Generalized Measure for Two Utility Distributions

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Abstract—The measure $I_\alpha(P:Q;U) = (\alpha-1)^{-1}\log\frac{\Sigma u_i p_i^\alpha q_i^{1-\alpha})}{\Sigma u_i p_i}$ is additive. In this paper the measure is find out, which are non-additive and satisfy the non-additive property by considering two probability distributions and two utility distributions attached with them. It is further shown that the characterize a non-additive generalized measure of relative useful information by using weighted sum property and two measures of J-divergence for U and V utility distributions.

***Index Terms*—Divergence measures; Relative information of type s; Relative J-divergence, Properties Of J^β.**

I. INTRODUCTION

The divergence measure is applied on numerous areas like probability measure, pattern recognition, signal processing, economics etc. These measures are normally used to find proper distance or difference between two probability distribution. These measures can be categorized as parametric, non-parametric and entropy-type measures of information. Parametric measures: These type of information, measure the amount of information about an unknown parameter α supplied by the data and are functions of α. The best known measure of this type is measure of information [1].
Non-parametric measures: These type of information, give the amount of information supplied by the data for discriminating in favor of one of a probability distribution against the another , or for measuring the distance or affinity between probability distribution. Kullback and Leibler [3] showed that this measure is the best known in this class.
Entropy-type Measures: These type of information express the amount of information contained in a distribution, that is, the amount of uncertainty associated with the outcome of an experiment.
New divergence measures and their relationships with the well known divergence measures are also studied by Kumar, Chhina [14], Kumar, Hunter [13] and Kumar, Johnson [15].
J-divergence equals the average of the two possible KL-distances between two probability distributions, although Jeffreys [2] did not develop it to symmetries the Kullback and Leibler [3] distance. Kumar and Taneja [16] discussed non-symmetric relative J-divergence measure and its properties.

Let $P = (p_1, p_2, \ldots, p_n)$, $0 < p_i \le 1$, $\sum_{i=1}^{n} p_i = 1$ be a finite probability distribution of set of 'n' events $E = (E_1, E_2, \ldots, E_n)$ and $Q = (q_1, q_2, \ldots, q_n)$, $0 < q_i \le 1$, $\sum_{i=1}^{n} q_i = 1$ be the revised probability distribution of set of 'n' set events $F = (F_1, F_2, \ldots, F_n)$.
Then Kullback and Leibler [3] in measures of directed divergence defined as

$$I(P/Q) = \sum_{i=1}^{n} p_i \log(p_i/q_i) \qquad \ldots\ldots(1.1)$$

Belis, Guiasu [6] have attached a utility distribution $U = (U_1, U_2, \ldots, U_n)$ to random experiment
$E = (E_1, E_2, \ldots, E_n)$ where $U_i > 0$ is the utility of the ith outcome E_i.
Bhaker, Hooda [12] characterized the following measure of 'useful' directed divergence or related information as

$$I(P:Q;U) = \frac{\sum_{i=1}^{n} u_i p_i \log(p_i/q_i)}{\sum_{i=1}^{n} u_i p_i} \qquad \ldots\ldots(1.2)$$

when $V = (v_1, v_2, \ldots, v_n)$ is the utility distribution function of events $F = (f_1, f_2, \ldots, f_n)$ of revised experiments of E. Then we define another new useful information measure as

$$I(P:Q;U;V) = \frac{\sum_{i=1}^{n} u_i p_i \log(p_i u_i / q_i v_i)}{\sum_{i=1}^{n} u_i p_i} \qquad \ldots\ldots(1.3)$$

when $u_i = v_i$ then (1.3)→ (1.2)

Bhaker, Hooda [12] also studied generalized mean value characterization of the following measure of 'useful' relative information of order α.

$$I_\alpha(P:Q;U) = (\alpha-1)^{-1}\log\frac{\sum u_i p_i^\alpha q_i^{1-\alpha})}{\sum u_i p_i} \qquad \ldots\ldots(1.4)$$

