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Multidimensional Matrix Mathematics: Solving Systems of Linear Equations and Multidimensional Matrix Calculus, Part 6 of 6

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***Abstract*—This is the first series of research papers to define multidimensional matrix mathematics, which includes multidimensional matrix algebra and multidimensional matrix calculus. These are new branches of math created by the author with numerous applications in engineering, math, natural science, social science, and other fields. Cartesian and general tensors can be represented as multidimensional matrices or vice versa. Some Cartesian and general tensor operations can be performed as multidimensional matrix operations or vice versa. However, many aspects of multidimensional matrix math and tensor analysis are not interchangeable. Part 6 of 6 describes the solution of systems of linear equations using multidimensional matrices. Also, the multidimensional matrix calculus operations for differentiation and integration are defined.**

***Index Terms*—multidimensional matrix math, multidimensional matrix algebra, multidimensional matrix calculus, matrix math, matrix algebra, matrix calculus, tensor analysis**

I. Introduction

Part 6 of 6 describes the solution of systems of linear equations using multidimensional matrices. Also, part 6 of 6 defines the multidimensional matrix calculus operations for differentiation and integration.

II. Solving Systems of Linear Equations with Multidimensional Matrices

Systems of linear equations can be represented and solved with multidimensional matrices.

Consider the following system of linear equations:

$x_{11} + 5x_{21} + 3x_{31} = 10$
$2x_{11} + 7x_{21} + 4x_{31} = 15$
$5x_{11} + 2x_{21} + 8x_{31} = 7$
$10x_{12} + 4x_{22} + 3x_{32} = 13$
$8x_{12} + 5x_{22} + x_{32} = 18$
$2x_{12} + 6x_{22} + 9x_{32} = 19$

This system of linear equations can be represented with the following multidimensional matrix equation composed of 3-D matrices:

$$\begin{bmatrix} \begin{bmatrix} 1 & 5 & 3 \\ 2 & 7 & 4 \\ 5 & 2 & 8 \end{bmatrix} \\ \begin{bmatrix} 10 & 4 & 3 \\ 8 & 5 & 1 \\ 2 & 6 & 9 \end{bmatrix} \end{bmatrix} * \begin{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} \\ \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 10 \\ 15 \\ 7 \end{bmatrix} \\ \begin{bmatrix} 13 \\ 18 \\ 19 \end{bmatrix} \end{bmatrix}$$

Any of the different methods used to solve systems of linear equations represented with classical matrices can be applied to each of the submatrices in multidimensional matrices. This includes graphing, the substitution method, the elimination method, Gaussian elimination, Gauss-Jordan elimination, Cramer's rule, LU decomposition, Cholesky decomposition, etc.

The preceding system of linear equations can be represented with the following augmented 3-D matrix:

$$\begin{bmatrix} \left[\begin{array}{ccc|c} 1 & 5 & 3 & 10 \\ 2 & 7 & 4 & 15 \\ 5 & 2 & 8 & 7 \end{array}\right] \\ \left[\begin{array}{ccc|c} 10 & 4 & 3 & 13 \\ 8 & 5 & 1 & 18 \\ 2 & 6 & 9 & 19 \end{array}\right] \end{bmatrix}$$

Using Gauss-Jordan elimination on each of the individual 2-D submatrices in the augmented 3-D matrix, the reduced row echelon form for this augmented multidimensional matrix can be found:

$$\begin{bmatrix} \left[\begin{array}{ccc|c} 1 & 5 & 3 & 10 \\ 2 & 7 & 4 & 15 \\ 5 & 2 & 8 & 7 \end{array}\right] \\ \left[\begin{array}{ccc|c} 10 & 4 & 3 & 13 \\ 8 & 5 & 1 & 18 \\ 2 & 6 & 9 & 19 \end{array}\right] \end{bmatrix} \sim \begin{bmatrix} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 37/25 \\ 0 & 1 & 0 & 51/25 \\ 0 & 0 & 1 & -14/25 \end{array}\right] \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & -13/112 \\ 0 & 1 & 0 & 31/8 \\ 0 & 0 & 1 & -25/56 \end{array}\right] \end{bmatrix}$$

Therefore, $x_{11} = \frac{37}{25}$; $x_{21} = \frac{51}{25}$; $x_{31} = \frac{-14}{25}$; $x_{12} = \frac{-13}{112}$; $x_{22} = \frac{31}{8}$; $x_{32} = \frac{-25}{56}$

Consider the following system of linear equations:

$a_{11111}x_{1111} + a_{12111}x_{2111} + a_{13111}x_{3111} = b_{1111}$
$a_{21111}x_{1111} + a_{22111}x_{2111} + a_{23111}x_{3111} = b_{2111}$
$a_{31111}x_{1111} + a_{32111}x_{2111} + a_{33111}x_{3111} = b_{3111}$
$a_{11121}x_{1121} + a_{12121}x_{2121} + a_{13121}x_{3121} = b_{1121}$
$a_{21121}x_{1121} + a_{22121}x_{2121} + a_{23121}x_{3121} = b_{2121}$

