Asymptotic Expansion for Thermal Flow through a Pipe

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*Abstract***We study the convection-diffusion equation in a thin or long pipe. The Reynolds number is chosen in a way that the effects of dispersion appear. We derive a complete asymptotic expansion leading to an approximation of arbitrary order.**

Keywords: asymptotic analysis, convection-diffusion, heat transfer, perturbation, mathematical modeling

1 Introduction

Motion of a fluid through a thin or long pipe, with heat exchange between the fluid and the surrounding medium is an important problem appearing in numerous engineering applications like air conditioners, refrigeration systems, central heating radiators and different kind of heat exchangers appearing in cars, ships, industrial facilities etc. Mathematically, such problem can be described by convection-diffusion equation, with Robin boundary condition describing the heat exchange between the fluid and it's surrounding. We cannot expect an exact solution of the model described by parabolic partial differential equation. Thus we use the perturbation technique with respect to the small parameter ε describing the ratio between the pipe's length and it's thickness. Our goal is to derive a complete asymptotic expansion giving very accurate approximation for the model. Different flow regimes lead to different models, as described in [2]. For low Reynolds number (slow flow) the convection term disappears from the model, for large Reynolds numbers the convection dominates the process and it can be described as a simple transport of heat. However, for "moderate" Reynolds numbers booth effects (diffusion and convection) should be taken into account. It turns out that the most interesting effects appear for Reynolds number Re^ε proportional to $1/\varepsilon$, i.e. for

$$Re^\varepsilon = \varepsilon^{-1} Re^0 \quad .$$

In that case the effects of dispersion appear, i.e. the effective diffusion coefficient appearing in a macroscopic model contains a term proportional to the convection velocity. That effect is known as the Taylor dispersion and it has been studied by different authors, mostly in context of a solute transport governed by Fick diffusion and simple convection, which is described by very similar model. The pioneer researcher G.I. Taylor [11] first discussed the dispersion of the solute flowing in a straight pipe with circular cross-section. The effective equations describing such problem are formally derived in [1] via the method of moments. Numerous researchers have examined the heat transfer and flow characteristics in pipes both theoretically and experimentally (see e.g. [9], [10], [7]). A nice review of the work done on this subject is given by Naphon and Wongwises [6] and we refer the reader to the large list of references therein. We also mention that, recently, rigorous derivation of the first order asymptotic approximation for a reactive solute transport in a narrow 2D channel is presented in [5]. We derive here higher order approximations.

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2 The problem

The pipe with thickness ε and a cross-section εS is denoted

$$\Omega_\varepsilon =]0,1[\times \varepsilon\, S \;\;,\;\; S \subset \mathbf{R}^2 \;\;,\;\; (x,y,z) \in \Omega_\varepsilon \;\;.$$

Assuming that the fluid flow has the Poisseuile form (classical parabolic profile flow of a viscous liquid through a pipe), we describe the process by equation in an adimensionalised form

$$\begin{aligned} &\frac{\partial \theta^\varepsilon}{\partial t} - \Delta\theta^\varepsilon + \\ &+ Re^\varepsilon\; Pr\, w^\varepsilon\; \frac{\partial \theta^\varepsilon}{\partial x} = 0 \;\text{ in }\; \Omega_\varepsilon^T && (1)\\ &\frac{\partial \theta^\varepsilon}{\partial \mathbf{n}} = Nu\;(G - \theta^\varepsilon) \;\text{ on }\Gamma_\varepsilon^T && (2)\\ &\Gamma_\varepsilon^T =]0,1[\times \partial(\varepsilon\, S) \times]0,T[\\ &\theta^\varepsilon = h_k \;\text{ for }\; x = k \;,\; k = 0,1 && (3)\\ &\theta^\varepsilon = \theta_0(x) \;\text{ for } t = 0 \;. && (4) \end{aligned}$$

$$w^\varepsilon(y,z,t) = w(y/\varepsilon, z/\varepsilon, t) \;\text{ Poiseuille velocity}$$

