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Analytical Treatment of System of Linear and Nonlinear PDEs by Homotopy-Perturbation Method

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Abstract—**In this paper, the homotopy-perturbation method (HPM) is employed to obtain approximate analytical solution to the linear and nonlinear systems of partial differential equations (PDEs). HPM yields solutions in convergent series forms with easily computable terms. Generally, the closed form of the exact solution or its expansion is obtained without any noise terms. Test examples demonstrate the efficiency of HPM.**

Keywords: ***Homotopy-perturbation method, Linear system of PDEs, Nonlinear system of PDEs***

1 Introduction

It is well-known that many physical and engineering phenomena such as wave propagation and shallow water waves can be modelled by systems of PDEs [9, 19, 20]. Finding accurate and efficient methods for solving nonlinear system of PDEs has long been an active research undertaking. Debnath [9] applied the characteristics method and Logan [19] used the Riemann invariants method to handle systems of PDEs. Vandewalle and Piessens [21] implemented a method based on a combination of the waveform relaxation method and multigrid to solve nonlinear systems. Wazwaz [22] used the Adomian decomposition method (ADM) to handle the systems of PDEs and reaction-diffusion Brusselator model. However, one notable difficulty inherent in ADM is the calculation of the so-called Adomian polynomials which can be cumbersome in general. Approximate solutions of nonlinear systems of PDEs were also obtained by the variational iteration method (VIM) [23]. Very recently, Belal et al. [2] obtained exact solutions of the nonlinear systems of PDEs studied in [23] directly via VIM.

In recent years, much attention has been devoted to the study of the homotopy-perturbation method (HPM) [11, 12, 13, 14, 15, 16, 17, 18] for solving a wide range of problems whose mathematical models yield differential equation or system of differential equations. HPM deforms a difficult problem into a set of problems which are easier to solve without any need to transform nonlinear terms. The applications of HPM in nonlinear problems have been demonstrated by many researchers, cf. [1, 3, 10]. Recently, HPM was employed for solving singular second-order differential equations [4], nonlinear population dynamics models [5] and time-dependent Emden-Fowler type equations [6], the Klein-Gordon and sine-Gordon equations [7]. Very recently, Chowdhury et al. [8] were the first to successfully apply the multistage homotopy-perturbation method (MHPM) to the chaotic Lorenz system.

The aim of this work is to present an alternative approach based on HPM for finding series solutions to linear and nonlinear systems of PDEs. The efficiency and accuracy of HPM are demonstrated through several test examples.

2 HPM for system of PDEs

To illustrate the basic idea of the HPM for system of PDEs, we consider the following non-homogeneous, nonlinear system of PDEs

$$\frac{\partial u_1}{\partial t} + g_1(t, u_1, u_2, \ldots, u_m) = f_1(t), \qquad (1)$$

$$\frac{\partial u_2}{\partial t} + g_2(t, u_1, u_2, \ldots, u_m) = f_2(t), \qquad (2)$$

$$\vdots$$

$$\frac{\partial u_m}{\partial t} + g_m(t, u_1, u_2, \ldots, u_m) = f_m(t), \qquad (3)$$

subject to the initial conditions

$$u_1(x, y, 0) = c_1,\ u_2(x, y, 0) = c_2,\ \ldots,\ u_m(x, y, 0) = c_m, \qquad (4)$$

where $u_m = u_m(x, y, t)$ and $f_m = f_m(x, y, t)$.

First write system (1)–(3) in the operator form:

$$L(u_1) + N_1(u_1, u_2, \ldots, u_m) - f_1 = 0, \qquad (5)$$

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$$L(u_2)+N_2(u_1,u_2,\ldots,u_m)-f_2 = 0, \quad (6)$$
$$\vdots$$
$$L(u_m)+N_m(u_1,u_2,\ldots,u_m)-f_m = 0, \quad (7)$$

subject to the initial conditions (4), where $L=\partial/\partial t$ is linear operator and $N_1,N_2,\ldots,N_m$ are nonlinear operators.

