Multidimensional Matrix Mathematics: Multidimensional Matrix Transpose, Symmetry, Antisymmetry, Determinant, and Inverse, Part 4 of 6

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***Abstract*—This is the first series of research papers to define multidimensional matrix mathematics, which includes multidimensional matrix algebra and multidimensional matrix calculus. These are new branches of math created by the author with numerous applications in engineering, math, natural science, social science, and other fields. Cartesian and general tensors can be represented as multidimensional matrices or vice versa. Some Cartesian and general tensor operations can be performed as multidimensional matrix operations or vice versa. However, many aspects of multidimensional matrix math and tensor analysis are not interchangeable. Part 4 of 6 defines the multidimensional matrix algebra operations for transpose, determinant, and inverse. Also, multidimensional matrix symmetry and antisymmetry are defined.**

***Index Terms*—multidimensional matrix math, multidimensional matrix algebra, multidimensional matrix calculus, matrix math, matrix algebra, matrix calculus, tensor analysis**

I. Introduction

Part 4 of 6 defines the multidimensional matrix algebra operations for transpose, determinant, and inverse. Also, part 4 of 6 defines multidimensional matrix symmetry and antisymmetry.

II. Multidimensional Matrix Transpose

A. Multidimensional Matrix Transpose Operation

A multidimensional matrix transpose is indicated by the letter "T" followed by the dimensions being transposed in parentheses, all of which is a superscript to the multidimensional matrix being transposed. The transpose of the xth dimension and yth dimension of multidimensional matrix **A** is indicated by $\mathbf{A}^{\mathrm{T}(x,\,y)}$.

A multidimensional matrix transpose can be done for any two dimensions. For each element in the initial multidimensional matrix, the two dimensions are transposed and the element is placed at the resulting location in the resulting multidimensional matrix.

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Let x and y be any two dimensions being transposed. If $\mathbf{B} = \mathbf{A}^{\mathrm{T}(x,\,y)}$, then $b_{ijk\,.\,.\,.\,q \text{ with } x \,\&\, y \text{ transposed}} = a_{ijk\,.\,.\,.\,q}$.

For example, if $\mathbf{B} = \mathbf{A}^{\mathrm{T}(1,\,5)}$, then $b_{mjklin\,.\,.\,.\,q} = a_{ijklmn\,.\,.\,.\,q}$.

Consider the 6-D matrix **A** with dimensions of 2 * 2 * 2 * 1 * 2 * 2:

$$\mathbf{A} = \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} a_{111111} & a_{121111} \\ a_{211111} & a_{221111} \end{bmatrix} \\ \begin{bmatrix} a_{112111} & a_{122111} \\ a_{212111} & a_{222111} \end{bmatrix} \end{bmatrix}, \begin{bmatrix} \begin{bmatrix} a_{111112} & a_{121112} \\ a_{211112} & a_{221112} \end{bmatrix} \\ \begin{bmatrix} a_{112112} & a_{122112} \\ a_{212112} & a_{222112} \end{bmatrix} \end{bmatrix} \\ \begin{bmatrix} \begin{bmatrix} a_{111121} & a_{121121} \\ a_{211121} & a_{221121} \end{bmatrix} \\ \begin{bmatrix} a_{112121} & a_{122121} \\ a_{212121} & a_{222121} \end{bmatrix} \end{bmatrix}, \begin{bmatrix} \begin{bmatrix} a_{111122} & a_{121122} \\ a_{211122} & a_{221122} \end{bmatrix} \\ \begin{bmatrix} a_{112122} & a_{122122} \\ a_{212122} & a_{222122} \end{bmatrix} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \end{bmatrix}, \begin{bmatrix} \begin{bmatrix} 9 & 10 \\ 11 & 12 \end{bmatrix} \\ \begin{bmatrix} 13 & 14 \\ 15 & 16 \end{bmatrix} \end{bmatrix} \\ \begin{bmatrix} \begin{bmatrix} 17 & 18 \\ 19 & 20 \end{bmatrix} \\ \begin{bmatrix} 21 & 22 \\ 23 & 24 \end{bmatrix} \end{bmatrix}, \begin{bmatrix} \begin{bmatrix} 25 & 26 \\ 27 & 28 \end{bmatrix} \\ \begin{bmatrix} 29 & 30 \\ 31 & 32 \end{bmatrix} \end{bmatrix} \end{bmatrix}$$

