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Subdivision Algorithm for the Triangular Spline Surface

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Abstract - Subdivision algorithms for rendering of box spline surfaces have been independently developed by Boehm, Cohen, Lyche, Riesenfeld, Dahmen, Michelli and Prautzsch. The algorithm refine the control net of any box spline surface so that the refined control nets converge to the spline. The aim of this paper is to study multivariate *B*-splines and triangular spline surface. The study will consider the divided difference of a function f which can be expressed in terms of a multiple integral. From here, we give a geometric interpretation of *B*-splines and the definition of triangular spline on uniform mesh. Subdivision algorithms for rendering of triangular spline surfaces are developed and the triangular spline surfaces are then generated.

Key words: Subdivision algorithm, triangular spline, uniform mesh.

1. INTODUCTION

Triangular polynomial patches were first considered by de Casteljau in Computer Aided Geometric Design (CAGD) (Farin, 1983), but these scarcely received any attention. The triangular patches were generated based on the Bezier polynomials defined over the arbitrary triangles. Sabin used triangular patches in Bernstein form to construct *B*-splines over regular triangular by convolution (Sabin, 1977). Later, the *B*-splines were found to be the triangular spline (de Boor and de Vore, 1983). Subdivision algorithms for rendering of triangular spline surfaces have been independently developed by Boehm, Cohen, Lyche, Riesenfeld, Dahmen, Micchelli and Prautzsch (Cohen, Lyche and Riesenfeld, 1984).

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2 TRIANGULAR SPLINES ON A 3-DIRECTION MESH

Let

$$V = \left\{ \utilde{e}^1, \utilde{e}^2, ..., \utilde{e}^k \right\} \in I\!R^2, \; k \geq 2,$$

where

$\utilde{e}^1 = (1, 0)$, $\utilde{e}^2 = (0, 1)$, $\utilde{e}^3, ..., \utilde{e}^k \in \left\{ \utilde{e}^1, \utilde{e}^2, \utilde{e}^2 - \utilde{e}^1 \right\}$

, and suppose span $\left[\utilde{e}^1, \utilde{e}^2 \right] = I\!R^2$.

Then, the triangular spline $M\left(\utilde{x} | \utilde{e}^1, \utilde{e}^2, ..., \utilde{e}^k \right)$, may be defined by

$$(1.2.1) \quad \begin{cases} M\left(\utilde{x} | \utilde{e}^1, \utilde{e}^2 \right) = \begin{cases} 1, \text{ if } \utilde{x} = \tau_1 \utilde{e}^1 + \tau_1 \utilde{e}^2, \tau_1, \tau_2 \in [0, 1] \\ 0, \text{ otherwise} \end{cases} \\ \text{and for } i = 3, 4, ..., k \\ M\left(\utilde{x} | \utilde{e}^1, \utilde{e}^2, ..., \utilde{e}^\ell \right) = \int_0^1 M\left(\utilde{x} - \tau \utilde{e}^1 | \utilde{e}^1, \utilde{e}^2, ..., \utilde{e}^{\ell - 1} \right) d\tau \end{cases}$$

Thus, the triangular spline $M\left(\utilde{x} | \utilde{e}^1, \utilde{e}^2, ..., \utilde{e}^k \right)$ is a piecewise polynomial of degree $l - 2$.
Now, let us see a few examples:

1. For $l = 2$ and let $\utilde{x} = (x, y)$.

By (1.2.1), triangular spline $M(\utilde{x} | \utilde{e}^1, \utilde{e}^2)$ is a characteristic function which forms a square $[0, 1]^2$, see Figure 2.1.

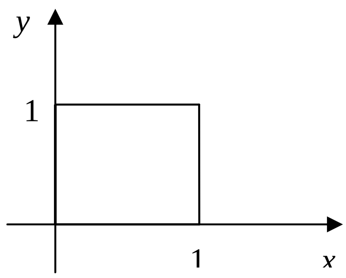

Figure 2. 1: Support $M(\utilde{x} | \utilde{e}^1, \utilde{e}^2)$

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2. For $l = 3$ and

let $\underset{\sim}{x} = (x,\ y),\ \underset{\sim}{e}^1 = (1,\ 0),\ \underset{\sim}{e}^2 = (0,\ 1),\ \underset{\sim}{e}^3 = (1,\ 1)$

$$M\left(\underset{\sim}{x}|\underset{\sim}{e}^1,\ \underset{\sim}{e}^2,\ \underset{\sim}{e}^3\right) = \int_0^1 M\left(\underset{\sim}{x} - \tau\underset{\sim}{e}^3\,|\underset{\sim}{e}^1,\ \underset{\sim}{e}^2\right)d\tau\ .$$