The measure (1.4) is also additive. So it is interesting to find out the measure, which are non-additive and satisfy the non-additivity property of the following form.

$$I(P*R:Q*S;U*V) = I(P:Q;U) + I(R;S;V) + KI(P;Q;U)\,I(R;S;V) \qquad \ldots\ldots(1.5)$$

Where P, Q ∈ Δ_n, R, S ∈ Δ_m, U and V are utility distributions and k (≠ 0) is any real number.
Now we define the 'useful' relative information of order 'α' as

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$$I_\alpha(P:Q;U:V)=(\alpha-1)^{-1}\log\frac{\sum u_i p_i (u_i/q_i^{\alpha-1})}{\sum u_i p_i}$$
…….(1.6)

The measure (1.6) is also additive.

We define another non-additive useful information measure of order 'β' as

$$I^\beta(P:Q;U:V)=\frac{1}{2^{\beta-1}-1}\left[\frac{\sum u_i p_i (u_i p_i/v_i q_i^{\beta-1})}{\sum u_i p_i}-1\right]$$
…….(1.7)

The measure (1.7) satisfy the no-additivity property of the following form

$$I(P*P:Q*Q;V*V)$$
$$=I(P:Q;U:V)+I(P:Q;U;V)+KI(P;Q;U:V)$$
…….(1.8)

Where P, Q $\in \Delta_n$, P', Q' $\in \Delta_m$, U, U' and V, V' are utility distributions and k ($\neq 0$) is any real number.

II. CHARACTRIZATION OF USEFUL DIRECTED DIVERGENCE

We consider the non-additivity of the type (1.8) let 'h' be real valued function defined or $R\times R$. We assume that

$$I(P:Q;U:V)=\frac{\sum u_i p_i h(u_i p_i v_i q_i)}{\sum u_i p_i}$$
…….(2.1)

When

$p=1, q=?$ and $u=v$; $h(1u,?\ u)-1$ and $h(u/2,u/2)=0$ …….(2.2)

The property (2.1) is called weighted sum property. Then functional equation (1.8) together with (2.1) reduces to

$$\frac{\sum\sum u_i p_i u'_i\, p'_i\, h(u_i p_i u_j p_j v_i q_i v_j q_j)}{\sum\sum u_i p_i u_j p_j}$$
$$=\frac{\sum u_i p_i h(u_i p_i v_i q_i)}{\sum u_i p_i}+\frac{\sum u'_j\, p'_j\, h(u'_j\, p'_j\, v'_j\, q'_j)}{\sum u_j p_j}$$
$$+\frac{\sum u_i p_i h(u_i p_i v_i q_i)}{\sum u_i p_i}\frac{\sum u'_j\, p'_j\, h(u'_j\, p'_j\, v'_j\, q'_j)}{\sum u'_j\, p'_j}$$
…….(2.3)

It can be easily proved Hooda [11] that the continuous function which satisfies the functional equation (2.3) is also continuous solution of the following functional equation.

$$h(up\,u'p';vq\,v'q')=h(up\,,vq\,)+h(u'p',v'q')$$
$$+kh(up\,;vq\,)h(u'p',v'q')$$
…….(2.4)

where pp′, qq′, uu′, vv′ ∈ R and 'k' is a non-zero real number.

Multiplying both sides of (2.4) by k and adding (1) we have

1+kh(up u′p′;vq v′q′)=1+kh(up, vq)+kh(u′p′, v′q′)+k2h(up, vq)h(u′p′, v′q′)

1+kh(up u′p′;vq v′q′)=[1+kh(up, vq)][1+kh(u′p′, v′q′)]

…….(2.5)

Putting 1+kh(up, vq)=f(up, vq) in (2.5)

F(up u′p′;vq v′q′) = f(up, vq)f(u′p′, v′q′)
…….(2.6)

Putting up = r, u′p′= r′, vq = s, v′q′ = s′ in (2.6)
f(rr′; ss′) = f(r, s) f(r′, s′) is continuous real valued functional equation.

Aczel [5] has given one of the continuous real solution is given by

f(r, s) = rβsα

f(up, vq) = (up)β(vq)α

Where α and β are real or complex numbers.