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$a_{31121}x_{1121} + a_{32121}x_{2121} + a_{33121}x_{3121} = b_{3121}$
$a_{11211}x_{1211} + a_{12211}x_{2211} + a_{13211}x_{3211} = b_{1211}$
$a_{21211}x_{1211} + a_{22211}x_{2211} + a_{23211}x_{3211} = b_{2211}$
$a_{31211}x_{1211} + a_{32211}x_{2211} + a_{33211}x_{3211} = b_{3211}$
$a_{11221}x_{1221} + a_{12221}x_{2221} + a_{13221}x_{3221} = b_{1221}$
$a_{21221}x_{1221} + a_{22221}x_{2221} + a_{23221}x_{3221} = b_{2221}$
$a_{31221}x_{1221} + a_{32221}x_{2221} + a_{33221}x_{3221} = b_{3221}$
$a_{11112}x_{1112} + a_{12112}x_{2112} + a_{13112}x_{3112} = b_{1112}$
$a_{21112}x_{1112} + a_{22112}x_{2112} + a_{23112}x_{3112} = b_{2112}$
$a_{31112}x_{1112} + a_{32112}x_{2112} + a_{33112}x_{3112} = b_{3112}$
$a_{11122}x_{1122} + a_{12122}x_{2122} + a_{13122}x_{3122} = b_{1122}$
$a_{21122}x_{1122} + a_{22122}x_{2122} + a_{23122}x_{3122} = b_{2122}$
$a_{31122}x_{1122} + a_{32122}x_{2122} + a_{33122}x_{3122} = b_{3122}$
$a_{11212}x_{1212} + a_{12212}x_{2212} + a_{13212}x_{3212} = b_{1212}$
$a_{21212}x_{1212} + a_{22212}x_{2212} + a_{23212}x_{3212} = b_{2212}$
$a_{31212}x_{1212} + a_{32212}x_{2212} + a_{33212}x_{3212} = b_{3212}$
$a_{11222}x_{1222} + a_{12222}x_{2222} + a_{13222}x_{3222} = b_{1222}$
$a_{21222}x_{1222} + a_{22222}x_{2222} + a_{23222}x_{3222} = b_{2222}$
$a_{31222}x_{1222} + a_{32222}x_{2222} + a_{33222}x_{3222} = b_{3222}$

This system of linear equations can be represented with this single multidimensional matrix equation composed of 5-D matrices:

$$\begin{bmatrix} \begin{bmatrix} \begin{bmatrix} a_{11111} & a_{12111} & a_{13111} \\ a_{21111} & a_{22111} & a_{23111} \\ a_{31111} & a_{32111} & a_{33111} \end{bmatrix}, \begin{bmatrix} a_{11121} & a_{12121} & a_{13121} \\ a_{21121} & a_{22121} & a_{23121} \\ a_{31121} & a_{32121} & a_{33121} \end{bmatrix} \\ \begin{bmatrix} a_{11211} & a_{12211} & a_{13211} \\ a_{21211} & a_{22211} & a_{23211} \\ a_{31211} & a_{32211} & a_{33211} \end{bmatrix}, \begin{bmatrix} a_{11221} & a_{12221} & a_{13221} \\ a_{21221} & a_{22221} & a_{23221} \\ a_{31221} & a_{32221} & a_{33221} \end{bmatrix} \end{bmatrix} \\ \begin{bmatrix} \begin{bmatrix} a_{11112} & a_{12112} & a_{13112} \\ a_{21112} & a_{22112} & a_{23112} \\ a_{31112} & a_{32112} & a_{33112} \end{bmatrix}, \begin{bmatrix} a_{11122} & a_{12122} & a_{13122} \\ a_{21122} & a_{22122} & a_{23122} \\ a_{31122} & a_{32122} & a_{33122} \end{bmatrix} \\ \begin{bmatrix} a_{11212} & a_{12212} & a_{13212} \\ a_{21212} & a_{22212} & a_{23212} \\ a_{31212} & a_{32212} & a_{33212} \end{bmatrix}, \begin{bmatrix} a_{11222} & a_{12222} & a_{13222} \\ a_{21222} & a_{22222} & a_{23222} \\ a_{31222} & a_{32222} & a_{33222} \end{bmatrix} \end{bmatrix} \end{bmatrix} * \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} x_{1111} \\ x_{2111} \\ x_{3111} \end{bmatrix}, \begin{bmatrix} x_{1121} \\ x_{2121} \\ x_{3121} \end{bmatrix} \\ \begin{bmatrix} x_{1211} \\ x_{2211} \\ x_{3211} \end{bmatrix}, \begin{bmatrix} x_{1221} \\ x_{2221} \\ x_{3221} \end{bmatrix} \end{bmatrix} \\ \begin{bmatrix} \begin{bmatrix} x_{1112} \\ x_{2112} \\ x_{3112} \end{bmatrix}, \begin{bmatrix} x_{1122} \\ x_{2122} \\ x_{3122} \end{bmatrix} \\ \begin{bmatrix} x_{1212} \\ x_{2212} \\ x_{3212} \end{bmatrix}, \begin{bmatrix} x_{1222} \\ x_{2222} \\ x_{3222} \end{bmatrix} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} b_{1111} \\ b_{2111} \\ b_{3111} \end{bmatrix}, \begin{bmatrix} b_{1121} \\ b_{2121} \\ b_{3121} \end{bmatrix} \\ \begin{bmatrix} b_{1211} \\ b_{2211} \\ b_{3211} \end{bmatrix}, \begin{bmatrix} b_{1221} \\ b_{2221} \\ b_{3221} \end{bmatrix} \end{bmatrix} \\ \begin{bmatrix} \begin{bmatrix} b_{1112} \\ b_{2112} \\ b_{3112} \end{bmatrix}, \begin{bmatrix} b_{1122} \\ b_{2122} \\ b_{3122} \end{bmatrix} \\ \begin{bmatrix} b_{1212} \\ b_{2212} \\ b_{3212} \end{bmatrix}, \begin{bmatrix} b_{1222} \\ b_{2222} \\ b_{3222} \end{bmatrix} \end{bmatrix} \end{bmatrix}$$