2.1 Asymptotic expansion

We rewrite the problem on the pipe with unit thickness by change of variables: $\xi_1 = y/\varepsilon$, $\xi_2 = z/\varepsilon$. Using the notations: $\Delta_\xi = \frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2}$, $\frac{\partial}{\mathbf{n}_\xi} = n_y \frac{\partial}{\partial \xi_1} + n_z \frac{\partial}{\partial \xi_2}$, $\Theta^\varepsilon(x,\xi,t) = \theta^\varepsilon(x, \varepsilon\,\xi_1, \varepsilon\,\xi_2, t)$

ISBN: 978-988-18210-8-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$$\Omega =]0,1[\times S \ , \ S \subset \mathbf{R}^2 \ , \ (x,\xi_1,\xi_2) \in \Omega$$

such form is suitable for asymptotic analysis:

$$\begin{aligned}
&\frac{\partial \Theta^\varepsilon}{\partial t} - \frac{1}{\varepsilon^2}\,\Delta_\xi \Theta^\varepsilon - \frac{\partial^2 \Theta^\varepsilon}{\partial x^2} + \\
&+\frac{1}{\varepsilon}\, Re^0\, Pr\, w\, \frac{\partial \Theta^\varepsilon}{\partial x} = 0 \ \text{ in } \ \Omega_\varepsilon^T && (5)\\
&\frac{\partial \Theta^\varepsilon}{\partial \mathbf{n}_\xi} = \varepsilon\, Nu\,(G - \Theta^\varepsilon) \ \text{ on } \Gamma^T && (6)\\
&\Gamma^T =]0,1[\times \partial S \times]0,T[\\
&\Theta^\varepsilon = h_k \ \text{ for } \ x = k \ , \ k = 0,1 && (7)\\
&\Theta^\varepsilon = \theta_0(x) \ \text{ for } t = 0 \ . && (8)
\end{aligned}$$

A standard approach to derive an approximation for such problem is to postulate an asymptotic expansion in powers of ε:

$$\Theta^\varepsilon(x,\xi,t) = \Theta_0(x,\xi,t) + \sum_{k=1}^{\infty} \varepsilon^k \Theta_k(x,\xi,t) \qquad (9)$$

Collecting equal powers of ε leads to a recursive sequence of equations:

$$\begin{aligned}
\tfrac{1}{\varepsilon^2} \ :& \quad \Delta_\xi\, \Theta_0 \ \text{ in } S \ , \ \frac{\partial \Theta_0}{\partial \mathbf{n}_\xi} = 0 \ \text{ on } \partial S \\
\tfrac{1}{\varepsilon} \ :& \quad -\Delta_\xi \Theta_1 + Re^0 Prw \frac{\partial \Theta_0}{\partial x} = 0 \ \text{ in } S \\
& \quad \frac{\partial \Theta_1}{\partial \mathbf{n}_\xi} = Nu\,(G - \Theta_0) \ \text{ on } \partial S \\
1 \ :& \quad \frac{\partial \Theta_0}{\partial t} - \Delta_\xi \Theta_2 - \frac{\partial^2 \Theta_0}{\partial x^2} + \\
& \quad +Re^0\ Pr\, w\, \frac{\partial \Theta_1}{\partial x} = 0 \ \text{ in } S \\
& \quad \frac{\partial \Theta_2}{\partial \mathbf{n}_\xi} + Nu\Theta_1 = 0 \ \text{ on } \partial S \\
\cdots \ & \quad \cdots \\
\varepsilon^k \ :& \quad \frac{\partial \Theta_k}{\partial t} - \Delta_\xi \Theta_{k+2} - \frac{\partial^2 \Theta_k}{\partial x^2} + \\
& \quad +Re^0 Prw \frac{\partial \Theta_{k+1}}{\partial x} = 0 \\
& \quad \frac{\partial \Theta_{k+2}}{\partial \mathbf{n}_\xi} = -Nu\Theta_{k+1}
\end{aligned}$$

