Multidimensional Matrix Mathematics: Multidimensional Matrix Equality, Addition, Subtraction, and Multiplication, Part 2 of 6

Ashu M. G. Solo

***Abstract*—This is the first series of research papers to define multidimensional matrix mathematics, which includes multidimensional matrix algebra and multidimensional matrix calculus. These are new branches of math created by the author with numerous applications in engineering, math, natural science, social science, and other fields. Cartesian and general tensors can be represented as multidimensional matrices or vice versa. Some Cartesian and general tensor operations can be performed as multidimensional matrix operations or vice versa. However, many aspects of multidimensional matrix math and tensor analysis are not interchangeable. Part 2 of 6 defines multidimensional matrix equality as well as the multidimensional matrix algebra operations for addition, subtraction, multiplication by a scalar, and multiplication of two multidimensional matrices. An alternative representation of the summation of quadratic terms using multidimensional matrix multiplication is described.**

***Index Terms*—multidimensional matrix math, multidimensional matrix algebra, multidimensional matrix calculus, matrix math, matrix algebra, matrix calculus, tensor analysis**

I. Introduction

Part 2 of 6 defines multidimensional matrix equality as well as the multidimensional matrix algebra operations for addition, subtraction, multiplication by a scalar, and multiplication of two multidimensional matrices. Also, part 2 of 6 describes an alternative representation of the summation of quadratic terms using multidimensional matrix multiplication.

II. Multidimensional Matrix Equality

Two multidimensional matrices **A** and **B** are considered equal if and only if **A** and **B** have the same number of elements in each dimension or can be simplified such that they have the same number of elements in each dimension and all corresponding elements of these two multidimensional matrices are equal. That is, $a_{ijk\,.\,.\,.\,q} = b_{ijk\,.\,.\,.\,q}$ for $1 \le i \le s$, $1 \le j \le t$, $1 \le k \le u$, and so on until $1 \le q \le z$.

Just like two tensors are equivalent when each corresponding component is equal, two multidimensional matrices are equal when each corresponding element is equal.

III. Multidimensional Matrix Addition

Multidimensional matrix addition is performed on an element-by-element basis. Two multidimensional matrices can only be added if they have the same number of elements in each dimension or can be simplified such that they have the same number of elements in each dimension. Consider the addition of two multidimensional matrices **A** and **B** where **A** + **B** = **C**. The resulting sum matrix **C** will have the same number of elements in each dimension as **A** and **B.** Its content is determined on an element-by-element basis with $c_{ijk\,.\,.\,.\,q} = a_{ijk\,.\,.\,.\,q} + b_{ijk\,.\,.\,.\,q}$ for $i = 1, 2, \ldots, s;$ $j = 1, 2, \ldots, t;$ $k = 1, 2, \ldots, u;$ and so on until $q = 1, 2, \ldots, z.$

In the following example, a 3-D matrix with dimensions of 3 * 2 * 2 is added to another 3-D matrix with dimensions of 3 * 2 * 2, and the result is a sum 3-D matrix with dimensions of 3 * 2 * 2:

$$\begin{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \\ \begin{bmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{bmatrix} \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} 2 & 8 \\ 4 & 10 \\ 6 & 12 \end{bmatrix} \\ \begin{bmatrix} 14 & 20 \\ 16 & 22 \\ 18 & 24 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 3 & 12 \\ 6 & 15 \\ 9 & 18 \end{bmatrix} \\ \begin{bmatrix} 21 & 30 \\ 24 & 33 \\ 27 & 36 \end{bmatrix} \end{bmatrix}$$

The addition of two tensors is equivalent to the addition of two multidimensional matrices with each corresponding component of each tensor being added like each corresponding element of each multidimensional matrix is added.

IV. Multidimensional Matrix Subtraction

Multidimensional matrix subtraction is performed on an element-by-element basis. Two multidimensional matrices can only be subtracted if they have the same number of elements in each dimension or can be simplified such that they have the same number of elements in each dimension. Consider the subtraction of two multidimensional matrices **A** and **B** where **A** - **B** = **C**. The resulting difference matrix **C** will have the same number of elements in each dimension as **A** and **B.** Its content is determined on an element-by-element basis with = $c_{ijk\,.\,.\,.\,q} = a_{ijk\,.\,.\,.\,q} - b_{ijk\,.\,.\,.\,q}$ for $i = 1, 2, \ldots, s;$ $j = 1, 2, \ldots, t;$ $k = 1, 2, \ldots, u;$ and so on until $q = 1, 2, \ldots, z.$

The subtraction of one tensor from another tensor is equivalent to the subtraction of one multidimensional matrix from another with each component of one tensor being

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subtracted from the corresponding component of another tensor like each element of one multidimensional matrix is subtracted from the corresponding element of another multidimensional matrix.