It implies 1 + kh(up, vq) = (up)β(vq)α

or

$$h(up,vq)=\frac{1}{k}[(up)^\beta(vq)^\alpha-1]$$ …….(2.7)

Applying suitable conditions (2.2) in (2.7), we get

α + β = 0 and k = 2-α-1 = 2β-1

$$\text{h(up, vq)}=\frac{1}{(2^\beta-1)}\left[\left(\frac{up}{vq}\right)^\beta-1\right]$$ …….(2.8)

Without the loss of generality we can replace β by β-1 in (2.8), we get

$$\text{h(up, vq)}=\frac{1}{(2^\beta-1)}\left[\left(\frac{up}{vq}\right)^{\beta-1}-1\right]$$ …….(2.9)

Substituting (2.9) in (2.1) we get

$$I_\beta(P:Q;U:V)=\frac{1}{2^{\beta-1}-1}\left[\frac{\sum u_i p_i (u_i p_i/V_i q_i)-1^{\beta-1}}{\sum u_i p_i}\right]$$
…….(2.10)

III. MEASURE OF 'USEFUL' J-DIVERGENCE

The 'Useful' J-divergence measure corresponding to (1.3) is given by

$$J(P:Q;U:V)=\frac{\sum u_i p_i (u_i p_i/V_i q_i)-1}{\sum u_i p_i}+\left[\frac{\sum v_i q_i \log(v_i q_i/u_i p_i)-1}{\sum u_i p_i}\right]$$
…….(3.1)

In case utilities are ignored or $u_i = v_i = 1$ for each (i) then (3.1) reduces to

$$J(P:Q)=\sum p_i\log(p_i/q_i)+\sum q_i\log(q_i/p_i)$$
…….(3.2)

The measure (3.2) was studied by Kullback [4] and has found wide applications in statistics and in pattern recognition [7]. The measure (3.1) can also be written as

$$J(P:Q;U:V)=I(P:Q;U:V)+I(P:Q;V:U)$$
…….(3.3)

Thus corresponding to (2.10) a measure of J-divergence is defined as

$$J^\beta(P:Q;U:V)$$
$$=\frac{1}{2^{\beta-1}-1}\left[\frac{\sum u_i p_i (u_i p_i/v_i q_i)^{\beta-1}}{\sum u_i p_i}+\frac{\sum v_i q_i \log(v_i q_i/u_i p_i)^{\beta-1}}{\sum u_i p_i}-2\right]$$
$$=\frac{1}{2^{\beta-1}-1}\left[\frac{\sum (u_i p_i)^\beta/(v_i q_i)^{1-\beta}}{\sum u_i p_i}+\frac{\sum (v_i q_i)^\beta (u_i p_i)^{\beta-1}}{\sum v_i q_i}-2\right]$$
…….(3.4)

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It may be noted that (3.4) reduces to (3.1) when $\beta \to 1$ and so it can be called generalized 'useful' J-divergence of degree β.

In case utilities are ignored, the measure (3.4) reduces to

$$J^{\beta}(P:Q) = \frac{1}{2^{\beta-1}}\left[\sum p_i q_i^{\beta} + \sum q_i p_i^{1-\beta} - 2\right], \quad \beta \neq 1$$

$$\{\sum p_i = \sum p_i = 1\} \qquad \text{.......(3.5)}$$

The measure (3.5) was studied by Sheng and Rathee, [8] and Burbea et al. [9] and was called J-divergence measure of degree β. Hence to call measure (3.4) as useful J-divergence measure of degree β justified.