According to HPM, we construct a homotopy for (5)–(7) which satisfies the following relations:

$$L(u_1)-L(v_1)+pL(v_1)+p[N_1(u_1,u_2,\ldots,u_m)-f_1]=0,$$

$$L(u_2)-L(v_2)+pL(v_2)+p[N_2(u_1,u_2,\ldots,u_m)-f_2]=0,$$

$$\vdots$$

$$L(u_m)-L(v_m)+pL(v_m)+p[N_m(u_1, u_2,\ldots,u_m)-f_m]=0, \quad (8)$$

where $p\in[0,1]$ is an embedding parameter and $v_1,v_2,\ldots,v_m$ are initial approximations which satisfying the given conditions. It is obvious that when the homotopy parameter $p=0$, the above Equations become a linear system of equations and when $p=1$ we get the original nonlinear system of equations. Consider the initial approximations as follows:

$$\begin{aligned} u_{1,0}(x,y,t) &= v_1(x,y,t)=u_1(x,y,0)=c_1,\\ u_{2,0}(x,y,t) &= v_2(x,y,t)=u_2(x,y,0)=c_2,\\ &\vdots\\ u_{m,0}(x,y,t) &= v_m(x,y,t)=u_m(x,y,0)=c_m,\end{aligned}$$

and

$$\begin{aligned} u_1(x,y,t) &= u_{1,0}(x,y,t)+pu_{1,1}(x,y,t)+\cdots,\\ u_2(x,y,t) &= u_{2,0}(x,y,t)+pu_{2,1}(x,y,t)+\cdots,\\ &\vdots\\ u_m(x,y,t) &= u_{m,0}(x,y,t)+pu_{m,1}(x,y,t)+\cdots, \quad (9)\end{aligned}$$

where $u_{i,j}$, $(i=1,2,\ldots,m;\ j=1,2,\ldots)$ are functions yet to be determined. Substituting (9) into (8) and arranging the coefficients of the same powers of p, obtain

$$\begin{aligned} &L(u_{1,1})+L(v_1)+N_1(u_{1,0},u_{2,0},\ldots,u_{m,0})-f_1=0,\\ &\quad u_{1,1}(x,y,0)=0,\\ &L(u_{2,1})+L(v_2)+N_2(u_{1,0},u_{2,0},\ldots,u_{m,0})-f_2=0,\\ &\quad u_{2,1}(x,y,0)=0,\\ &\vdots\\ &L(u_{m,1})+L(v_m)+N_m(u_{1,0},u_{2,0},\ldots,u_{m,0})\\ &\quad -f_m=0, u_{m,1}(x,y,0)=0,\end{aligned}$$

$$\begin{aligned} &L(u_{1,2})+N_1(u_{1,1},u_{2,1},\ldots,u_{m,1})=0,\\ &\quad u_{1,2}(x,y,0)=0,\\ &L(u_{2,2})+N_2(u_{1,1},u_{2,1},\ldots,u_{m,1})=0,\\ &\quad u_{2,2}(x,y,0)=0,\\ &\vdots\\ &L(u_{m,2})+N_m(u_{1,1},u_{2,1},\ldots,u_{m,1})=0,\\ &\quad u_{m,2}(x,y,0)=0,\end{aligned}$$

etc.

Now solve the above systems of equations for the unknowns $u_{i,j}$ $(i=1,2,\ldots,m;\ j=1,2,\ldots)$. Therefore, according to HPM the n-term approximations for the solutions of (5)–(7) can be expressed as

$$\begin{aligned} \phi_{1,n}(x,y,t) &= u_1(x,y,t)\\ &= \lim_{p\to 1} u_1(x,y,t)=\sum_{k=0}^{n-1} u_{1,k}(x,y,t),\\ \phi_{2,n}(x,y,t) &= u_2(x,y,t)\\ &= \lim_{p\to 1} u_2(x,y,t)=\sum_{k=0}^{n-1} u_{2,k}(x,y,t),\\ &\vdots\\ \phi_{m,n}(x,y,t) &= u_m(x,y,t)\\ &= \lim_{p\to 1} u_m(x,y,t)=\sum_{k=0}^{n-1} u_{m,k}(x,y,t).\end{aligned}$$

3 Applications of HPM

In this section, we shall demonstrate the efficiency and accuracy of HPM to systems of linear and nonlinear PDEs through two examples. The HPM algorithm is coded in the computer algebra package Maple.