V. Multiplication Of a Multidimensional Matrix By A Scalar

Multiplication of a multidimensional matrix by a scalar results in multiplying every element of the multidimensional matrix by the scalar.

In the following example, a 4-D matrix with dimensions of 3 * 2 * 1 * 2 is multiplied by the scalar 5 with the resulting 4-D matrix with dimensions of 3 * 2 * 1 * 2 as shown:

$$5 * \begin{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}, \begin{bmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 5 & 20 \\ 10 & 25 \\ 15 & 30 \end{bmatrix}, \begin{bmatrix} 35 & 50 \\ 40 & 55 \\ 45 & 60 \end{bmatrix} \end{bmatrix}$$

The multiplication of a tensor by a scalar is equivalent to the multiplication of a multidimensional matrix by a scalar with each component of the tensor being multiplied by the scalar like each element of the multidimensional matrix is multiplied by the scalar.

VI. Multidimensional Matrix Product

In multidimensional matrix algebra, any two dimensions of two multidimensional matrices can be multiplied together. This section describes the multidimensional matrix product, which is an extension and generalization of the matrix product in classical matrix algebra. Note that the multidimensional matrix outer product and multidimensional matrix inner product are described in separate sections in this series of research papers.

A. Notation for Multidimensional Matrix Multiplication

In the equations below, **A** represents the first multidimensional matrix being multiplied, **B** represents the second multidimensional matrix being multiplied, and **C** represents the multidimensional matrix product.

$N_d(\mathbf{M})$ is the number of elements in the *d*th dimension of multidimensional matrix **M.**

The variable *da* refers to the first and lower dimension being multiplied. The variable *db* refers to the second and higher dimension being multiplied. Therefore, $db > da$.

Therefore, the variable $N_{da}(\mathbf{A})$ refers to the number of elements in the first dimension being multiplied in multidimensional matrix **A,** the variable $N_{db}(\mathbf{A})$ refers to the number of elements in the second dimension being multiplied in multidimensional matrix **A,** the variable $N_{da}(\mathbf{B})$ refers to the number of elements in the first dimension being multiplied in multidimensional matrix **B,** and the variable $N_{db}(\mathbf{B})$ refers to the number of elements in the second dimension being multiplied in multidimensional matrix **B.**

The variable *n* is the number of elements in the second dimension being multiplied of the first multidimensional matrix **A** or the number of elements in the first dimension being multiplied of the second multidimensional matrix **B.** That is, $n = N_{db}(\mathbf{A}) = N_{da}(\mathbf{B})$.

B. Conformability Requirements for Multidimensional Matrix Multiplication

For the following conformability requirements, for any 1-D column vector, there is considered to be a second dimension having a size of one. By the rules of multidimensional matrix simplification, these are equivalent multidimensional matrices.

The conformability requirements for multiplication of two multidimensional matrices are as follows:

1. The number of elements in the second dimension being multiplied in the first multidimensional matrix must equal the number of elements in the first dimension being multiplied of the second multidimensional matrix. That is, $N_{db}(\mathbf{A}) = N_{da}(\mathbf{B})$. Alternatively, one or both of the multidimensional matrices can be simplified such that the number of elements in the second dimension being multiplied of the first multidimensional matrix must equal the number of elements in the first dimension being multiplied of the second multidimensional matrix.

   For example, if the first dimension and second dimension of multidimensional matrix **A** and multidimensional matrix **B** are being multiplied, then $N_2(\mathbf{A}) = N_1(\mathbf{B})$. If the third dimension and fourth dimension of multidimensional matrix **A** and multidimensional matrix **B** are being multiplied, then $N_4(\mathbf{A}) = N_3(\mathbf{B})$. If the first dimension and fourth dimension of multidimensional matrix **A** and multidimensional matrix **B** are being multiplied, then $N_4(\mathbf{A}) = N_1(\mathbf{B})$.
2. The number of elements in each dimension not being multiplied in the first multidimensional matrix must equal the number of elements in the same dimension not being multiplied in the second multidimensional matrix. Alternatively, one or both of the multidimensional matrices can be simplified such that the number of elements in each dimension not being multiplied in the first multidimensional matrix must equal the number of elements in the same dimension not being multiplied in the second multidimensional matrix.

