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An M/G/1 Retrial Queue with Negative Arrivals and Unreliable Server

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Abstract— **We study an M/G/1 retrial queue with negative arrivals and repeated attempts. This model is motivated by several practical applications. In multiprocessor computer systems, negative arrivals represent commands to delete some transactions. In Neural networks, primary and negative arrivals represent excitatory and inhibitory signals respectively. Such models can be used in relation with some problems of virus infection.**

We obtain the generating function of the number of primary customers in the system in stationary regime.

Index Terms— **Reliability, Security, Retrial Queues, Negative Arrivals.**

I. Introduction

Consider an M/G/1 retrial queue with unreliable server and two types of arrivals, regular and negative.

Regular arrivals correspond to primary customers who join the system with the intention of getting served and then leaving the system. They are treated in the normal way:

(i) If the server is free and available, an arriving primary customer (regular) begins to be served and leaves the system after service completion (if no breakdowns had occurred during his service time).

(ii) If the server is busy or out of order, the arriving customer joins the orbit and seeks service again at subsequent epochs until he finds the server free and available.

A negative arrival has the effect of remaining a customer from the orbit, if customers are present. This model is motivated by several applications. Negative arrivals can represent commands to delete some transactions as in distributed computer systems or databases, in which some operations become impossible because of locking of data or because of inconsistency. Negative and positive customers may also represent inhibitory and excitatory signals, respectively, in mathematical models of Neural Networks, while queue length represents the input potential to a neuron.

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In this paper we consider an M/G1 retrial queueing model with negative arrivals when the server is subject to random breakdowns and repairs. In section 2 we give the mathematical formulation of the model. In section 3, we study the model with constant retrial policy without breakdowns. In section 4, we give an extension to the model with breakdowns and repairs.

II. Mathematical Model

We assume that arrival times form two independent homogeneous Poisson streams with rate $\lambda > 0$ and $\delta > 0$, corresponding to primary (regular) and negative arrivals respectively.

The constant retrial policy for access to the server from the orbit can be described as follows. If the orbit is not empty at time t and the next attempt finds the server free, then a random customer (or the customer at the head of the orbit) is chosen to occupy the server after an exponentially distributed amount of time with rate $\nu > 0$.

The source of customers sends negative signals which have the effect of deleting one customer of the orbit (if any), who is selected according to some specified killing strategy. The service time of primary or secondary orders is a nonnegative random variable with distribution function $H(x)$, $H(+0) = 0$ and Laplace-Stieltjes transform $h(s), \mathrm{Re}(s) \geq 0$. Denote by h_k the k th order moment of the service time, $k \geq 1$.

Let $D_0(x)$ be the probability of a server breaking down during the interval $(t, t+x)$ given that it is idle at time t and no arrivals (primary or secondary) during the period x. Similarly, let $D_1(x)$ be the probability of a server breaking down during $(t, t+x)$ given that it is rendering service at time t. In this note we assume that $D_0(x) = 1 - e^{-\theta_0 x}$, $D_1(x) = 1 - e^{-\theta_1 x}, x \geq 0$

After the occurrence of a breakdown, a random renewal period begins in which the service is interrupted. We denote by $R_0(x)$ and $R_1(x)$ the distribution functions of the corresponding stationary renewal times, with Laplace-Stieltjes transforms $r_0(s)$ and $r_1(s)$, $\mathrm{Re}(s) \geq 0$ and first order moments $r_i, i = 0,1$. When a breakdown occurs during the service of a certain customer, then this customer enters orbit with probability $1 - p = q$ or leaves the system without completion of the service with probability p.

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III. DISTRIBUTION OF THE ORBIT SIZE

In this section we assume that the server is absolutely reliable, so that $\theta_0 \equiv 0, \theta_1 \equiv 0$. We first derive the joint distribution of the server state and the orbit size, from which we deduce the marginal distribution of the orbit size.

Define the indicator of activity of the server at time t : $C(t)=0$ if the server is free at time t ; $C(t)=1$, if it is busy at this time. At time t, let $R(t)$ be the number of customers in orbit and $N(t)$, be the number of customers in the system. Next, define the continuous random variable $\xi(t)$ as the residual time of the current service at time t , if $C(t) \neq 0$.