If we take the transpose of the first dimension and second dimension of multidimensional matrix **A,** the resulting 6-D matrix **C** has dimensions of 2 * 2 * 2 * 1 * 2 * 2. If $\mathbf{C} = \mathbf{A}^{\mathrm{T}(1,\,2)}$**,** then $c_{jiklmn} = a_{ijklmn}$.

$$\mathbf{C} = \mathbf{A}^{\mathrm{T}(1,\,2)} = \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} a_{111111} & a_{211111} \\ a_{121111} & a_{221111} \end{bmatrix} \\ \begin{bmatrix} a_{112111} & a_{212111} \\ a_{122111} & a_{222111} \end{bmatrix} \end{bmatrix}, \begin{bmatrix} \begin{bmatrix} a_{111112} & a_{211112} \\ a_{121112} & a_{221112} \end{bmatrix} \\ \begin{bmatrix} a_{112112} & a_{212112} \\ a_{122112} & a_{222112} \end{bmatrix} \end{bmatrix} \\ \begin{bmatrix} \begin{bmatrix} a_{111121} & a_{211121} \\ a_{121121} & a_{221121} \end{bmatrix} \\ \begin{bmatrix} a_{112121} & a_{212121} \\ a_{122121} & a_{222121} \end{bmatrix} \end{bmatrix}, \begin{bmatrix} \begin{bmatrix} a_{111122} & a_{211122} \\ a_{121122} & a_{221122} \end{bmatrix} \\ \begin{bmatrix} a_{112122} & a_{212122} \\ a_{122122} & a_{222122} \end{bmatrix} \end{bmatrix} \end{bmatrix}$$

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$$= \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \\ \begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix} \end{bmatrix}, \begin{bmatrix} \begin{bmatrix} 9 & 11 \\ 10 & 12 \end{bmatrix} \\ \begin{bmatrix} 13 & 15 \\ 14 & 16 \end{bmatrix} \end{bmatrix} \\ \begin{bmatrix} \begin{bmatrix} 17 & 19 \\ 18 & 20 \end{bmatrix} \\ \begin{bmatrix} 21 & 23 \\ 22 & 24 \end{bmatrix} \end{bmatrix}, \begin{bmatrix} \begin{bmatrix} 25 & 27 \\ 26 & 28 \end{bmatrix} \\ \begin{bmatrix} 29 & 31 \\ 30 & 32 \end{bmatrix} \end{bmatrix} \end{bmatrix}$$

In classical matrix algebra, the transpose operation always involves the first dimension and second dimension because there are no other dimensions involved in classical matrix algebra. In multidimensional matrix algebra, when the first dimension and second dimension of multidimensional matrices are being transposed, the subscripted dimensions in parentheses "(1, 2)" by the multiplication sign can be omitted. That is, $\mathbf{A}^{T(1,\,2)} = \mathbf{A}^{T}$. This makes the notation for transpose of 2-D matrices in multidimensional matrix algebra consistent with classical matrix algebra.

If we take the transpose of the third dimension and fourth dimension of multidimensional matrix **A,** the resulting 6-D matrix **D** has dimensions of 2 * 2 * 1 * 2 * 2 * 2. If $\mathbf{D} = \mathbf{A}^{T(3,\,4)}$**,** then $d_{ijlkmn} = a_{ijklmn}$.