   For example, if the first dimension and second dimension of multidimensional matrix **A** and multidimensional matrix **B** are being multiplied, then $N_d(\mathbf{A}) = N_d(\mathbf{B})$ for $d >= 3$. If the third dimension and fourth dimension of multidimensional matrix **A** and multidimensional matrix **B** are being multiplied, then $N_d(\mathbf{A}) = N_d(\mathbf{B})$ for $d = 1$, $d = 2$, and $d >= 5$. If the first dimension and fourth dimension of multidimensional matrix **A** and multidimensional matrix **B** are being multiplied, then $N_d(\mathbf{A}) = N_d(\mathbf{B})$ for $d = 2$, $d = 3$, and $d >= 5$.

If conformability requirements are not met, two multidimensional matrices cannot be multiplied and their product is nonexistent.

If conformability requirements are met, in the resulting multidimensional matrix product, the number of dimensions is equal to the number of dimensions in the first or second matrix being multiplied.

C. Number of Elements in Each Dimension of Multidimensional Matrix Product

The number of elements in the first dimension being multiplied in the first multidimensional matrix provides the

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number of elements in the corresponding dimension of the multidimensional matrix product. That is, N_{da}**(C)** = N_{da}**(A).** The number of elements in the second dimension being multiplied in the second multidimensional matrix provides the number of elements in the corresponding dimension of the multidimensional matrix product. That is, N_{db}**(C)** = N_{db}**(B).** The number of elements in each other dimension of the multidimensional matrix product is equal to the number of elements in the same dimension of the first or second multidimensional matrix being multiplied.

For example, consider the case in which the first and second dimensions of multidimensional matrix **A** and multidimensional matrix **B** are being multiplied. Then the number of elements in the first dimension of multidimensional matrix product **C** is equal to the number of elements in the first dimension of multidimensional matrix **A.** That is, N_1**(C)** = N_1**(A).** The number of elements in the second dimension of multidimensional matrix product **C** is equal to the number of elements in the second dimension of multidimensional matrix **B.** That is, N_2**(C)** = N_2**(B).** The number of elements in other dimensions of multidimensional matrix product **C** is equal to the number of elements in corresponding dimensions of multidimensional matrix **A** or multidimensional matrix **B.** That is, N_d**(C)** = N_d**(A)** = N_d**(B)** for $d >= 3$.

Consider the case in which the second and fifth dimensions of multidimensional matrix **A** and multidimensional matrix **B** are being multiplied. Then the number of elements in the second dimension of multidimensional matrix product **C** is equal to the number of elements in the second dimension of multidimensional matrix **A.** That is, N_2**(C)** = N_2**(A).** The number of elements in the fifth dimension of multidimensional matrix product **C** is equal to the number of elements in the fifth dimension of multidimensional matrix **B.** That is, N_5**(C)** = N_5**(B).** The number of elements in other dimensions of multidimensional matrix product **C** is equal to the number of elements in corresponding dimensions of multidimensional matrix **A** or multidimensional matrix **B.** That is, N_d**(C)** = N_d**(A)** = N_d**(B)** for d = 1, 3, 4 and for $d >= 6$.

D. Multidimensional Matrix Multiplication

In classical matrix algebra, each individual element of a matrix product is determined as follows:

$$c_{ij} = \sum_{x=1}^{n} a_{ix} * b_{xj}$$

In this equation, n is the number of elements in the second dimension of the first matrix or the number of elements in the first dimension of the second matrix.

The equation for multiplication of classical matrices can be extended to multiplication of multidimensional matrices. Each individual element in the multidimensional matrix product is determined as follows:

$$c_{ijk\ldots q} = \sum_{x=1}^{n} a_{ijk\ldots q} \text{ where } x \text{ replaces index of } db * b_{ijk\ldots q} \text{ where } x \text{ replaces index of } da$$

In this equation, for $a_{ijk\ldots q}$, the variable x replaces one of the indices, $i, j, k, \ldots, q$, that corresponds to the second dimension being multiplied, db. Also, in this equation, for $b_{ijk\ldots q}$, the variable x replaces one of the indices, $i, j, k, \ldots, q$, that corresponds to the first dimension being multiplied, da.