Consider now the following random process:

$$\zeta(t) = \{0, R(t)\} \qquad \text{If } C(t)=0,$$
$$\zeta(t) = \{1, R(t); \xi(t)\} \qquad \text{If } C(t) \neq 0 .$$

Defined on the state-space $\Theta = \{0,1\} \otimes IN \otimes \Re^+$, where IN is the set of nonnegative integers and $\Re^+$ the set of nonnegative real numbers. If the stationary regime exists, we can introduce the steady-state probabilities of the process $\{\zeta(t), t \geq 0\}$ as follows:

$$P_0(m) = \lim_{t\to\infty} P\{C(t)=0, R(t)=m\}, m \geq 0.$$
$$P_1(m,x) = \lim_{t\to\infty} P\{C(t)=1, R(t)=m; \xi(t)<x\}, \; m \geq 0, x \geq 0.$$

By considering all possible transitions from state to state over the interval $(t, t+h), h>0$, and letting $h \to 0$, we derive the system of Kolmogorov forward equations:

$$[\lambda + \delta(1-\delta_{0m}) + \nu(1-\delta_{0m})]P_0(m) =$$
$$\frac{dP_1(m,0)}{dx} + \delta P_0(m+1) \; , \tag{1}$$

$$[\lambda + \delta(1-\delta_{0m})]P_1(m,x) =$$
$$\frac{dP_1(m,x)}{dx} - \frac{dP_1(m,0)}{dx} + \lambda(1-\delta_{0m})P_1(m-1,x) +$$
$$+\lambda H(x)P_0(m) + \delta P_1(m+1,x) + \nu P_0(m+1)H(x) . \tag{2}$$

Define the partial generating functions

$$Q_0(z) = \lim_{t\to\infty} E\{z^{R(t)}; C(t)=0\} = \sum_{m=0}^{\infty} z^m P_0(m),$$
$$Q_1(z,x) = \lim_{t\to\infty} E(z^{R(t)}; C(t)=1; \xi(t)<x)$$
$$= \sum_{m=0}^{\infty} z^m P_1(m,x),$$

which converges at least in the disk $|z|<1$. From (1) and (2), we obtain the following system of equation

$$\Lambda(z)Q_0(z) = \frac{\partial Q_1(z,0)}{\partial x} + \omega(z)P_0(0) \, , \tag{3}$$

$$\Delta(z)Q_1(z,x) = \frac{\partial Q_1(z,x)}{\partial x} - \frac{\partial Q_1(z,0)}{\partial x} +$$
$$+\beta(z)Q_0(z)H(x) + \gamma(z)P_1(0,x) - \frac{\nu}{z}P_0(0)H(x) \, , \tag{4}$$

where $\Lambda(z) = \lambda + \nu + \gamma(z)$; $\quad \gamma(z) = \delta - \frac{\delta}{z}$,

$$\Delta(z) = \lambda - \lambda z + \gamma(z) \, ; \, \beta(z) = \lambda + \frac{\nu}{z} \, ,$$
$$\omega(z) = \nu + \gamma(z) \, .$$

If we introduce the Laplace transform

$$f_1(z,s) = \int_0^{\infty} e^{-sx} Q_1(z,x)dx \, , \quad |z|<1 \, , \quad \mathrm{Re}(s)>0, \text{ then}$$

the equation (4) becomes

$$s(s-\Delta(z))\, f_1(z,s) = \frac{\partial Q_1(z,0)}{\partial x} -$$
$$-\beta(z)Q_0(z)h(s) - \gamma(z)a(s) + \frac{\nu}{Z}P_0(0)h(s).$$
$$-\beta(z)Q_0(z)h(s) - \gamma(z)a(s) + \frac{\nu}{Z}P_0(0)h(s). \tag{5}$$

We denote $a(s) = \int_0^{\infty} e^{-sx} dP_1(0,x).$

By assumptions, $f_1(z,s)$ is an analytic function for each $z, |z|<1$ in the domain $\mathrm{Re}(s) \geq 0$, so for $s=\Delta(z)$ the right hand side of equation (5) must be zero and then

$$\frac{\partial Q_1(z,0)}{\partial x} = \beta(z)Q_0(z)h(\Delta(z)) + \gamma(z)a(\Lambda(z)) -$$
$$-\frac{\nu}{z}P_0(0)h(\Delta(z)). \tag{6}$$

Substituting (6) back into (5), we obtain:

$$f_1(z,s) =$$
$$= \frac{\Delta(z)[h(\Delta(z)) - h(s)] + \gamma(z)[a(\Delta(z)) - a(s)]}{s(s-\Delta(z))}, \tag{7}$$

where $\Delta(z) = \beta(z)Q_0(z) - \frac{\nu}{z}P_0(0)$.