By (1.2.1), we have

$$(1.\ 2.\ 2)\quad M\left(\underset{\sim}{x} - \tau\underset{\sim}{e}^3\,|\underset{\sim}{e}^1,\ \underset{\sim}{e}^2\right) = \begin{cases} 1, & \underset{\sim}{x} - \tau\underset{\sim}{e}^3 = \tau_1\underset{\sim}{e}^1 + \tau_2\underset{\sim}{e}^2;\ \tau_1,\ \tau_2 \in [0,\ 1] \\ 0, & \text{otherwise.} \end{cases}$$

Thus

$$\underset{\sim}{x} - \tau\underset{\sim}{e}^3 = \tau_1\underset{\sim}{e}^1 + \tau_2\underset{\sim}{e}^2,$$

$$(1.\ 2.\ 3)\quad \begin{cases} \Leftrightarrow\ \ x - \tau = \tau_1, & y - \tau = \tau_2, \\ \Leftrightarrow\ \ 0 \le x - \tau \le 1, & 0 \le y - \tau \le 1, \\ \Leftrightarrow\ \ x - 1 \le \tau \le x, & y - 1 \le \tau \le y. \end{cases}$$

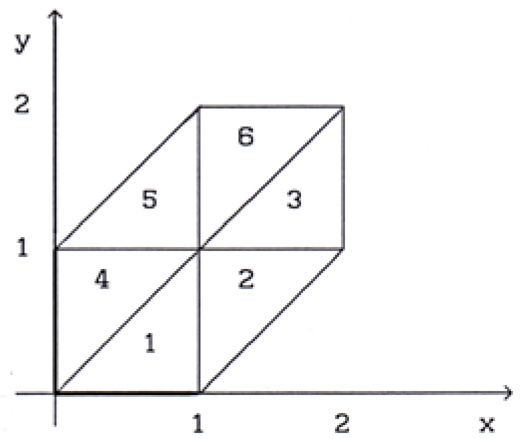

Figure 2.2: Support $M(\underset{\sim}{x}|\underset{\sim}{e}^1,\ \underset{\sim}{e}^2,\ \underset{\sim}{e}^3)$

We can evaluate triangular spline $M(\underset{\sim}{x}|\underset{\sim}{e}^1,\ \underset{\sim}{e}^2,\ \underset{\sim}{e}^3)$ based on inequality (1.2.3) and values of x and y on each triangular patch in figure 2.2.

By (1.2.2), we have

$M(\underset{\sim}{x}|\underset{\sim}{e}^1,\ \underset{\sim}{e}^2,\ \underset{\sim}{e}^3) = 0$, for $0 > x > 2$ or $0 > y > 2$

or $0 \le y \le x - 1$ and $1 \le x \le 2$,

or $0 \le x \le y - 1$ and $1 \le y \le 2$,

We shall consider for each region:

Region 1, for

$$0 \le x \le 1 \text{ and } 0 \le y \le x.$$

By (1.2.3), in order that for (x, y) in region 1,
ee have

$$0 \le \tau \le y,$$

thus

$$M(\underset{\sim}{x}|\underset{\sim}{e}^1,\ \underset{\sim}{e}^2,\ \underset{\sim}{e}^3) = \int_0^y 1\,d\tau = y.$$

Region 2, for

$$1 \le x \le 2 \text{ and } x - 1 \le y \le 1.$$

By (1.2.3), in order that for (x, y) in region 2,
we have

$$x - 1 \le \tau \le y,$$

thus,

$$M(\underset{\sim}{x}|\underset{\sim}{e}^1,\ \underset{\sim}{e}^2,\ \underset{\sim}{e}^3) = \int_{x-1}^y 1\,d\tau = y - x + 1.$$

Region 3, for

$$1 \le x \le 2 \text{ and } 1 \le y \le x.$$

By (1.2.3), in order that for (x, y) in region 3,
we have

$$x - 1 \le \tau \le 1,$$

thus,

$$M(\underset{\sim}{x}|\underset{\sim}{e}^1,\ \underset{\sim}{e}^2,\ \underset{\sim}{e}^3) = \int_{x-1}^1 1\,d\tau = 2 - x.$$