$\mathbf{D} = \mathbf{A}^{T(3,\,4)} =$

$$\begin{bmatrix} \begin{bmatrix} \begin{bmatrix} a_{111111} & a_{121111} \\ a_{211111} & a_{221111} \end{bmatrix}, \begin{bmatrix} a_{112111} & a_{122111} \\ a_{212111} & a_{222111} \end{bmatrix} \end{bmatrix}, \begin{bmatrix} \begin{bmatrix} a_{111112} & a_{121112} \\ a_{211112} & a_{221112} \end{bmatrix}, \begin{bmatrix} a_{112112} & a_{122112} \\ a_{212112} & a_{222112} \end{bmatrix} \end{bmatrix} \\ \begin{bmatrix} \begin{bmatrix} a_{111121} & a_{121121} \\ a_{211121} & a_{221121} \end{bmatrix}, \begin{bmatrix} a_{112121} & a_{122121} \\ a_{212121} & a_{222121} \end{bmatrix} \end{bmatrix}, \begin{bmatrix} \begin{bmatrix} a_{111122} & a_{121122} \\ a_{211122} & a_{221122} \end{bmatrix}, \begin{bmatrix} a_{112122} & a_{122122} \\ a_{212122} & a_{222122} \end{bmatrix} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \end{bmatrix}, \begin{bmatrix} \begin{bmatrix} 9 & 10 \\ 11 & 12 \end{bmatrix}, \begin{bmatrix} 13 & 14 \\ 15 & 16 \end{bmatrix} \end{bmatrix} \\ \begin{bmatrix} \begin{bmatrix} 17 & 18 \\ 19 & 20 \end{bmatrix}, \begin{bmatrix} 21 & 22 \\ 23 & 24 \end{bmatrix} \end{bmatrix}, \begin{bmatrix} \begin{bmatrix} 25 & 26 \\ 27 & 28 \end{bmatrix}, \begin{bmatrix} 29 & 30 \\ 31 & 32 \end{bmatrix} \end{bmatrix} \end{bmatrix}$$

If we take the transpose of the first dimension and fourth dimension of multidimensional matrix **A,** the resulting 6-D matrix **E** has dimensions of 1 * 2 * 2 * 2 * 2 * 2. If $\mathbf{E} = \mathbf{A}^{T(1,\,4)}$, then $e_{ljkimn} = a_{ijklmn}$.

$\mathbf{E} = \mathbf{A}^{T(1,\,4)} =$

$$\begin{bmatrix} \begin{bmatrix} \begin{bmatrix} a_{111111} & a_{121111} \end{bmatrix}, \begin{bmatrix} a_{211111} & a_{221111} \end{bmatrix} \\ \begin{bmatrix} a_{112111} & a_{122111} \end{bmatrix}, \begin{bmatrix} a_{212111} & a_{222111} \end{bmatrix} \end{bmatrix}, \begin{bmatrix} \begin{bmatrix} a_{111112} & a_{121112} \end{bmatrix}, \begin{bmatrix} a_{211112} & a_{221112} \end{bmatrix} \\ \begin{bmatrix} a_{112112} & a_{122112} \end{bmatrix}, \begin{bmatrix} a_{212112} & a_{222112} \end{bmatrix} \end{bmatrix} \\ \begin{bmatrix} \begin{bmatrix} a_{111121} & a_{121121} \end{bmatrix}, \begin{bmatrix} a_{211121} & a_{221121} \end{bmatrix} \\ \begin{bmatrix} a_{112121} & a_{122121} \end{bmatrix}, \begin{bmatrix} a_{212121} & a_{222121} \end{bmatrix} \end{bmatrix}, \begin{bmatrix} \begin{bmatrix} a_{111122} & a_{121122} \end{bmatrix}, \begin{bmatrix} a_{211122} & a_{221122} \end{bmatrix} \\ \begin{bmatrix} a_{112122} & a_{122122} \end{bmatrix}, \begin{bmatrix} a_{212122} & a_{222122} \end{bmatrix} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 4 \end{bmatrix} \\ \begin{bmatrix} 5 & 6 \end{bmatrix}, \begin{bmatrix} 7 & 8 \end{bmatrix} \end{bmatrix}, \begin{bmatrix} \begin{bmatrix} 9 & 10 \end{bmatrix}, \begin{bmatrix} 11 & 12 \end{bmatrix} \\ \begin{bmatrix} 13 & 14 \end{bmatrix}, \begin{bmatrix} 15 & 16 \end{bmatrix} \end{bmatrix} \\ \begin{bmatrix} \begin{bmatrix} 17 & 18 \end{bmatrix}, \begin{bmatrix} 19 & 20 \end{bmatrix} \\ \begin{bmatrix} 21 & 22 \end{bmatrix}, \begin{bmatrix} 23 & 24 \end{bmatrix} \end{bmatrix}, \begin{bmatrix} \begin{bmatrix} 25 & 26 \end{bmatrix}, \begin{bmatrix} 27 & 28 \end{bmatrix} \\ \begin{bmatrix} 29 & 30 \end{bmatrix}, \begin{bmatrix} 31 & 32 \end{bmatrix} \end{bmatrix} \end{bmatrix}$$