Another generalization of (3.1) can be considered of the following form

$$J_{\alpha}(P:Q;U:V) = (\alpha-1)^{-1}\log\sum\frac{1}{2}\left[\frac{\sum(u_i p_i)^{\alpha}(v_i q_i)^{1-\alpha}}{\sum u_i p_i} + \frac{(v_i q_i)^{\alpha} u_i p_i)^{1-\alpha}}{\sum v_i q_i}\right] \qquad \text{.......(3.6)}$$

We see that (3.6) reduces to (3.1) in case utilities are ignored or ui = pi = 1 for each i, (3.6) becomes

$$J_{\alpha}(P:Q) = (\alpha-1)^{-1}\log\sum\frac{1}{2}\left[p_i^{\alpha} q_i^{1-\alpha} + q_i^{\alpha} p_i^{1-\alpha}\right], \alpha \neq 1 \qquad \text{.......(3.7)}$$

which was characterized by Taneja [10].

Let

$$W^{\beta}(P:Q;U:V) = \left[\frac{\sum(u_i p_i)^{\beta}(v_i q_i)^{1-\beta}}{\sum u_i p_i} + \frac{(v_i q_i)^{\beta}(u_i p_i)^{1-\beta}}{\sum v_i q_i}\right] \qquad \text{.......(3.8)}$$

(1) Then (3.4) and (3.6) can be written as

$$J^{\beta}(P:Q;U:V) = (2^{\beta-1}-1)^{-1}\left[W^{\beta}(P:Q;U:V) - 2\right] \qquad \text{.......(3.9)}$$

$$J_{\alpha}(P:Q;U:V) = (\alpha-1)^{-1}\log\left[W^{\beta}(P:Q;U:V)/2\right] \qquad \text{.......(3.10)}$$

(2) The measure $W^{\beta}(P:Q;U:V)$ given by (3.8) plays an important role in characterization of the above two generalized measures of J- divergence. So we characterized $W^{\beta}(P:Q;U:V)$. Which can be written as

$$W^{\beta}(P:Q;U:V) = W_1^{\beta}(P:Q;U:V) + W_2^{\beta}(P:Q;U:V)$$

we see that $W_1^{\beta}(P:Q;U:V)$ and $W_2^{\beta}(P:Q;U:V)$ satisfy the following relation

$$W(P*P':Q*Q';U*U':V*V') = W(P:Q;U:V)\,W(P':Q';U':V') \qquad \text{.......(3.11)}$$

Let 'h' be a real valued continuous function on $R \times R$ such that

$$W_1(P:Q;U:V) = \frac{\sum u_i p_i\ g(u_i p_i, v_i q_i)}{\sum u_i p_i} \qquad \text{.......(3.12)}$$

and $$W_2(Q:P;V:U) = \frac{\sum v_i q_i\ g(v_i q_i, u_i p_i)}{\sum v_i q_i} \qquad \text{.......(3.13)}$$

The functional eqn (3.11) together with (3.12) and (3.13) gives

$$\frac{\sum\sum u_i u_j'\ p_i p_j'\ g(p_i p_j', q_i q_j'; u_i u_j', v_i v_j')}{\sum u_i u_j'\ p_i p_j'} = \frac{\sum u_i p_i\ g(u_i p_i, v_i q_i)}{\sum u_i p_i}\cdot\frac{\sum u_j' p_j'\ g(u_j' p_j', v_j' q_j')}{\sum u_j' p_j'} \qquad \text{.......(3.14)}$$

Theorem 1: One of the continuous solution of the functional equation (3.14) under boundary conditions
$p_i = q_i = ½$ and u_i, v_i i.e. g(u/2, u/2) = 1 is given by
g(up, vq) = (up)β-1 (vq)1-β β ≠ 0, β > 0.

Proof: It can be easily verified that continuous function which satisfy the functional equation (3.14) is the continuous solution of the following functional equation

g(pp′, qq′ ; uu′, vv′) = g(p, q; u, v) g(p′, q′; u′, v′) ....(3.15)

Where p, p′, q, q′ u, u′, v, v′ ∈ R

The most general continuous solution of the functional equation (3.15) is given by

g(up, vq) = (up)β(vq)α ....(3.16)

where α, β are real or complex numbers.

Now (3.16) together with condition g(u/2, u/2) = 1 and thus we get

g(up, vq) = (up)β(vq)β , β > 0

Without loss of generality, we can replace β by β-1 and thus we get

g(up, vq) = (up)β-1(vq)β-1, β ≠ 0 β > 0 ....(3.17)

This proves the theorem.