It follows that

$$\begin{aligned}
&\Theta_0 = \Theta_0(x,t) \\
&\Theta_1 = A^1_{1,0}(\xi)\, \frac{\partial \Theta_0}{\partial x} + B^1_{0,0}(\xi)\ (G - \Theta_0) \\
&\begin{cases} \Delta_\xi A^1_{1,0} = Re^0\, Pr(w - \langle w \rangle) \ \text{ in } S \\ \frac{\partial A^1_{1,0}}{\partial \mathbf{n}_\xi} = 0 \ \text{ on } \partial S \ , \ \langle f \rangle = \frac{1}{|S|}\int_S f\, d\xi \end{cases} \\
&\begin{cases} \Delta_\xi B^1_{0,0} = Nu \frac{|\partial S|}{|S|} \ \text{ in } S \\ \frac{\partial B^1_{0,0}}{\partial \mathbf{n}_\xi} = Nu \ \text{ on } \partial S \end{cases}
\end{aligned}$$

$$\begin{aligned}
\Theta_2 =\ & A^2_{1,0}(\xi) \frac{\partial \Theta_0}{\partial x} + A^2_{2,0}(\xi) \frac{\partial^2 \Theta_0}{\partial x^2} + \\
&+B^2_{0,0}(\xi)(G - \Theta_0) + B^2_{1,0}(\xi) \frac{\partial}{\partial x}(G - \Theta_0) + \\
&+A^2_{1,1}(\xi) \frac{\partial}{\partial x}\frac{\partial}{\partial t}\,\Theta_0 + B^2_{0,1}(\xi)\,\frac{\partial}{\partial t}\,(G - \Theta_0) \\
\Theta_3 =\ & A^3_{1,0}(\xi) \frac{\partial \Theta_0}{\partial x} + A^3_{2,0}(\xi) \left(\frac{\partial}{\partial x}\right)^2 \Theta_0 + \\
&+A^3_{3,0}(\xi) \left(\frac{\partial}{\partial x}\right)^3 \Theta_0 + B^3_{0,0}(\xi)(G - \Theta_0) + \\
&+B^3_{1,0}(\xi) \frac{\partial}{\partial x}(G - \Theta_0) + B^3_{2,0}(\xi) \left(\frac{\partial}{\partial x}\right)^2 (G - \Theta_0) + \\
&+A^3_{1,1}(\xi) \frac{\partial}{\partial x}\frac{\partial}{\partial t}\Theta_0 + A^3_{2,1}(\xi) \left(\frac{\partial}{\partial x}\right)^2 \frac{\partial}{\partial t}\Theta_0 + \\
&+B^3_{0,1}(\xi) \frac{\partial}{\partial t}(G - \Theta_0) + B^3_{1,1}(\xi)\,\frac{\partial}{\partial x}\,\frac{\partial}{\partial t}(G - \Theta_0)
\end{aligned}$$

In general

$$\begin{aligned}
\Theta_k =& \sum_{1 \le m+\ell \le k} A^k_{m,\ell}(\xi) \left(\frac{\partial}{\partial x}\right)^m \left(\frac{\partial}{\partial t}\right)^\ell \Theta_0 + \\
&+ \sum_{0 \le m+\ell \le k-1} B^k_{m,\ell}(\xi) \left(\frac{\partial}{\partial x}\right)^m \left(\frac{\partial}{\partial t}\right)^\ell (G - \Theta_0)
\end{aligned}$$

We take $A^0_{0,0} = 1$, $B^0_{i,j} = 0$, $A^k_{i,j} = 0$ and $B^k_{i,j} = 0$ if any of the indices k,i,j is negative. We have, in addition, $A^k_{0,\ell} = 0$ for any k and ℓ. Furthermore, $A^k_{m,\ell} = 0$ if $m + \ell > k$ and $B^k_{m,\ell} = 0$ if $m + \ell \ge k$. Else, we define