By symmetry,
region 4, for

$$0 \le x \le 1 \text{ and } x \le y \le 1,$$

thus,

$$M(\underset{\sim}{x}|\underset{\sim}{e}^1,\ \underset{\sim}{e}^2,\ \underset{\sim}{e}^3) = x,$$

region 5, for

$$0 \le x \le 1 \text{ and } 1 \le y \le x + 1,$$

thus,

$$M(\underset{\sim}{x}|\underset{\sim}{e}^1,\ \underset{\sim}{e}^2,\ \underset{\sim}{e}^3) = x - y + 1,$$

region 6, for

$$1 \le x \le 2 \text{ and } x \le y \le 2,$$

thus,

$$M(\underset{\sim}{x}|\underset{\sim}{e}^1,\ \underset{\sim}{e}^2,\ \underset{\sim}{e}^3) = 2 - y.$$

The triangular spline $M(\underset{\sim}{x}|\underset{\sim}{e}^1,\ \underset{\sim}{e}^2,\ \underset{\sim}{e}^3)$ is also roof function, a piecewise polynomial of degree 1 on triangular patch which is generated by three vector $\underset{\sim}{e}^1,\ \underset{\sim}{e}^2$ and $\underset{\sim}{e}^3 = \underset{\sim}{e}^1 + \underset{\sim}{e}^2$. Support $M(\underset{\sim}{x}|\underset{\sim}{e}^1,\ \underset{\sim}{e}^2,\ \underset{\sim}{e}^3)$ is a hexagon with vertices (0, 0), (1, 0), (2, 1), (2, 2), (1, 2), (0, 1). Therefore

(1.2.4)

$$M((i,\ j)|\underset{\sim}{e}^1,\ \underset{\sim}{e}^2,\ \underset{\sim}{e}^3) = \begin{cases} 1 & ,\ (i,\ j) = (1,\ 1) \\ 0 & ,\ (i,\ j) \in \mathbb{Z}^2 \setminus (1,\ 1). \end{cases}$$

Now, given control points $a_{(i,j)} \in \mathbb{R}^3$, $(i,\ j) \in \mathbb{Z}^2$, i = 0, 1, …, m and j = 0, 1, …, n, the triangular spline is a surface {S} in $\mathbb{R}^3$ can be parameterized as

(1.2.5)

$$S(x,\ y) = \sum_{(i,j)} a_{(i,j)}\,M(x - i, y - j|e^1,\ e^2,\ ..., e^k)\ .$$

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By using (1.2.4), a control polyhedron {P} for {S} can be parameterized as

(1.2.6)

$$P(x,\ y) = \sum_{(i,j)} a_{(i,j)}\, M(x-i, y-j|\underset{\sim}{e}^1,\ \underset{\sim}{e}^2,\ \underset{\sim}{e}^3)$$

This is because for $\underset{\sim}{\gamma} \in \mathbb{Z}^2$, $\underset{\sim}{\gamma} = (\gamma_1,\ \gamma_2)$, $\gamma_1 = 0, 1, ..., m,$ and $\gamma_2 = 0,1,...,n$, $P(x,\ y)$ is a piecewise linear polynomial, i.e.

(1.2.7)

$$P(\gamma_1,\ \gamma_2) = \sum_{(i,j)} a_{(i,j)}\, M(\gamma_1 - i, \gamma_2 - j\,|\,\underset{\sim}{e}^1, \underset{\sim}{e}^2, \underset{\sim}{e}^3)\,.$$

$$= a_{(\gamma_1,\gamma_2)}.$$

Thus $P: \mathbb{R}^2 \to \mathbb{R}^3$ interpolates vectors $a_{(i,\ j)}$ on the lattice points $(i,\ j) \in \mathbb{Z}^2$. Hence, the surface {P} is a union of triangular faces with vertices $a_{(i,\ j)}$, $a_{(i+1,\ j)}$, $a_{(i+1,\ j+1)}$ and triangular faces with vertices $a_{(i,\ j)}$, $a_{(i,\ j+1)}$, $a_{(i+1,j+1)}$. Moreover, in order to obtain a smooth surface we shall use subdivision algorithm.

3 SUBDIVISION ALGORITHM FOR A TRIANGULAR SPLINE SURFACE

Let $a_{(i,j)} \in \mathbb{R}^3$, $(i, j) \in \mathbb{Z}^2$, i = 0, 1, …, m, j = 0, 1, …, n and $M_{r,s,t}(x,y)$ be the triangular spline on a three direction mesh in $\mathbb{R}^2$. A surface $\{S\} \in \mathbb{R}^3$ which is parameterized by

(1.3.1)

$$S(x,\ y) = \sum_{(i,j)} a_{(i,j)}\, M_{r,s,t}(x-i, y-j), \qquad (x,\ y) \in \mathbb{R}^2,$$

is called a triangular spline surface. By using (1.2.8), a control polyhedron {P} for the surface {S} can be parameterized as

(1.3.2)

$$P(x,y) = \sum_{(i,j)} a_{(i,j)}\, M_{1,1,1}(x-i, y-j), \qquad (x,\ y) \in \mathbb{R}^2$$

This is because for $\underset{\sim}{\gamma} \in \mathbb{Z}^2$, $\underset{\sim}{\gamma} = (\gamma_1,\ \gamma_2)$, $\gamma_1 = 0, 1, ..., m,$ and $\gamma_2 = 0, 1, ..., n$, $P(x,\ y)$ is a piecewise linear polynomial, i.e.