For example, let $da = 1$ and $db = 4$. Then in this equation, for $a_{ijk\ldots q}$, the variable x replaces the index l. Also, in this equation, for $b_{ijk\ldots q}$, the variable x replaces the index i. The resulting equation follows:

$$c_{ijk\ldots q} = \sum_{x=1}^{n} a_{ijkx\ldots q} * b_{xjkl\ldots q}$$

It is important to note that the equation for multiplication of classical matrices is just a special case of the equation for multiplication of multidimensional matrices:

$$c_{ij} = \sum_{x=1}^{n} a_{ix} * b_{xj}$$

According to the equation for multiplication of multidimensional matrices, when the first dimension and second dimension of multidimensional matrices are multiplied, corresponding 2-D submatrices, occupying the same relative positions with respect to third and higher dimensions in both matrices being multiplied, are multiplied together using the matrix multiplication method of classical matrix algebra.

Consider the case where the first dimension and second dimension are multiplied for a 4-D matrix **A** with dimensions of 2 * 3 * 2 * 2 and a 4-D matrix **B** with dimensions of 3 * 2 * 2 * 2. This meets the conformability requirements for multidimensional matrix multiplication because N_2**(A)** = N_1**(B),** N_3**(A)** = N_3**(B),** and N_4**(A)** = N_4**(B).** As shown below, this multidimensional matrix operation results in a 4-D matrix **C** with dimensions of 2 * 2 * 2 * 2.

The following equation is used for multiplication of the first dimension and second dimension of these two 4-D matrices:

$$c_{ijkl} = \sum_{x=1}^{n} a_{ixkl} * b_{xjkl}$$

where n = N_{db}**(A)** = N_{da}**(B)** = 3

When the first dimension and second dimension of multidimensional matrices are being multiplied, this is equivalent to multiplying corresponding 2-D submatrices using the matrix multiplication method of classical matrix algebra as shown:

$$\begin{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}, \begin{bmatrix} 13 & 15 & 17 \\ 14 & 16 & 18 \end{bmatrix} \\ \begin{bmatrix} 7 & 9 & 11 \\ 8 & 10 & 12 \end{bmatrix}, \begin{bmatrix} 19 & 21 & 23 \\ 20 & 22 & 24 \end{bmatrix} \end{bmatrix} *_{(1,\,2)} \begin{bmatrix} \begin{bmatrix} 24 & 21 \\ 23 & 20 \\ 22 & 19 \end{bmatrix}, \begin{bmatrix} 12 & 9 \\ 11 & 8 \\ 10 & 7 \end{bmatrix} \\ \begin{bmatrix} 18 & 15 \\ 17 & 14 \\ 16 & 13 \end{bmatrix}, \begin{bmatrix} 6 & 3 \\ 5 & 2 \\ 4 & 1 \end{bmatrix} \end{bmatrix}$$

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$$= \begin{bmatrix} \begin{bmatrix} 1*24+3*23+5*22 & 1*21+3*20+5*19 \\ 2*24+4*23+6*22 & 2*21+4*20+6*19 \end{bmatrix}, & \begin{bmatrix} 13*12+15*11+17*10 & 13*9+15*8+17*7 \\ 14*12+16*11+18*10 & 14*9+16*8+18*7 \end{bmatrix} \\ \begin{bmatrix} 7*18+9*17+11*16 & 7*15+9*14+11*13 \\ 8*18+10*17+12*16 & 8*15+10*14+12*13 \end{bmatrix}, & \begin{bmatrix} 19*6+21*5+23*4 & 19*3+21*2+23*1 \\ 20*6+22*5+24*4 & 20*3+22*2+24*1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 203 & 176 \\ 272 & 236 \end{bmatrix}, & \begin{bmatrix} 491 & 356 \\ 524 & 380 \end{bmatrix} \\ \begin{bmatrix} 455 & 374 \\ 506 & 416 \end{bmatrix}, & \begin{bmatrix} 311 & 122 \\ 326 & 128 \end{bmatrix} \end{bmatrix}$$

In classical matrix algebra, matrix multiplication always involves the first dimension and second dimension because there are no other dimensions involved in classical matrix algebra. In multidimensional matrix algebra, when the first dimension and second dimension of multidimensional matrices are being multiplied, the subscripted dimensions in parentheses "(1, 2)" by the multiplication sign can be omitted. And if the subscripted dimensions in parentheses "(1, 2)" can be omitted, then the multiplication sign can be omitted. That is, $\mathbf{A} *_{(1,\,2)} \mathbf{B} = \mathbf{A} * \mathbf{B} = \mathbf{AB}$. This makes the notation for multiplication of 2-D matrices in multidimensional matrix algebra consistent with classical matrix algebra.