In view of Tauberian theorems

$$Q_1(z,\infty) = \lim_{s\to 0} sf_1(z,s), |z| \leq 1,$$

Then, we obtain from (7)

$$Q_1(z,\infty) =$$
$$= \frac{\left[\beta(z)Q_0(z) - \frac{\nu}{z}P_0(0)\right][1-h(\Delta(z))] + \gamma(z)[a(0)-a(\Delta(z))]}{\Delta(z)}. \tag{8}$$

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Substituting now (6) into (5), we get

$$Q_0(z) = \frac{\gamma(z)a(\Delta(z)) + P_0(0)\left[\gamma(z) + \nu - \frac{\nu}{z}h(\Delta(z))\right]}{\gamma(z) + \lambda + \nu - \beta(z)h(\Delta(z))}. \tag{9}$$

Now, denote the generating function of the number of customers in the system in stationary regime by

$$Q(z) = \lim_{t\to\infty} E\left(z^{R(t)}\right) = Q_0(z) + zQ_1(z,\infty) .$$

Then after elementary computations, we obtain

$$Q(z) = \left[1 + (\lambda z + \nu)\frac{1 - h(\Delta(z))}{\Delta(z)}\right] \times$$

$$\times \frac{\gamma(z)a(\Delta(z)) + P_0(0)\left[\gamma(z) + \nu - \frac{\nu}{z}h(\Delta(z))\right]}{\gamma(z) + \lambda + \nu - \beta(z)h(\Delta(z))} -$$

$$- \frac{\nu P_0(0)[1 - h(\Delta(z))] - (\delta z - \delta)[a(0) - a(\Delta(z))]}{\Delta(z)}. \tag{10}$$

By using the normalization condition $Q(1) = 1$ and L'Hospital's rule, whenever necessary we have

$$Q_0(1) = \frac{\delta a(0) + P_0(0)[\delta + \nu + \nu(\delta - \lambda)h_1]}{\delta + \nu + (\lambda + \nu)(\delta - \lambda)h_1}, \tag{11}$$

$$P_0(0) = \frac{\sigma + \lambda(\delta - \lambda)h_1 - [1 - (\lambda + \nu)h_1]\delta a(0)}{[1 + (\lambda + \nu)h_1]\sigma + \nu h_1[\sigma + \lambda(\delta - \lambda)h_1]}, \tag{12}$$

where $h_1 = -h'(0)$ is the mean service time of an arbitrary customer and

$$a(0) = P_1(0,\infty) = Q_1(0,\infty),$$
$$\sigma = \delta + \nu[1 + (\delta - \lambda)h_1]. \tag{13}$$

Recall that from (7)

$$\begin{aligned} sf_1(0,s) &= s\int_0^\infty e^{-sx}Q_1(0,x)dx \\ &= s\int_0^\infty e^{-sx}(P_1(0,x))dx \\ &= s\int_0^\infty e^{-sx} \lim_{t\to\infty} P(C(t) = 1, R(t) = 0; \xi(t) < x)dx = a(s) \end{aligned} \tag{14}$$

Now, by arguments of Renewal Theory, the conditional limiting distribution of the residual service time provided that the server is busy and the orbit idle is equal to

$$\lim_{t\to\infty} P(\xi(t) < x / C(t) = 1, R(t) = 0) = .$$

$$\lim_{t\to\infty} \frac{P(\xi(t) < x, C(t) = 1, R(t) = 0)}{P(C(t) = 1, R(t) = 0)} = \frac{\int_0^x [1 - H(x)]}{\int_0^\infty [1 - H(x)]} . \tag{15}$$

Taking the Laplace transform of (15), we have that

$$a(s) = f_1(1,s) = Q_1(0,\infty)\frac{1 - h(s)}{sh_1}. \tag{16}$$

Now, the function $Q(z)$ given by formula (10) is entirely determined.

IV. THE MODEL WITH BREAKDOWNS

We consider now the case when the server is subject to random breakdowns (active and/or passive) and repairs under the assumptions of section 2, ($\theta_0 \ge 0, \theta_1 \ge 0$).

Define the indicator of the availability of the server at time t : $E(t) = 0$, if the server is available at time t and $E(t) = 1$, if it is out of order at this time. As before, let $R(t)$ be the number of customers in orbit at time t, $N(t)$ be the number of customers in the system at time t, and $C(t)$ be the indicator of business of the server at this time.

In order to embed $R(t)$ into a Markov process, let us introduce the continuous random variable $\xi(t)$ as follows. If $C(t) = 0$ and $E(t) \ne 0$, then $\xi(t)$ is the residual renewal time of the current breakdown at time t. If $C(t) \ne 0$ and $E(t) = 0$, then $\xi(t)$ is the residual time of the current service at this time.

