at the t-th outcome) or (2) the last occurred consecutive failures

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are less than k, or (3) the last occurred consecutive failures are more than or equal to $[\frac{n-r}{2}]+1$ or (4) it is the last outcome of a success run with length greater than or equal to the length of the immediately preceded failure run (of length greater than $k-1$) plus r.

- $j=i$, $i=1,2,\ldots,[\frac{n-r}{2}]$, if the last i outcomes, the $t-i+1,...,t-1,t$ are i consecutive failures which are preceded by a success
- $j=[\frac{n-r}{2}]+1$, if the t-th outcome is the last failure of a failure run with length greater than or equal to $[\frac{n-r}{2}]+1$.
- $j=-i$, $i=1,2,\ldots,[\frac{n-r}{2}]+r-1$, if the t-th outcome is the last success of a success run which is preceded by a failure run with length greater than $k-1$ (and less than or equal to $[\frac{n-r}{2}]$) and i more successes are required to "cover" the preceding failure run (i.e. the length of the success run to be equal in number plus r to the length of the preceding failure run).

We note that the number of patterns $\mathcal{E}_{k,r}$ increases by 1 when the number of consecutive failures becomes equal to k and reduces by 1 when a run of consecutive successes "covers" the preceding failure run. Therefore, the transition matrices $A_t(x)$, $B_t(x)$ and $D_t(x)$ become as they are presented in case (a) before Theorem 2. For $k=1$, the transition matrices A, B and D are obtained using similar concepts. Next, let Γ_n be as in A.1, then it is evident that

$$R_1(k,r,n;\mathbf{p})=P(N_{n,k,r}<1)=\mathbf{f}_n(0)\mathbf{1}^{'}$$

and for $m\geq 2$,

$$\begin{aligned} R_m(k,r,n;\mathbf{p}) &= P((N_{n,k,r}<m)\cap\Gamma_n^c) \\ &= \sum_{x=0}^{m-1}\mathbf{f}_n(x)\mathbf{1}^{'}-\prod_{i=0}^{n}(1-p_i). \end{aligned}$$

The results follow using the recursive relations of Lemma 2.

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