Putting (3.17) and (3.12) we have

$$\omega_1^{\beta}(P:Q;U:V) = \frac{\sum(u_i p_i)\beta(v_i q_i)^{1-\beta}}{\sum(u_i p_i)} \qquad \text{.......(3.18)}$$

Similarly, we can prove

$$\omega_2^{\beta}(Q:P;V:U) = \frac{\sum(v_i q_i)\beta(u_i p_i)^{1-\beta}}{\sum(u_i p_i)} \qquad \text{.......(3.19)}$$

and we have

$$\omega^{\beta}(P:Q;U:V) = \frac{\sum(u_i p_i)^{\beta}(v_i q_i)^{1-\beta}}{\sum(u_i p_i)} + \frac{\sum(v_i q_i)^{\beta}(u_i p_i)^{1-\beta}}{\sum(v_i q_i)} \qquad \text{.......(3.20)}$$

Substituting (3.20) in (3.9) we get required generalized measures J_{β}(P:Q;U:V) and Jα(P:Q;U:V) respectively.

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IV. PROPERTIES OF Jβ(P:Q;U:V)

(i) Non-Negativity

Theorem 2 : $J^{\beta}(P:Q;U:V)$ is non-negative and vanishes iff $p_i = q_i$ & $u_i = v_i$ for all
i = 1, 2, ………n.

Proof:

$J^{\beta}(P:Q;U:V)$

$$= \frac{1}{2^{\beta-1}-1}\left[\frac{\sum(u_ip_i)^{\beta}(v_iq_i)^{1-\beta}}{\sum(u_ip_i)^{\beta}} + \frac{\sum(v_iq_i)^{\beta}(u_ip_i)^{1-\beta}}{\sum(v_iq_i)^{\beta}} - 2\right], \beta \neq 0 \quad \text{…….(4.1)}$$

for β > 1 by Holder's inequality

$$\frac{\sum(u_ip_i)^{\beta}(v_iq_i)^{1-\beta}}{\sum(u_ip_i)^{\beta}} \geq 1 \text{ and } \frac{\sum(v_iq_i)^{\beta}(u_ip_i)^{1-\beta}}{\sum(v_iq_i)^{\beta}} \geq 1$$

and also $\frac{1}{2^{\beta-1}-1} > 1$

Therefore $J^{\beta}(P:Q;U:V)$ is non-negative for β > 1.

for β < 1 by Holder's inequality

$$\frac{\sum(u_ip_i)^{\beta}(v_iq_i)^{1-\beta}}{\sum(u_ip_i)^{\beta}} \leq 1 \text{ and } \frac{\sum(v_iq_i)^{\beta}(u_ip_i)^{1-\beta}}{\sum(v_iq_i)^{\beta}} \leq 1$$

and also $\frac{1}{2^{\beta-1}-1} < 1$

Therefore $J^{\beta}(P:Q;U:V)$ is non-negative for β > 1.

It is trivial that if $p_i = q_i$, $u_i = v_i$ the $J_{\beta}(P:Q;U:V)$ vanishes conversely. let $J^{\beta}(P:Q;U:V) = 0$

Then $\frac{\sum(u_ip_i)^{\beta}(v_iq_i)^{1-\beta}}{\sum(u_ip_i)^{\beta}} + \frac{\sum(v_iq_i)^{\beta}(u_ip_i)^{1-\beta}}{\sum(v_iq_i)^{\beta}} = 2$

$$\Rightarrow \frac{\sum(u_ip_i)^{\beta}(v_iq_i)^{1-\beta}}{\sum(u_ip_i)^{\beta}} = 1 \text{ and } \frac{\sum(v_iq_i)^{\beta}(u_ip_i)^{1-\beta}}{\sum(v_iq_i)^{\beta}} = 1$$

for β ≠ 1

⇒ $p_i = q_i$ and $u_i = v_i$ for i = 1, 2, …n since β ≠ 1.
Hence $J^{\beta}(P:Q;U:V)$ vanishes iff pi = qi and ui = vi for i = 1, 2, …n.