$$\begin{aligned}
&\Delta_\xi A^{k+2}_{m\ell} = Re^0 Pr(w\, A^{k+1}_{(m-1),\ell} - \langle w\, A^{k+1}_{(m-1),\ell}\rangle) - \\
&-A^k_{(m-2),\ell} + A^k_{m,(\ell-1)} - Nu\, \overline{A^{k+1}_{m,\ell}} \ \text{ in } S \\
&\frac{\partial A^{k+2}_{m,\ell}}{\partial \mathbf{n}_\xi} + Nu A^{k+1}_{m,\ell} = 0 \ \text{ on } \ \partial S && (10)\\
&\text{for } \ \ 2 \le m + \ell \le k+1 \ \text{ and } \ k \ge 0 \\
&\Delta_\xi A^{k+2}_{m,\ell} = -A^k_{(m-2),\ell} \ \text{ in } \ S && (11)\\
&\frac{\partial A^{k+2}_{m,\ell}}{\partial \mathbf{n}_\xi} = 0 \ \text{ on } \ \partial S \ \ , \text{ if } \ m + \ell = k+2 && (12)\\
&\Delta_\xi B^{k+2}_{m\ell} = Re^0 Pr(w\, B^{k+1}_{(m-1),\ell} - \langle w\, B^{k+1}_{(m-1),\ell}\rangle) - \\
&-B^k_{(m-2),\ell} + B^k_{m,(\ell-1)} - Nu\, \overline{B^{k+1}_{m,\ell}} \ \text{ in } S \\
&\frac{\partial B^{k+2}_{m,\ell}}{\partial \mathbf{n}_\xi} + Nu\ B^{k+1}_{m,\ell} = 0 \ \text{ on } \ \partial S \ \ , && (13)\\
&\text{for } \ 1 \le m + \ell \le k \ \text{ and } \ k \ge 0 && (14)\\
&\Delta_\xi B^{k+2}_{m\ell} = B^k_{m-2,\ell} \ \text{ in } S \\
&\frac{\partial B^{k+2}_{m,\ell}}{\partial \mathbf{n}_\xi} = 0 \ \text{ on } \ \partial S \ \ , \ \text{ if } \ m + \ell = k+1 && (15)
\end{aligned}$$

where we denote $\overline{f} = \frac{1}{|S|}\int_{\partial S} f\ dS_\xi$. Now (9) can be written as

ISBN: 978-988-18210-8-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$$\Theta^\varepsilon(x,\xi,t) = \Theta_0(x,t) + \quad (16)$$
$$+\sum_{k=1}^{\infty}\varepsilon^k\Big\{\sum_{1\le m+\ell\le k} A^k_{m\ell}(\xi)\left(\frac{\partial}{\partial x}\right)^m\left(\frac{\partial}{\partial t}\right)^\ell\Theta_0 +$$
$$+\sum_{0\le m+\ell\le k-1} B^k_{m\ell}(\xi)\left(\frac{\partial}{\partial x}\right)^m\left(\frac{\partial}{\partial t}\right)^\ell (G-\Theta_0)\Big\}$$

Plugging (16) in (5) we obtain a formal "differential equation of infinite order" for Θ_0 in the form

$$\varepsilon^{-1}\left(\langle w\rangle Re^0 Pr\,\frac{\partial\Theta_0}{\partial x} + Nu\frac{|\partial S|}{|S|}(\Theta_0 - G)\right) + \quad (17)$$
$$+\Bigg[\frac{\partial\Theta_0}{\partial t} - (1 + Re^0\,Pr\langle w\,A^1_{1,0}\rangle)\frac{\partial^2\Theta_0}{\partial x^2} +$$
$$+Re^0 Pr\,\langle w\,B^1_{0,0}\rangle\frac{\partial}{\partial x}(G-\Theta_0)\Bigg] +$$
$$+\sum_{k=2}^{\infty}\varepsilon^{k-1}\Bigg[\sum_{2\le m+\ell\le k}(Re^0\,Pr\langle wA^k_{m-1,\ell}\rangle +$$
$$+Nu\overline{A^k_{m,\ell}})\,\frac{\partial^{m+\ell}\Theta_0}{\partial x^m\partial t^\ell} + \quad (18)$$
$$+\sum_{1\le m+\ell\le k-1}(Re^0\,Pr\,\langle wB^k_{m-1,\ell}\rangle +$$
$$+\overline{B^k_{m,\ell}})\,\frac{\partial^{m+\ell}(G-\Theta_0)}{\partial x^m\partial t^\ell}\Bigg] = 0$$