B. Transpose Laws of Multidimensional Matrix Algebra

The multidimensional matrix transpose laws follow:

Multidimensional Matrix Transpose Law #1:
$(\mathbf{A}^{T(x,\,y)})^{T(x,\,y)} = \mathbf{A}$

Multidimensional Matrix Transpose Law #2:
$(\alpha\mathbf{A})^{T(x,\,y)} = \alpha\mathbf{A}^{T(x,\,y)}$

Multidimensional Matrix Transpose Law #3:
$(\mathbf{A} + \mathbf{B})^{T(x,\,y)} = \mathbf{A}^{T(x,\,y)} + \mathbf{B}^{T(x,\,y)}$

C. Proofs of Multidimensional Matrix Transpose Laws

The following definitions are used in the following transpose laws and their proofs:

$E_{ijk\ldots q}(\mathbf{A})$ refers to the element with indices $i, j, k, \ldots, q$ in multidimensional matrix **A**.

$E_{ijk\ldots q \text{ with } x \,\&\, y \text{ transposed}}(\mathbf{A}^{T(x,\,y)})$ refers to the element with indices $i, j, k, \ldots, q,$ but with the xth dimension and yth dimension transposed. The xth dimension and yth dimension can be any two different dimensions.

For example, if x=2 and y=4, then the second and fourth dimensions of multidimensional matrix **A** are being transposed and for any element of A, $E_{ijkl\ldots q \text{ with } x \,\&\, y \text{ transposed}}(\mathbf{A}^{T(x,\,y)}) = E_{ilkj\ldots q}(\mathbf{A}^{T(2,\,4)}) = E_{ijkl\ldots q}(\mathbf{A})$.

Multidimensional matrix transpose law #1, $(\mathbf{A}^{T(x,\,y)})^{T(x,\,y)} = \mathbf{A}$**,** is proven as follows:

For any element of **A** and for any dimensions x and y:

$E_{ijk\ldots q}[(\mathbf{A}^{T(x,\,y)})^{T(x,\,y)}] = E_{ijk\ldots q \text{ with } x \,\&\, y \text{ transposed}}(\mathbf{A}^{T(x,\,y)})$

$= E_{ijk\ldots q}(\mathbf{A})$

Because $E_{ijk\ldots q}[(\mathbf{A}^{T(x,\,y)})^{T(x,\,y)}] = E_{ijk\ldots q}(\mathbf{A})$ for any element of **A** and for any dimensions x and y, then $(\mathbf{A}^{T(x,\,y)})^{T(x,\,y)} = \mathbf{A}$**.**

Multidimensional matrix transpose law #2, $(\alpha\mathbf{A})^{T(x,\,y)} = \alpha\mathbf{A}^{T(x,\,y)}$, is proven as follows:

For any element of **A** and for any dimensions x and y:

$E_{ijk\ldots q}[(\alpha\mathbf{A})^{T(x,\,y)}] = E_{ijk\ldots q \text{ with } x \,\&\, y \text{ transposed}}(\alpha\mathbf{A})$

$= \alpha * E_{ijk\ldots q \text{ with } x \,\&\, y \text{ transposed}}(\mathbf{A})$

$= \alpha * E_{ijk\ldots q}(\mathbf{A}^{T(x,\,y)})$

Because $E_{ijk\ldots q}[(\alpha\mathbf{A})^{T(x,\,y)}] = \alpha * E_{ijk\ldots q}(\mathbf{A}^{T(x,\,y)})$ for any element of **A** and for any dimensions x and y, then $(\alpha\mathbf{A})^{T(x,\,y)} = \alpha\mathbf{A}^{T(x,\,y)}$.

Multidimensional matrix transpose law #3, $(\mathbf{A} + \mathbf{B})^{T(x,\,y)} = \mathbf{A}^{T(x,\,y)} + \mathbf{B}^{T(x,\,y)}$, is proven as follows:

For any element of **A** and for any dimensions x and y:

$E_{ijk\ldots q}[(\mathbf{A} + \mathbf{B})^{T(x,\,y)}] = E_{ijk\ldots q \text{ with } x \,\&\, y \text{ transposed}}(\mathbf{A} + \mathbf{B})$

$= E_{ijk\ldots q \text{ with } x \,\&\, y \text{ transposed}}(\mathbf{A}) + E_{ijk\ldots q \text{ with } x \,\&\, y \text{ transposed}}(\mathbf{B})$

$= E_{ijk\ldots q}(\mathbf{A}^{T(x,\,y)}) + E_{ijk\ldots q}(\mathbf{B}^{T(x,\,y)})$

$= E_{ijk\ldots q}(\mathbf{A}^{T(x,\,y)} + \mathbf{B}^{T(x,\,y)})$

Because $E_{ij}[(\mathbf{A} + \mathbf{B})^{T(x,\,y)}] = E_{ij}(\mathbf{A}^{T(x,\,y)} + \mathbf{B}^{T(x,\,y)})$ for any element of **A** and for any dimensions x and y, then $(\mathbf{A} + \mathbf{B})^{T(x,\,y)} = \mathbf{A}^{T(x,\,y)} + \mathbf{B}^{T(x,\,y)}$.

III. Symmetry and Antisymmetry of Multidimensional Matrices

Just like a tensor can be symmetric or antisymmetric with respect to any two indices, a multidimensional matrix can be symmetric or antisymmetric with respect to any two indices.

Let da and db be any two dimensions being transposed. A multidimensional matrix **A** is symmetric with respect to the two indices da and db if $a_{ijklmnop} = a_{ijklmnop \text{ with } da \,\&\, db \text{ transposed}}$. A multidimensional matrix **A** is antisymmetric with respect to the two indices da and db if

$a_{ijklmnop} = -\, a_{ijklmnop \text{ with } da \,\&\, db \text{ transposed}}$.

For example, a multidimensional matrix **A** is symmetric with respect to the two indices i and j if $a_{ijklmnop} = a_{jiklmnop}$. Also, for example, a multidimensional matrix **A** is antisymmetric with respect to the two indices i and j if $a_{ijklmnop} = -a_{jiklmnop}$.