The following example shows how the subscripted dimensions "(1, 2)" in parentheses by the multiplication sign can be omitted when the first dimension and second dimension of multidimensional matrices are being multiplied:

$$\begin{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} \\ \begin{bmatrix} 3 & 4 \end{bmatrix} \end{bmatrix} *_{(1,\,2)} \begin{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ \begin{bmatrix} 3 \\ 4 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} \\ \begin{bmatrix} 3 & 4 \end{bmatrix} \end{bmatrix} * \begin{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ \begin{bmatrix} 3 \\ 4 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} \\ \begin{bmatrix} 3 & 4 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ \begin{bmatrix} 3 \\ 4 \end{bmatrix} \end{bmatrix}$$

Consider the case where the first dimension and fourth dimension are multiplied for a 4-D matrix **A** with dimensions of 1 * 2 * 2 * 3 and a 4-D matrix **B** with dimensions of 3 * 2 * 2 * 1. This meets the conformability requirements for multidimensional matrix multiplication because $N_2(\mathbf{A}) = N_1(\mathbf{B})$, $N_3(\mathbf{A}) = N_3(\mathbf{B})$, and $N_4(\mathbf{A}) = N_4(\mathbf{B})$. As shown below, this multidimensional matrix operation results in a 4-D matrix **C** with dimensions of 1 * 2 * 2 * 1 or dimensions of 1 * 2 * 2 by the rules of multidimensional matrix simplification.

The following formula is used for multiplication of the third dimension and fourth dimension of these two 4-D matrices:

$$c_{ijkl} = \sum_{x=1}^{n} a_{ijkx} * b_{xjkl}$$

where $n = N_{db}(\mathbf{A}) = N_{da}(\mathbf{B}) = 3$

$$\begin{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 6 \end{bmatrix} \\ \begin{bmatrix} 7 & 8 \end{bmatrix}, \begin{bmatrix} 9 & 10 \end{bmatrix}, \begin{bmatrix} 11 & 12 \end{bmatrix} \end{bmatrix} *_{(3,\,4)} \begin{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \\ \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 35 & 56 \end{bmatrix} \\ \begin{bmatrix} 251 & 308 \end{bmatrix} \end{bmatrix}$$

VII. Multidimensional Matrix Product

A summation of quadratic terms can be alternatively represented using multidimensional matrix multiplication.

The following equation is used for neural network applications. Variables x_i and x_j are inputs and w_{ij} are weights for these inputs:

$$\sum_{i=1}^{3} \sum_{j=1}^{3} w_{ij} x_i x_j$$

$= w_{11}x_1^2 + w_{12}x_1x_2 + w_{13}x_1x_3 + w_{21}x_2x_1 + w_{22}x_2^2 + w_{23}x_2x_3 + w_{31}x_3x_1 + w_{32}x_3x_2 + w_{33}x_3^2$

$= w_{11}x_1^2 + w_{22}x_2^2 + w_{33}x_3^2 + x_1x_2\,(w_{12} + w_{21}) + x_1x_3\,(w_{13} + w_{31}) + x_2x_3\,(w_{23} + w_{32})$

This weighted summation is easily represented using classical matrices multiplied together:

$$\sum_{i=1}^{3} \sum_{j=1}^{3} w_{ij} x_i x_j = \begin{bmatrix} x1 & x2 & x3 \end{bmatrix} * \begin{bmatrix} w11 & w12 & w13 \\ w21 & w22 & w23 \\ w31 & w32 & w33 \end{bmatrix} * \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix}$$

It is extremely useful to express these weighted summations as matrices multiplied together to eliminate unnecessary terms in neural network designs. Because both the weights w_{ij} and w_{ji} in the matrix above correspond to the same second-order term x_ix_j, it is sufficient to use only an upper triangular or lower triangular matrix. For instance, instead of separately determining values for w_{12} and w_{21}, both of which are weights for x_1x_2, one can eliminate one of these weights and determine a value for either w_{12} or w_{21} that would be as much as both of these combined if they were computed separately. The same applies for other redundant weights. This saves time in the neural network's intensive procedure of computing weights.

However, the following equation and more complicated equations used for neural network applications cannot be expressed using classical matrices. Variables x_i, x_j, and x_k are inputs and w_{ijk} are weights for these inputs.