(ii) Symmetry

$J^{\beta}((p_1, p_2, \dots p_n) : (q_1, q_2, \dots q_n); (u_1, u_2, \dots u_n): (v_1, v_2, \dots v_n))$
$= J^{\beta}((pa_1, pa_2,\dots,pa_n):(qa_1, qa_2,\dots,qa_n);(ua_1, ua_2,\dots,ua_n):(va_1, va_2,\dots,va_n))$

Where $(a_1, a_2, \dots, a_n)$ is an arbitrary permutation of (1, 2, …n).

This means that the permutation of pair-wise labeling of events does not change the value of useful J-divergence of degree β.

(iii) Expansibility

$J^{\beta}\{(p_1, p_2,\dots,p_n, 0):(q_1, q_2,\dots,q_n, 0);(u_1, u_2,\dots,u_n, u_n+1):(v_1, v_2,\dots,v_n, v_n-1))$
$= J^{\beta}((p_1, p_2,\dots,p_n : (q_1, q_2,\dots,q_n; u_1, u_2,\dots,u_n: (v_1, v_2,\dots,v_n)$

Thus incorporating an event of probability zero does not change the value of measure.

(iv) Recursivity

Let A_i, A_j be two events having probabilities pi, pj and utilities u_i, u_j respectively and β_i, β_j be another two events having probabilities q_i, q_j and utilities v_i, v_j. Then we defined utility U of the compound event $Ai \cup Aj$ and utility V of the compound event $\beta_i \cup \beta_j$ as

$$U(A_i \cup A_j) = \frac{u_ip_i + u_jp_j}{p_i + p_j} \quad \text{and}$$

$$V(B_i \cup B_j) = \frac{v_iq_i + v_jq_j}{q_i + q_j}$$

Theorem 3: Under the composition law (4.1) the following holds

$$J^{\beta}_{n+1}\{(p_1, p_2, \dots\dots, p_{n-1}, p', p''):(q_1, q_2, \dots\dots, q_{n-1}, q', q''); (u_1, u_2, \dots\dots, u_{n-1}, u', u''):(v_1, v_2, \dots\dots, v_{n-1}, v', v'')\}$$
$$= J^{\beta}_n\{(P:Q;U:V)$$
$$+(p'+p'')^{\beta-1}(q',q'')^{1-\beta} I^{\beta}_2\left[\frac{p'}{p'+p''}, \frac{p''}{p'+p''}, \frac{q'}{q'+q''}, \frac{q''}{q'+q''}, u', u'', v', v''\right]$$
$$+(q',q'')^{\beta-1}(p'+p'')^{1-\beta} I^{\beta}_2\left[\frac{q'}{q'+q''}, \frac{q''}{q'+q''}, \frac{p'}{p'+p''}, \frac{p''}{p'+p''}, u', u'', v', v''\right]$$

Where pn = p′+p″, qn = q′+q″

and $u_n = \frac{p'u' + p''u''}{p' + p''}$, $v_n = \frac{q'v' + q''v''}{q' + q''}$

Proof:

$$\text{L.H.S.} = \frac{1}{2^{\beta-1}-1}\left[\frac{\sum_{c=1}^{n-1}(u_ip_i)^{\beta}(v_iq_i)^{1-\beta}}{\sum_{c=1}^{n-1}(u_ip_i)} + \frac{\sum_{c=1}^{n-1}(v_iq_i)^{\beta}(u_ip_i)^{1-\beta}}{\sum_{c=1}^{n-1}(v_iq_i)} + \right.$$

$$\left.\frac{(v'q')^{\beta}(u'p')^{1-\beta}(v''q'')^{\beta}(u''p'')^{1-\beta}}{(v'q') + (v''q'')} - 2\right]$$