Consider now the following random Markov process:

If $C(t) = E(t) = 0$, then $\zeta(t) = \{C(t), E(t), R(t)\}$.
If $C(t) \ne 0$ or $E(t) \ne 0$, then

$$\zeta(t) = \{C(t), E(t), R(t); \xi(t)\}.$$

This process is defined on the state-space

$$\Theta = \{0,1\} \otimes \{0,1\} \otimes IN \otimes \Re^+,$$

where IN is the set of nonnegative integers and $\Re^+$ the set of nonnegative real numbers.

If the stationary regime exists, we can introduce the steady-state probabilities of the process $\{\zeta(t); t \ge 0\}$ as follows:

$$P_{00}(m) = \lim_{t\to\infty} P\{E(t) = 0, C(t) = 0, R(t) = m\},$$

$$i, j \in \{0,1\}, m \ge 0, x \ge 0,$$

$$P_{ij}(m,x) = \lim_{t\to\infty} P\{E(t) = i, C(t) = j, R(t) = m; \xi(t) < x\}.$$

It is not difficult to show that the steady-state probabilities are solutions of the following system of ordinary differential equations

$$[\lambda + \delta(1 - \delta_{0m}) + \nu(1 - \delta_{0m}) + \theta_0]P_{00}(m) =$$

$$\frac{dP_{10}(m,x)}{dx} - \frac{dP_{01}(m,x)}{dx} + \delta P_{00}(m+1) \tag{18}$$

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$$[\lambda+\delta(1-\delta_{0m})]P_{10}(m)=\frac{dP_{10}(m,x)}{dx}-$$
$$-\frac{dP_{10}(m,0)}{dx}+\lambda(1-\delta_{0m})P_{10}(m-1,x)+$$
$$+\delta P_{10}(m+1,x)+\theta_0 P_{00}(m)R_0(x)+\theta_1\times$$
$$\times P_{01}(m-1,\infty)R_1(x)(1-q)+\theta_1 qP_{01}(m,\infty)R_1(x), \qquad (19)$$

$$[\lambda+\delta(1-\delta_{0m})+\theta_1]P_{01}(m)=\frac{dP_{01}(m,x)}{dx}-$$
$$-\frac{dP_{01}(m,0)}{dx}+\lambda(1-\delta_{0m})P_{01}(m-1,x)+\lambda\times$$
$$\times hH(x)P_{00}(m)+\delta P_{01}(m+1,x)+\nu P_{00}(m+1)H(x). \qquad (20)$$

Define the partial generating functions:

$$Q_{00}(z)=\lim_{t\to\infty}E\{z^{R(t)};E(t)=0,C(t)=0\}=$$
$$=\sum_{m=0}^{\infty}z^m P_{00}(m), \qquad (21)$$

$$Q_{ij}(z)=E\left(z^{R(t)};E(t)=i,C(t)=j,\xi(t)<x\right)$$
$$=\sum_{m=0}^{\infty}z^m P_{ij}(m,x),(i,j)=(0,1),(1,0), \qquad (22)$$

From (18)-(20), we obtain the following system of equation

$$\tilde{\Lambda}(z)Q_{00}(z)=\frac{\partial Q_{10}(z,0)}{\partial x}+\frac{\partial Q_{01}(z,0)}{\partial x}+\omega(z)P_{00}(0) \qquad (23)$$

$$\Delta(z)Q_{10}(z,x)=\frac{\partial Q_{10}(z,x)}{\partial x}-\frac{\partial Q_{10}(z,x)}{\partial x}+$$
$$+\theta_0 Q_{00}(z)R_0(x)+\theta_1[(1-q)+qz]\times$$
$$\times Q_{01}(z,\infty)R_1(x)+\gamma(z)P_{10}(0,x), \qquad (24)$$

$$\Delta(z)Q_{01}(z,x)=\frac{\partial Q_{01}(z,x)}{\partial x}-\frac{\partial Q_{01}(z,0)}{\partial x}+$$
$$+\left[\lambda+\frac{\nu}{z}\right]Q_{00}(z)H(x)++\gamma(z)P_{01}(0,x)-\frac{\nu}{z}P_{00}(0)H(x), \qquad (25)$$

where $\tilde{\Lambda}(z)=\theta_0+\Lambda(z)$, $\tilde{\Delta}(z)=\theta_1+\Delta(z)$

Define now the Laplace transform

$$f_{ij}(z,s)=\int_0^{\infty}e^{-sx}F_{ij}(z,x)dx\,,i,j=0,1,$$
$$|z|<1,\mathrm{Re}(s)\geq 0 \qquad (26)$$