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$$\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} w_{ijk} x_i x_j x_k$$

$= w_{111}x_1^{\ 3} + w_{112}x_1^{\ 2}x_2 + w_{121}x_1^{\ 2}x_2 + w_{122}x_1x_2^{\ 2} + w_{211}x_1^{\ 2}x_2 + w_{212}x_1x_2^{\ 2} + w_{221}x_1x_2^{\ 2} + w_{222}x_2^{\ 3}$

$= w_{111}x_1^{\ 3} + x_1^{\ 2}x_2\,(w_{112} + w_{121} + w_{211}) + x_1x_2^{\ 2}\,(w_{122} + w_{212} + w_{221}) + w_{222}x_2^{\ 3}$

This weighted summation can be alternatively represented using multidimensional matrices multiplied together. Premultiply the 2 * 2 * 2 weight matrix by a 1 * 2 * 2 input matrix in the first dimension and second dimension. Then postmultiply the 2 * 2 * 2 weight matrix by a 2 * 1 * 2 input matrix in the first dimension and second dimension. Premultiply this entire product by a 1 * 2 input matrix in the first dimension and second dimension. Note that because the first dimension and second dimension of these multidimensional matrices are being multiplied, this does not need to be indicated in the equations below.

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} w_{ijk} x_i x_j x_k$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} * \left(\begin{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix} \\ \begin{bmatrix} x_1 & x_2 \end{bmatrix} \end{bmatrix} * \begin{bmatrix} \begin{bmatrix} w_{111} & w_{121} \\ w_{211} & w_{221} \end{bmatrix} \\ \begin{bmatrix} w_{112} & w_{122} \\ w_{212} & w_{222} \end{bmatrix} \end{bmatrix} * \begin{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{bmatrix} \right)$$

The multidimensional matrix product of the first dimension and second dimension of the 1 * 2 * 2 input matrix and the 2 * 2 * 2 weight matrix results in a 1 * 2 * 2 matrix.

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} w_{ijk} x_i x_j x_k$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} * \left(\begin{bmatrix} \begin{bmatrix} w_{111}x_1 + w_{211}x_2 & w_{121}x_1 + w_{221}x_2 \end{bmatrix} \\ \begin{bmatrix} w_{112}x_1 + w_{212}x_2 & w_{122}x_1 + w_{222}x_2 \end{bmatrix} \end{bmatrix} * \begin{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{bmatrix} \right)$$

The multidimensional matrix product of the first dimension and second dimension of the 1 * 2 * 2 matrix and the 2 * 1 * 2 input matrix results in a 1 * 1 * 2 matrix.

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} w_{ijk} x_i x_j x_k$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} * \begin{bmatrix} \begin{bmatrix} w_{111}x_1^2 + w_{211}x_1x_2 + w_{121}x_1x_2 + w_{221}x_2^2 \end{bmatrix} \\ \begin{bmatrix} w_{112}x_1^2 + w_{212}x_1x_2 + w_{122}x_1x_2 + w_{222}x_2^2 \end{bmatrix} \end{bmatrix}$$

The 1 * 1 * 2 matrix can be simplified into a 1-D matrix with 2 elements, so it can be premultiplied by the 1 * 2 input matrix in the first dimension and second dimension.

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} w_{ijk} x_i x_j x_k$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} * \begin{bmatrix} w_{111}x_1^2 + w_{211}x_1x_2 + w_{121}x_1x_2 + w_{221}x_2^2 \\ w_{112}x_1^2 + w_{212}x_1x_2 + w_{122}x_1x_2 + w_{222}x_2^2 \end{bmatrix}$$

$= w_{111}x_1^{\ 3} + w_{112}x_1^{\ 2}x_2 + w_{121}x_1^{\ 2}x_2 + w_{122}x_1x_2^{\ 2} + w_{211}x_1^{\ 2}x_2 + w_{212}x_1x_2^{\ 2} + w_{221}x_1x_2^{\ 2} + w_{222}x_2^{\ 3}$

$= w_{111}x_1^{\ 3} + x_1^{\ 2}x_2\,(w_{112} + w_{121} + w_{211}) + x_1x_2^{\ 2}\,(w_{122} + w_{212} + w_{221}) + w_{222}x_2^{\ 3}$

Thus, this multidimensional matrix multiplication yields the same result as the summation of quadratic terms above.

VIII. Conclusion

Part 2 of 6 defined multidimensional matrix equality as well as the multidimensional matrix algebra operations for addition, subtraction, multiplication by a scalar, and multiplication of two multidimensional matrices. Also, part 2 of 6 described an alternative representation of the summation of quadratic terms using multidimensional matrix multiplication.

Part 3 of 6 defines the multidimensional null matrix and multidimensional identity matrix. Also, part 3 of 6 defines the multidimensional matrix algebra operations for outer product and inner product.

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