We can "solve" it by expanding Θ_0 in powers of ε as

$$\Theta_0(x,t) = \varphi^0(x,t) + \sum_{\ell=1}^{\infty}\varepsilon^\ell\varphi^\ell(x,t) \quad . \quad (19)$$

That way we get for each φ^ℓ an ordinary differential equation of first order:

$$Re^0 Pr\,\langle w\rangle\frac{\partial\varphi^0}{\partial x} + Nu\frac{|\partial S|}{|S|}(\varphi^0 - G) = 0 \quad (20)$$
$$Re^0 Pr\,\langle w\rangle\frac{\partial\varphi^1}{\partial x} + Nu\frac{|\partial S|}{|S|}\varphi^1 +$$
$$+\Bigg[\frac{\partial\varphi^0}{\partial t} - (1 + Re^0\,Pr\langle w\,A^1_{1,0}\rangle)\frac{\partial^2\varphi^0}{\partial x^2} -$$
$$-Re^0 Pr\langle w\,B^1_{0,0}\rangle\frac{\partial\varphi^0}{\partial x}\Bigg] = 0 \quad (21)$$

In general, denoting

$$\mathbf{K}^k_{m,\ell} = Re^0\;Pr\,\langle w\,B^k_{m-1,\ell}\rangle + Nu\;\overline{B^k_{m,\ell}}$$
$$\mathbf{H}^k_{m,\ell} = Re^0\;Pr\,\langle w\,(A^k_{m-1,\ell} - B^k_{m-1,\ell})\rangle +$$
$$+Nu\;\overline{(B^k_{m,\ell} - B^k_{m,\ell})}\quad .$$

we get

$$Re^0 Pr\,\langle w\rangle\frac{\partial\varphi^{n+1}}{\partial x} + Nu\frac{|\partial S|}{|S|}\varphi^{n+1} + \quad (22)$$
$$+\frac{\partial\varphi^n}{\partial t} - (1 + Re^0\,Pr\langle w\,A^1_{1,0}\rangle)\frac{\partial^2\varphi^n}{\partial x^2} -$$
$$-Re^0\,Pr\langle w\,B^1_{0,0}\rangle\frac{\partial\varphi^n}{\partial x} +$$
$$+\sum_{k=0}^{n}\sum_{m=1}^{k+1}\sum_{\ell=0}^{k+1-m}\mathbf{H}^{k+1}_{m,\ell}\,\frac{\partial^{m+\ell}\varphi^{n-k}}{\partial x^m\partial t^\ell} +$$
$$\sum_{m=1}^{n}\sum_{\ell=0}^{n-m}\mathbf{K}^{k+1}_{m,\ell}\,\frac{\partial^{m+\ell}G}{\partial x^m\partial t^\ell} = 0$$

Those equations are of the type

$$\langle w\rangle\;Re^0 Pr\,\frac{\partial\varphi^k}{\partial x} + Nu\,\frac{|\partial S|}{|S|}\,\varphi^k = F_k \quad .$$

It can be solved by quadratures and

$$\varphi^k(x,t) = \varphi^k(0,t)\,e^{-\frac{Nu}{Re^0\,Pr}\,x} + \quad (23)$$
$$+\frac{Nu\,|\partial S|}{|S|\langle w\rangle\,Re^0\,Pr}\int_0^x e^{\frac{Nu}{Re^0\,Pr}\,(s-x)}F_k(s)\,ds \quad .$$

Thus, denoting

$$b = \frac{Nu\,|\partial S|}{\langle w\rangle\;Re^0\;Pr\;|S|}\quad,\quad d = Re^0\;Pr\;\langle w\,B^1_{0,0}\rangle$$
$$c = 1 + Re^0\;Pr\;\langle w\,A^1_{1,0}\rangle$$

we obtain

$$\varphi^0(x,t) = h_0(t)\,e^{-b\,x} - \int_0^x e^{b\,(s-x)}G(s,t)\,ds \quad .$$

and for the next term

$$\begin{aligned}\varphi^1(x,t) \;&= e^{-bx}\Bigg\{x\,[b\,h_0(t)\,(bc-d) - h_0'(t)] + \\ &+\int_0^x e^{bs}\left[c\,\frac{\partial G}{\partial x}(s,t) - (bc-d)\,G(s,t)\right]ds + \\ &+\int_0^x\int_0^p e^{bs}\Big[\frac{\partial G}{\partial t}(s,t) + \\ &+b(bc-d)\,G(s,t)\Big]\,ds\,dp\Bigg\}\end{aligned}$$

Term $\langle w\,A^1_{1,0}\rangle$ appearing in (21) and (22) is the Taylor dispersion term.