(1.3.3)

$$P(\gamma_1,\ \gamma_2) = \sum_{(i,j)} a_{(i,j)}\ M_{1,1,1}(\gamma_1 - i,\ \gamma_2 - j) = a_{(\gamma_1,\gamma_2)}.$$

Thus, $P: \mathbb{R}^2 \to \mathbb{R}^3$ interpolates vectors $a_{(i,\ j)}$ on the lattice points $(i, j) \in \mathbb{Z}^2$. Hence, the surface {P} is a union of triangular faces with vertices $a_{(i,\ j)}$, $a_{(i+1,j)}$, $a_{(i+1,\ j+1)}$ and triangular faces with vertices $a_{(i,\ j)}$, $a_{(i,\ j+1)}$, $a_{(i+1,\ j+1)}$. Since triangular spline is a generalization of uniform B-spline, the subdivision algorithm for uniform B-spline curve can be easily extended to triangular spline surface. The idea is to apply the subdivision algorithm for uniform B-spline curve along each of the three directions $\underset{\sim}{e}^1 = (1,\ 0)$, $\underset{\sim}{e}^2 = (0,\ 1)$, $\underset{\sim}{e}^3 = \underset{\sim}{e}^1 + \underset{\sim}{e}^2 = (1,\ 1)$.

Then, for a surface {P} as parameterized by (1.3.2) we can obtain a smooth surface $\{P^1\}$ which is parameterized by

(1.3.4)

$$P^1(x,y) = \sum_{(i,j)} a^{(1)}_{(i,j)}\, M_{1,1,1}(x-i, y-j) : (x,y) \in \mathbb{R}^2,$$

where $a^{(1)}_{(i,j)}$ are obtained by using the subdivision algorithm. This algorithm can be written in the following steps.

(I) $$b^{1,1,1}_{(2i+\nu,2j+\mu)} = \frac{1}{2}\left\{a_{(i,j)} + a_{(i+\nu,\ j+\mu)}\right\}$$

$(\nu, \mu) = (0,\ 0),\ (1,\ 0),\ (0,\ 1),\ (1,\ 1)$

(II) For l = 2, 3, …, r,

$$b^{1,1,1}_{(i,j)} = \frac{1}{2}\left\{b^{\ell-1,1,1}_{(i,j)} + b^{\ell-1,1,1}_{(i+1,j)}\right\}.$$

For m = 2, 3, …, s,

$$b^{r,m,1}_{(i,j)} = \frac{1}{2}\left\{b^{r,m-1,1}_{(i,j)} + b^{r,m-1,1}_{(i,j+1)}\right\}.$$

For n = 2, 3, …, t,

$$b^{r,s,n}_{(i,j)} = \frac{1}{2}\left\{b^{r,s,n-1}_{(i,j)} + b^{r,s,n-1}_{(i+1,j+1)}\right\}.$$

Set $a^{(1)}_{(i,j)} = b^{r,s,t}_{(i,j)}$.

We then have a new control polyhedron P^1 consisting of triangular faces with vertices $a^{(1)}_{(i,j)}$, $a^{(1)}_{(i+1,j)}$, $a^{(1)}_{(i+1,j+1)}$ and triangular faces with vertices $a^{(1)}_{(i,j)}$, $a^{(1)}_{(i,j+1)}$, $a^{(1)}_{(i+1,j+1)}$. We may call this control polyhedron P^1 as triangular spline of one iteration.
We then repeat step I and II to obtain triangular spline of two iterations P^{11}, triangular spline of three iterations P^{111} and so forth.

Figure 3.1 shows the points $b^{1,1,1}_{(2i+\nu,2j+\mu)}$ for $(\nu,\ \mu)$ = (0, 0), (1, 0), (0, 1), (1, 1), i = 0, 1, …, m and j = 0, 1, …, n, which are obtained by using (I). These points are denoted by the small circles. We observe that for $(\nu,\ \mu) \neq (0,\ 0)$ the points $b^{1,1,1}_{(2i+\nu,2j+\mu)}$ are the midpoints for each side of the triangles which form the surface {P}.