The following multidimensional matrix **B** is symmetric with respect to the two indices i and j because $b_{ijkl} = b_{jikl}$ for all $i, j, k,$ and l:

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$$\mathbf{B} = \begin{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix} \\ \begin{bmatrix} 7 & 8 \\ 8 & 9 \end{bmatrix}, \begin{bmatrix} 10 & 11 \\ 11 & 12 \end{bmatrix} \end{bmatrix}$$

The following multidimensional matrix **C** is symmetric with respect to the two indices *j* and *k* because $c_{ijkl} = c_{ikjl}$ for all *i, j, k,* and *l:*

$$\mathbf{C} = \begin{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \\ \begin{bmatrix} 2 & 9 \\ 4 & 10 \end{bmatrix}, \begin{bmatrix} 6 & 11 \\ 8 & 12 \end{bmatrix} \end{bmatrix}$$

The following multidimensional matrix **D** is antisymmetric with respect to the two indices *i* and *j* because $d_{ijkl} = -d_{jikl}$ for all *i, j, k,* and *l:*

$$\mathbf{D} = \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix} \end{bmatrix}$$

For multidimensional matrix **D** to be antisymmetric ($d_{ijkl} = -d_{jikl}$), it is necessary that $d_{iikl} = 0$ for all *i, k,* and *l*. That is, $d_{11kl} = 0$, $d_{22kl} = 0$ for all *k* and *l.*

The following multidimensional matrix **E** is antisymmetric with respect to the two indices *j* and *k* because $e_{ijkl} = -e_{ikjl}$ for all *i, j, k,* and *l:*

$$\mathbf{E} = \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 0 & 4 \end{bmatrix} \\ \begin{bmatrix} -1 & 0 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} -3 & 0 \\ -4 & 0 \end{bmatrix} \end{bmatrix}$$

For multidimensional matrix **E** to be antisymmetric ($e_{ijkl} = -e_{ikjl}$), it is necessary that $e_{ijjl} = 0$ for all *i, j,* and *l*. That is, $e_{i11l} = 0$, $e_{i22l} = 0$ for all *i* and *l.*

IV. Multidimensional Matrix Determinant

The multidimensional matrix determinant for a 1-D matrix is undefined. The multidimensional matrix determinant of a 2-D square matrix results in a scalar and is calculated using the standard means in classical matrix algebra. The multidimensional matrix determinant of a 2-D nonsquare matrix is undefined. For the multidimensional matrix determinant of multidimensional matrices with three or more dimensions, each 2-D square submatrix is replaced by its scalar determinant and each 2-D nonsquare submatrix is replaced by an undefined element.

Consider two multidimensional matrices **A** and **B** where **B** is the multidimensional matrix determinant of **A:**

$$\mathbf{B} = |\mathbf{A}|$$

Multidimensional matrix determinant **B** has two dimensions less than multidimensional matrix **A:**

$q(\mathbf{B}) = q(\mathbf{A}) - 2$ for $q(\mathbf{A}) \geq 3$ where *q* represents the number of dimensions of the matrix in parentheses.

$N_d(\mathbf{B}) = N_{d+2}(\mathbf{A})$ for $d+2 = 3, \ldots, q(\mathbf{A})$ where N represents the number of elements in the subscripted dimension for the matrix in parentheses and $q(\mathbf{A})$ is the number of dimensions of multidimensional matrix **A.**

The multidimensional matrix determinant of the following 5-D matrix with dimensions of 2 * 2 * 2 * 2 * 2 results in a 3-D matrix with dimensions of 2 * 2 * 2 and is calculated as follows:

$$\left| \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} 1 & 5 \\ 8 & 0 \end{bmatrix}, \begin{bmatrix} 6 & 2 \\ 4 & 5 \end{bmatrix} \\ \begin{bmatrix} 2 & 3 \\ 7 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 7 \\ 9 & 4 \end{bmatrix} \end{bmatrix} \\ \begin{bmatrix} \begin{bmatrix} 9 & 2 \\ 8 & 1 \end{bmatrix}, \begin{bmatrix} 8 & 0 \\ 5 & 2 \end{bmatrix} \\ \begin{bmatrix} 6 & 7 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 6 \\ 3 & 2 \end{bmatrix} \end{bmatrix} \end{bmatrix} \right| = \begin{bmatrix} \begin{bmatrix} 1*0-5*8 & 6*5-2*4 \\ 2*6-3*7 & 0*4-7*9 \end{bmatrix} \\ \begin{bmatrix} 9*1-2*8 & 8*2-0*5 \\ 6*0-7*3 & 1*2 - 6*3 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} -40 & 22 \\ -9 & -63 \end{bmatrix} \\ \begin{bmatrix} -7 & 16 \\ -21 & -16 \end{bmatrix} \end{bmatrix}$$