3.1 Example 1

The second system we shall study is the nonhomogeneous linear system of PDEs,

$$\begin{aligned} u_t-v_x-u+v &= -2,\\ v_t+u_x-u+v &= -2, \quad (10)\end{aligned}$$

subject to the initial conditions

$$u(x,0)=1+\mathrm{e}^x,\quad v(x,0)=-1+\mathrm{e}^x. \quad (11)$$

According to the HPM, we can construct a homotopy of system (10) which satisfies the following relation:

$$\begin{aligned} u_t-(y_0)_t+p\left[(y_0)_t-v_x-u+v+2\right] &= 0,\\ v_t-(z_0)_t+p\left[(z_0)_t+u_x-u+v+2\right] &= 0. \quad (12)\end{aligned}$$

Let us choose the initial approximations as

$$\begin{aligned} u_0(x,t) = y_0(x,t) = u(x,0) = 1 + \mathrm{e}^x, \\ v_0(x,t) = z_0(x,t) = v(x,0) = -1 + \mathrm{e}^x. \end{aligned}$$

and

$$\begin{aligned} u(x,t) &= u_0(x,t) + pu_1(x,t) + p^2u_2(x,t) + \cdots, \\ v(x,t) &= v_0(x,t) + pv_1(x,t) + p^2v_2(x,t) + \cdots, \end{aligned} \quad (13)$$

Substituting (13) into (12) and equating terms of the same powers of p, we have

$$\begin{aligned} &(u_1)_t + (y_0)_t - (v_0)_x - u_0 + v_0 + 2 = 0, u_1(x,0) = 0, \\ &(v_1)_t + (z_0)_t + (u_0)_x - u_0 + v_0 + 2 = 0, v_1(x,0) = 0, \\ &(u_2)_t - (v_1)_x - u_1 + v_1 = 0, \qquad u_2(x,0) = 0, \\ &(v_2)_t + (u_1)_x - u_1 + v_1 = 0, \qquad v_2(x,0) = 0, \\ &(u_3)_t - (v_2)_x - u_2 + v_2 = 0, \qquad u_3(x,0) = 0, \\ &(v_3)_t + (u_2)_x - u_2 + v_2 = 0, \qquad v_3(x,0) = 0, \\ &(u_4)_t - (v_3)_x - u_3 + v_3 = 0, \qquad u_4(x,0) = 0, \\ &(v_4)_t + (u_3)_x - u_3 + v_3 = 0, \qquad v_4(x,0) = 0, \\ &(u_5)_t - (v_4)_x - u_4 + v_4 = 0, \qquad u_5(x,0) = 0, \\ &(v_5)_t + (u_4)_x - u_4 + v_4 = 0, \qquad v_5(x,0) = 0, \end{aligned}$$

etc. Solving the above differential equations we obtain,

$$\begin{aligned} &u_1(x,t) = \mathrm{e}^x t, \quad v_1(x,t) = -\mathrm{e}^x t, \\ &u_2(x,t) = \frac{1}{2}\mathrm{e}^x t^2, \quad v_2(x,t) = \frac{1}{2}\mathrm{e}^x t^2, \\ &u_3(x,t) = \frac{1}{6}\mathrm{e}^x t^3, \quad v_3(x,t) = -\frac{1}{6}\mathrm{e}^x t^3, \\ &u_4(x,t) = \frac{1}{24}\mathrm{e}^x t^4, \quad v_4(x,t) = \frac{1}{24}\mathrm{e}^x t^4, \\ &u_5(x,t) = \frac{1}{120}\mathrm{e}^x t^5, \quad v_5(x,t) = -\frac{1}{120}\mathrm{e}^x t^5, \end{aligned}$$

etc.

Hence, the series solutions are

$$\begin{aligned} u(x,t) &= 1 + \mathrm{e}^x\left(1 + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right), \\ v(x,t) &= 1 + \mathrm{e}^x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right), \end{aligned}$$

which again converge to the closed-form solutions,

$$u(x,t) = 1 + \mathrm{e}^{x+t}, \quad v(x,t) = -1 + \mathrm{e}^{x-t}.$$

3.2 Example 2

Now we shall study the following nonhomogeneous nonlinear system of PDEs,

$$\begin{aligned} u_t + vu_x + u = 1, \\ v_t - uv_x - v = 1, \end{aligned} \quad (14)$$

subject to the initial conditions

$$u(x,0) = \mathrm{e}^x, \quad v(x,0) = \mathrm{e}^{-x}. \quad (15)$$

According to the HPM, we can construct a homotopy of system (14) which satisfies the following relation:

$$\begin{aligned} u_t - (y_0)_t + p\left[(y_0)_t + vu_x + u - 1\right] &= 0, \\ v_t - (z_0)_t + p\left[(z_0)_t - uv_x - v - 1\right] &= 0. \end{aligned} \quad (16)$$