3 Boundary layer in time

As our expansion does not satisfy the initial condition we need to correct it in vicinity of $t = 0$. We introduce the fast time variable $\tau = t/\varepsilon^2$ and we build the time boundary layer corrector in the form

$$\mathscr{Z}^\varepsilon(x,\tau,\xi) = \sum_{k=0}^{\infty}\varepsilon^k\Big\{\sum_{1\le m+\ell\le k} C^k_{m\ell}(\tau,\xi)\frac{\partial^{m+\ell}\Theta_0(x,0)}{\partial x^m\partial t^\ell} +$$
$$+\sum_{0\le m+\ell\le k-1} D^k_{m\ell}(\tau,\xi)\frac{\partial^{m+\ell}(G-\Theta_0(x,0))}{\partial x^m\partial t^\ell}\Big\} \quad (24)$$

The auxiliary functions $C^k_{m\ell}$ and $D^k_{m\ell}$ are defined by the auxiliary mixed problems posed on $S\times]0,+\infty[$

$$0 = \frac{\partial C^{k+2}_{m,\ell}}{\partial \tau} - \Delta_\xi C^{k+2}_{m,\ell} + \\ + Re^0 Pr(w\, C^{k+1}_{(m-1),\ell} - \langle w\, C^{k+1}_{(m-1),\ell}\rangle) - \\ - C^k_{(m-2),\ell} + C^k_{m,(\ell-1)} - Nu\, \overline{C^{k+1}_{m,\ell}} \text{ in } S\times]0,+\infty[$$

$$\frac{\partial C^{k+2}_{m,\ell}}{\partial \mathbf{n}_\xi} + Nu C^{k+1}_{m,\ell} = 0 \text{ on } \partial S\times]0,+\infty[\tag{25}$$

$$C^{k+2}_{m\ell}(0,\xi) + A^{k+2}_{m\ell}(\xi) = 0\ ,\ \xi \in S \tag{26}$$

$$0 = \frac{\partial D^{k+2}_{m,\ell}}{\partial \tau} - \Delta_\xi D^{k+2}_{m,\ell} + \\ + Re^0 Pr(w\, D^{k+1}_{(m-1),\ell} - \langle w\, D^{k+1}_{(m-1),\ell}\rangle) - \\ - D^k_{(m-2),\ell} + D^k_{m,(\ell-1)} - Nu\, \overline{D^{k+1}_{m,\ell}} = 0 \text{ in } S\times]0,+\infty[$$

$$\frac{\partial D^{k+2}_{m,\ell}}{\partial \mathbf{n}_\xi} + Nu D^{k+1}_{m,\ell} \text{ on } \partial S\times]0,+\infty[\tag{27}$$

$$D^{k+2}_{m\ell}(0,\xi) + B^{k+2}_{m\ell}(\xi) = 0\ ,\ \xi \in S\,. \tag{28}$$

The boundary layers near the ends of the pipe $x = 0, 1$ are neglected in our analysis.

4 Conclusion

We have derived a formal asymptotic expansion for a heat transfer in a capillary, in terms of the capillary thickness ε, assumed to be a small parameter. We identify all terms in the expansion and reduce their computation to a simple auxiliary problems posed on a cross-section of the pipe. A recursive sequence of problems is defined giving an asymptotic approximation with arbitrary order of accuracy. Correctors for the time boundary layer are also computed. Derived approximation is easy to compute (in fact for circular tube all terms can be computed explicitly) and accurate, at least outside of the boundary layers near the ends of the pipe $x = 0, 1$.

Acknowledgement. Supported by the Ministry of Science, Education and Sports, Republic of Croatia, Grant 037 - 0372787 - 2797.

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