When each element a_{ijklm} of the coefficient matrix and each element b_{jklm} of the product matrix is defined, each element x_{jklm} of the variable matrix can be calculated. Any of the different methods used to solve systems of linear equations represented with classical matrices can be applied to each of the submatrices in multidimensional matrices. The system of linear equations can be solved like in the preceding example.

III. Multidimensional Matrix Differentiation

In multidimensional matrix calculus, multidimensional matrices are differentiated by finding the derivative of each element in the multidimensional matrix. The result is a multidimensional matrix of derivatives. The resulting multidimensional matrix has the same number of dimensions and same number of elements in each dimension as the multidimensional matrix that is differentiated.

A 3-D matrix with dimensions of 2 * 2 * 2 is differentiated as follows:

$$\frac{d}{dx}\begin{bmatrix} \begin{bmatrix} 5 & \cos x \\ 3x^2 & e^{-x} \end{bmatrix} \\ \begin{bmatrix} -e^{-x} & 2x \\ \sec x & -\tan x \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & -\sin x \\ 6x & -e^{-x} \end{bmatrix} \\ \begin{bmatrix} e^{-x} & 2 \\ \sec x \tan x & -\sec^2 x \end{bmatrix} \end{bmatrix}$$

IV. Multidimensional Matrix Integration

In multidimensional matrix calculus, multidimensional matrices are integrated by finding the integral of each element in the multidimensional matrix. The result is a multidimensional matrix of integrals. The resulting multidimensional matrix has the same number of dimensions and same number of elements in each dimension as the multidimensional matrix that is integrated.

A 3-D matrix with dimensions of 2 * 2 * 2 is integrated as follows:

$$\int \begin{bmatrix} \begin{bmatrix} 0 & -\sin x \\ 6x & -ye^{-xy} \end{bmatrix} \\ \begin{bmatrix} e^{-x} & 2y \\ \sec x \tan x & -\sec^2 x \end{bmatrix} \end{bmatrix} dx = \begin{bmatrix} \begin{bmatrix} K & \cos x \\ 3x^2 & e^{-xy} \end{bmatrix} \\ \begin{bmatrix} -e^{-x} & 2xy \\ \sec x & -\tan x \end{bmatrix} \end{bmatrix}$$

V. Conclusion

Part 6 of 6 described the solution of systems of linear equations using multidimensional matrices. Also, part 6 of 6 defined the multidimensional matrix calculus operations for differentiation and integration.

Classical matrix math offers many benefits not present in tensor analysis for a first or second order tensor, and tensor analysis for a first or second order tensor offers many benefits not present in classical matrix math. Similarly, multidimensional matrix math offers many benefits not present in tensor analysis for tensors of any order, and tensor analysis for tensors of any order offers many benefits not present in multidimensional matrix math.

The author predicts that multidimensional matrix math will replace classical matrix math in the future when this subject is taught in university courses. The author has made many more developments in multidimensional matrix math and developed many more innovative applications of multidimensional matrix math that will soon be published.

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Numerous applications have been developed for classical matrix math and its subsets, classical matrix algebra and classical matrix calculus, in many extremely diverse fields. Similarly, numerous applications in many extremely diverse fields will emerge for multidimensional matrix math and its subsets, multidimensional matrix algebra and multidimensional matrix calculus. These new branches of math will make it easier to solve many problems than before and even solve problems that couldn't be solved before.

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