Figure 3.2 shows that a control polyhedron P^1 consisting of triangular faces with vertices

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$a^{(1)}_{(i,j)}$, $a^{(1)}_{(i+1,j)}$, $a^{(1)}_{(i+1,j+1)}$ and triangular faces with vertices $a^{(1)}_{(i,j)}$, $a^{(1)}_{(i,j+1)}$, $a^{(1)}_{(i+1,j+1)}$, which are obtained using (I) and (II).

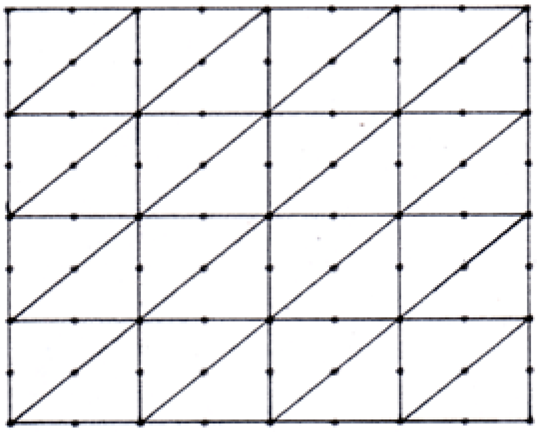

Figure 3.1: The points $b^{1,1,1}_{(2i+\nu,2j+\mu)}$, i = 0, 1, 2, 3, 4, j = 0, 1, 2, 3, 4 and (ν,μ) = (0, 0), (1, 0), (0, 1), (1, 1), which are obtained by using step I.

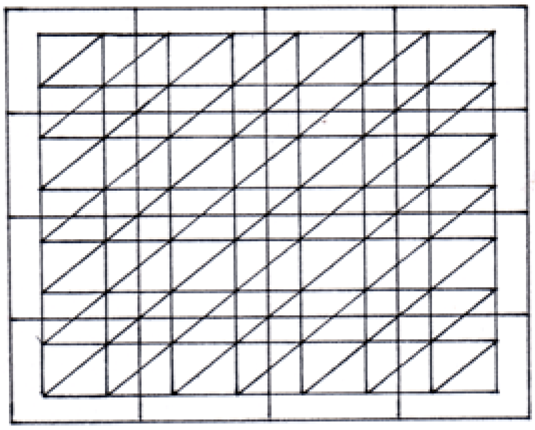

Figure 3.2: Triangular spline surface, which is obtained by using 1 iteration of subdivision algorithm for triangular spline, i.e. steps (I) and (II).

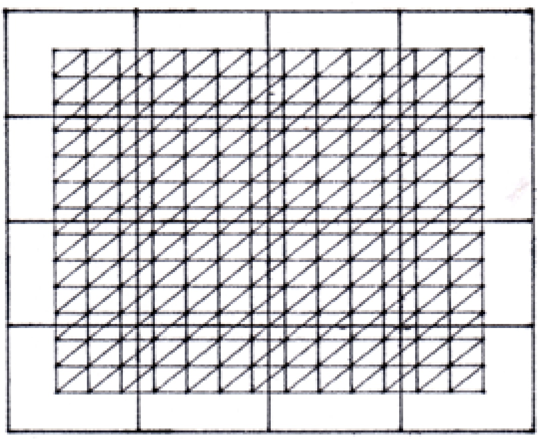

Figure 3.3: Triangular spline surface, which is obtained by using 2 iterations of subdivision algorithm for triangular spline, i.e. steps (I) and (II).

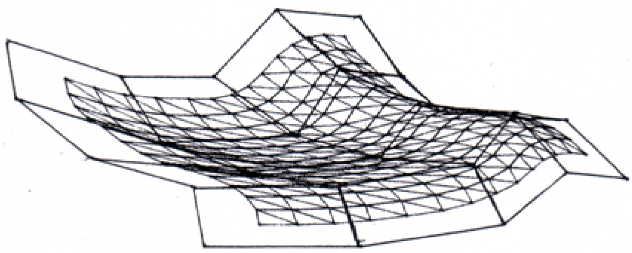

Figure 3.4: Triangular spline surface, which is obtained by using 2 iterations of subdivision algorithm for triangular spline.

4 CONCLUSION

In this paper, we have considered geometric interpretation of B-splines and the definition of triangular spline on the uniform mesh. Apart from that, we have developed subdivision algorithm for rendering of triangular spline surfaces. The triangular spline surfaces, which are obtained by using 1 and 2 iterations of subdivision algorithm are then generated.

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