The multidimensional matrix determinant of the following 3-D matrix with dimensions of 2 * 4 * 2 results in a 1-D matrix with two undefined elements:

$$\left| \begin{bmatrix} 1 & 6 & 4 & 7 \\ 4 & 0 & 9 & 6 \end{bmatrix} \\ \begin{bmatrix} 5 & 3 & 1 & 7 \\ 3 & 2 & 2 & 8 \end{bmatrix} \right| = \begin{bmatrix} \text{undefined} \\ \text{undefined} \end{bmatrix}$$

A multidimensional matrix determinant is simply a multidimensional matrix of determinants for each 2-D square submatrix within the multidimensional matrix.

The properties of determinants in classical matrix algebra apply to each 2-D square submatrix within the multidimensional matrix:

For a 2-D square submatrix **A,** $|\mathbf{A}| = |\mathbf{A}^T|$

If all the elements of any row or column are zero in a 2-D square submatrix **A,** then $|\mathbf{A}| = 0$.

If one row is proportional to another row of a 2-D square submatrix **A,** then $|\mathbf{A}| = 0$.

If one column is proportional to another column of a 2-D square submatrix **A,** then $|\mathbf{A}| = 0$.

If one row is a linear combination of one or more other rows of a 2-D square submatrix **A**, then $|\mathbf{A}| = 0$.

If one column is a linear combination of one or more other columns of a 2-D square submatrix **A**, then $|\mathbf{A}| = 0$.

If two rows of a 2-D square submatrix **A** are interchanged, the sign of the determinant of submatrix **A** is changed.

If two columns of a 2-D square submatrix **A** are interchanged, the sign of the determinant of submatrix **A** is changed.

If all elements of a row of a 2-D square submatrix **A** are multiplied by a scalar α, the determinant of submatrix **A** is multiplied by α|.

If all elements of a column of a 2-D square submatrix **A** are multiplied by a scalar α, the determinant of submatrix **A** is multiplied by α.

Any multiple of a row of a 2-D square submatrix **A** can be added to any other row without changing the value of the determinant for submatrix **A.**

Any multiple of a column of a 2-D square submatrix **A** can be added to any column without changing the value of the determinant for submatrix **A.**

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V. MULTIDIMENSIONAL MATRIX INVERSE

The multidimensional matrix inverse for a 1-D matrix is undefined. The multidimensional matrix inverse of a 2-D matrix exists if it is a square matrix and has a nonzero determinant, and is calculated using the standard means in classical matrix algebra. For the multidimensional matrix inverse of multidimensional matrices with three or more dimensions, each 2-D square submatrix with a nonzero determinant is replaced by its inverse and any other 2-D submatrices or 1-D submatrices are replaced by an undefined element.

Consider the following 4-D matrix **A** with dimensions of 2 * 2 * 2 * 2:

$$\mathbf{A} = \begin{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, & \begin{bmatrix} 9 & 11 \\ 10 & 12 \end{bmatrix} \\ \begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix}, & \begin{bmatrix} 13 & 15 \\ 14 & 16 \end{bmatrix} \end{bmatrix}$$

The four 2-D square submatrices of **A** are

$$\mathbf{A_{11}} = \begin{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \end{bmatrix};\ \mathbf{A_{21}} = \begin{bmatrix} \begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix} \end{bmatrix};\ \mathbf{A_{12}} = \begin{bmatrix} \begin{bmatrix} 9 & 11 \\ 10 & 12 \end{bmatrix} \end{bmatrix};$$

$$\mathbf{A_{22}} = \begin{bmatrix} \begin{bmatrix} 13 & 15 \\ 14 & 16 \end{bmatrix} \end{bmatrix}$$