$$= J^{\beta}_n(p_1, p_2, \dots\dots, p_n : q_1, q_2, \dots\dots, q_n; u_1, u_2, \dots\dots, u_n, v_1, v_2, \dots\dots, v_n)$$
$$+\frac{1}{2^{\beta-1}-1}\left[\frac{(u'p')^{\beta}(v'q')^{1-\beta}(u''p'')^{\beta}(v''q'')^{1-\beta}}{u'p' + u''v''}\right.$$
$$\left.+\frac{(v'q')^{\beta}(u'p')^{1-\beta}(v''q'')^{\beta}(u''p'')^{1-\beta}}{u'p' + u''v''}\right]$$
$$-\frac{1}{2^{\beta-1}-1}\left[\frac{(u_np_n)^{\beta}(v_nq_n)^{1-\beta}}{u_np_n} + \frac{(v_nq_n)^{\beta}(u_np_n)^{1-\beta}}{v_nq_n}\right]$$
$$= J^{\beta}_n(P:Q;U:V)$$
$$+\frac{1}{2^{\beta-1}-1}\left[\frac{(u'p')^{\beta}(v'q')^{1-\beta} + (u''p'')^{\beta}(v''q'')^{1-\beta}}{u'p' + u''p''} - (u_np_n)^{\beta-1}(v_nq_n)^{1-\beta}\right]$$
$$+\frac{1}{2^{\beta-1}-1}\left[\frac{(v'q')^{\beta}(u'p')^{1-\beta} + (v''q'')^{\beta}(u''p'')^{1-\beta}}{v'q' + v''q''} - (v_nq_n)^{\beta-1}(u_np_n)^{1-\beta}\right]$$
$$= J^{\beta}_n(P:Q;U:V)$$
$$+\frac{1}{2^{\beta-1}-1}\Big[\Big\{(u_np_n)^{\beta-1}(v_nq_n)^{1-\beta}\Big\{(u'p'/u_np_n)^{\beta}(v'q'/v_nq_n)^{1-\beta}$$
$$+\frac{(u'p'/u_np_n)^{\beta}(v'q'/v_nq_n)^{1-\beta}}{u'p' + u''p''/u_np_n} - 1\Big\} \quad +(v_nq_n)^{\beta-1}(u_np_n)^{1-\beta}$$

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$$\left\{\frac{(v'q'/v_nq_n)^{\beta}(u'p'/u_np_n)^{1-\beta}(v''q''/v_nq_n)^{\beta}(u''p''/u_np_n)^{1-\beta}}{v'q'+v''q''/v_nq_n}-1\right\}\Bigg]$$

$$=J_n^{\beta}(P:Q;U:V)$$

$$+(p'+p'')^{\beta-1}(q'+q'')^{1-\beta}I_2^{\beta}\left[\frac{p'}{p'+p''},\frac{p''}{p'+p''},\frac{q'}{q'+q''},\frac{q''}{q'+q''},u',u'',v',v''\right]$$

$$+(q'+q'')^{\beta-1}(p'+p'')^{1-\beta}I_2^{\beta}\left[\frac{q'}{q'+q''},\frac{q''}{q'+q''},\frac{p'}{p'+p''},\frac{p''}{p'+p''},u',u'',v',v''\right]$$

(v) Permutational symmetry of distributions

For P, Q, $\in \Delta_n$ we have

$$J_n^{\beta}(P:Q;U:V)=J_n^{\beta}(Q:P;U:V)$$

(vi) Continuity

$J_n^{\beta}(P:Q;U:V)$ is a continuous function of Un variables V_{1z}, $p_1, p_2, \ldots, p_n$; $q_1, q_2, \ldots, q_n$, $u_1, u_2, \ldots, u_n$, $v_1, v_2, \ldots, v_n$.

V. CONCLUSION

In this paper we had defined $I(P:Q;U;V)=\dfrac{\sum_{i=1}^{n}u_ip_i\log(p_iu_i/q_iv_i)}{\sum_{i=1}^{n}u_ip_i}$, the non additive measure which also satisfies the non additive property. Further we defined the 'useful' relative information of order 'α'. Another non-additive useful information measure of order 'β' is also defined which also satisfies the non-additive property.

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