Let us choose the initial approximations as

$$\begin{aligned} u_0(x,t) = y_0(x,t) = u(x,0) = \mathrm{e}^x, \\ v_0(x,t) = z_0(x,t) = v(x,0) = \mathrm{e}^{-x}. \end{aligned} \quad (17)$$

Substituting (13) and (17) into (16) and equating terms of the same powers of p, obtain

$$\begin{aligned} &(u_1)_t + (y_0)_t + v_0(u_0)_x + u_0 - 1 = 0, \quad u_1(x,0) = 0, \\ &(v_1)_t + (z_0)_t - u_0(v_0)_x - v_0 - 1 = 0, \quad v_1(x,0) = 0, \\ &(u_2)_t + (u_0)_x v_1 + (u_1)_x v_0 + u_1 = 0, \quad u_2(x,0) = 0, \\ &(v_2)_t - (v_0)_x u_1 - (v_1)_x u_0 - v_1 = 0, \quad v_2(x,0) = 0, \\ &(u_3)_t + (u_0)_x v_2 + (u_1)_x v_1 + (u_2)_x v_0 + u_2 = 0, \\ &\quad u_3(x,0) = 0, \\ &(v_3)_t - (v_0)_x u_2 - (v_1)_x u_1 - (v_2)_x u_0 - v_2 = 0, \\ &\quad v_3(x,0) = 0, \\ &(u_4)_t + (u_0)_x v_3 + (u_2)_x v_1 + (u_1)_x v_2 + (u_3)_x v_0 \\ &\quad + u_3 = 0, \quad u_4(x,0) = 0, \\ &(v_4)_t - (v_0)_x u_3 - (v_2)_x u_1 - (v_1)_x u_2 - (v_3)_x u_0 \\ &\quad - v_3 = 0, \quad v_4(x,0) = 0, \\ &(u_5)_t + (u_2)_x v_2 + (u_3)_x v_1 + (u_0)_x v_4 + (u_1)_x v_3 \\ &+ (u_4)_x v_0 + u_4 = 0, \qquad u_5(x,0) = 0, \\ &(v_5)_t - (v_2)_x u_2 - (v_3)_x u_1 - (v_0)_x u_4 - (v_1)_x u_3 \\ &\qquad - (v_4)_x u_0 - v_4 = 0, \qquad v_5(x,0) = 0, \end{aligned}$$

etc. Solving the above differential equations we obtain,

$$\begin{aligned} &u_1(x,t) = -\mathrm{e}^x t, \quad v_1(x,t) = \mathrm{e}^{-x} t, \\ &u_2(x,t) = \frac{1}{2}\mathrm{e}^x t^2, \quad v_2(x,t) = \frac{1}{2}\mathrm{e}^{-x} t^2, \\ &u_3(x,t) = -\frac{1}{6}\mathrm{e}^x t^3, \quad v_3(x,t) = \frac{1}{6}\mathrm{e}^{-x} t^3, \\ &u_4(x,t) = \frac{1}{24}\mathrm{e}^x t^4, \quad v_4(x,t) = \frac{1}{24}\mathrm{e}^{-x} t^4, \\ &u_5(x,t) = -\frac{1}{120}\mathrm{e}^x t^5, \quad v_5(x,t) = \frac{1}{120}\mathrm{e}^{-x} t^5, \end{aligned}$$

etc.

Hence, the series solutions are

$$\begin{aligned} u(x,t) &= \mathrm{e}^x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right), \\ v(x,t) &= \mathrm{e}^{-x}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right), \end{aligned}$$

which converge to the closed-form solutions,

$$u(x,t) = \mathrm{e}^{x-t}, \quad v(x,t) = \mathrm{e}^{-x+t}.$$

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4 Conclusion

The homotopy-perturbation method (HPM) was employed successfully for solving linear and nonlinear system of partial differential equations. HPM avoids the difficulties arising in finding the Adomian polynomials and transformation formulas. In addition, the calculations involved in HPM are very simple and straightforward. It is demonstrated that HPM is a promising tool for systems of PDEs.

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