The inverse of each 2-D square submatrix is calculated using standard methods in classical matrix algebra:

$$\mathbf{A_{11}}^{-1} = \frac{\mathrm{Adj}(\mathbf{A_{11}})}{|\mathbf{A_{11}}|} = \begin{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \end{bmatrix}^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}^{T}$$

$$= \begin{bmatrix} -2 & \frac{3}{2} \\ 1 & \frac{-1}{2} \end{bmatrix}$$

The inverses of the other three 2-D square submatrices in **A** are as follows:

$$\mathbf{A_{21}}^{-1} = \begin{bmatrix} -4 & \frac{7}{2} \\ 3 & \frac{-5}{2} \end{bmatrix};\ \mathbf{A_{12}}^{-1} = \begin{bmatrix} -6 & \frac{11}{2} \\ 5 & \frac{-9}{2} \end{bmatrix};\ \mathbf{A_{22}}^{-1} = \begin{bmatrix} -8 & \frac{15}{2} \\ 7 & \frac{-13}{2} \end{bmatrix}$$

The inverse of multidimensional matrix **A** consists of the four inverted 2-D square submatrices:

$$\mathbf{A}^{-1} = \begin{bmatrix} \begin{bmatrix} -2 & \frac{3}{2} \\ 1 & \frac{-1}{2} \end{bmatrix}, & \begin{bmatrix} -6 & \frac{11}{2} \\ 5 & \frac{-9}{2} \end{bmatrix} \\ \begin{bmatrix} -4 & \frac{7}{2} \\ 3 & \frac{-5}{2} \end{bmatrix}, & \begin{bmatrix} -8 & \frac{15}{2} \\ 7 & \frac{-13}{2} \end{bmatrix} \end{bmatrix}$$

In classical matrix algebra, if a matrix has an inverse and that matrix is multiplied by its inverse, the product is an identity matrix with the same dimensions. That is, $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

The same applies to each 2-D submatrix in a multidimensional matrix. Because multidimensional matrices are a concatenation of 2-D submatrices, if a multidimensional matrix has an inverse and that multidimensional matrix is multiplied by its inverse in the first dimension and second dimension, then the product will be a multidimensional identity matrix with the same dimensions. That is, $\mathbf{A} *_{(1,2)} \mathbf{A}^{-1} = \mathbf{A}^{-1} *_{(1,2)} \mathbf{A} = \mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{UNIT}$.

Multiplying the first dimension and second dimension of multidimensional matrix **A** by multidimensional matrix $\mathbf{A}^{-1}$ results in a multidimensional identity matrix $\mathbf{UNIT}_{(2*2*2*2,\,1,\,2)}$:

$$\mathbf{AA}^{-1} = \begin{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, & \begin{bmatrix} 9 & 11 \\ 10 & 12 \end{bmatrix} \\ \begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix}, & \begin{bmatrix} 13 & 15 \\ 14 & 16 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} -2 & \frac{3}{2} \\ 1 & \frac{-1}{2} \end{bmatrix}, & \begin{bmatrix} -6 & \frac{11}{2} \\ 5 & \frac{-9}{2} \end{bmatrix} \\ \begin{bmatrix} -4 & \frac{7}{2} \\ 3 & \frac{-5}{2} \end{bmatrix}, & \begin{bmatrix} -8 & \frac{15}{2} \\ 7 & \frac{-13}{2} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} = \mathbf{UNIT}_{(2*2*2*2,\,1,\,2)}$$

VI. CONCLUSION

Part 4 of 6 defined the multidimensional matrix algebra operations for transpose, determinant, and inverse. Also, part 4 of 6 defined multidimensional matrix symmetry and antisymmetry.

Part 5 of 6 describes the commutative, associative, and distributive laws of multidimensional matrix algebra.

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