Applying these operators to the system (24)-(25), we obtain

$$s(s-\Delta(z))f_{10}(z,s)=$$
$$=\frac{\partial Q_{10}(z,0)}{\partial x}-\theta_0 Q_{00}(z)r_0(s)-$$
$$-\theta_1(p+qz)Q_{01}(z,\infty)r_1(s)-\gamma(z)a_{10}(s) \qquad (27)$$

$$s(s-\theta_1-\Delta(z))f_{01}(z,s)=$$
$$=\frac{\partial Q_{01}(z,0)}{\partial x}-\left(\lambda+\frac{\nu}{z}\right)Q_{00}(z)h(s)-$$
$$-\gamma(z)a_{01}(s)+\frac{\nu}{z}P_{00}(0)h(s),$$
(28)

where $a_{ij}(s)=\int_0^{\infty}e^{-sx}dP_{ij}(0,x)$.

Since $f_{10}(z,s)$ is an analytic function in s in the domain $\mathrm{Re}(s)\geq 0$, and for $s=\Delta(z)$ the right hand side of equation (27) vanishes, so

$$\frac{\partial Q_{10}(z,0)}{\partial x}=\theta_0 Q_{00}(z)r_0(\Delta(z))+$$
$$\theta_1[p+qz]Q_{01}(z,\infty)r_1(\Delta(z))+\gamma(z)a_{10}(\Delta(z)). \qquad (29)$$

Similarly, for $s=\theta_1+\Delta(z)$, the right hand-side of (28) vanishes, and

$$\frac{\partial Q_{01}(z,0)}{\partial x}=\left(\lambda+\frac{\nu}{z}\right)Q_{00}(z)h(\tilde{\Delta}(z))+\gamma(z)a_{01}(\tilde{\Delta}(z)).$$
$$-\frac{\nu}{z}P_{00}(0)h(\tilde{\Delta}(z)), \qquad (30)$$

Substituting now (29), (30) in (27), (28) respectively, we obtain

$$s(s-\Delta(z))f_{10}(z,s)=\theta_0[r_0(\Delta(z))-r_0(s)]Q_{00}(z)+$$
$$+\theta_1(p+qz)Q_{01}(z,\infty)[r_1(\Delta(z))-r_1(s)]$$
$$+\gamma(z)[a_{10}(\Delta(z))-a_{10}(s)], \qquad (31)$$

$$s(s-\tilde{\Delta}(z))f_{01}(z,s)=\left[\lambda+\frac{\nu}{z}\right]Q_{00}(z)\left[h(\tilde{\Delta}(z))-h(s)\right]$$
$$+\gamma(z)\left[a_{01}(\tilde{\Delta}(z))-a_{01}(s)\right]$$
$$+\frac{\nu}{z}\left[h(s)-h(\tilde{\Delta}(z))\right]P_{00}(0). \qquad (32)$$

In view of Tauberian theorems, we obtain from (31) and (32):

$$Q_{10}(z,\infty)=\frac{\theta_0[1-r_0(\Delta(z))]}{\Delta(z)}Q_{00}(z)$$
$$+\frac{\theta_1(p+qz)[1-r_1(\Delta(z))]}{\Delta(z)}Q_{01}(z,\infty)$$
$$+\frac{\lambda(z)[1-a_{10}(\Delta(z))]}{\Delta(z)}, \qquad (33)$$

$$Q_{01}(z,\infty)=\frac{\left(\lambda+\frac{\nu}{z}\right)[1-h(\Delta(z)+\theta_1)]}{\Delta(z)+\theta_1}Q_{00}(z)$$
$$+\frac{\gamma(z)[1-a_{01}(\Delta(z)+\theta_1]}{\Delta(z)+\theta_1}$$
$$+\frac{\nu}{z}\frac{h(\Delta(z)+\theta_1)-1}{\Delta(z)+\theta_1}P_{00}(0). \qquad (34)$$

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Substituting now (29), (30) in (23 and taking into account (34), we have:

$$Q_{00}(z) = P_{00}(0)\gamma(z) \times \frac{\left\{a_{10}(\Delta(z)) + U_{01}(\tilde{\Delta}(z))\right\} + \left\{\gamma(z) + \nu - \frac{\nu}{z}U(\tilde{\Delta}(z))\right\}}{(\lambda+\nu) - \left(\lambda + \frac{\nu}{z}\right)U(\tilde{\Delta}(z)) + \theta_0 - \theta_0 r_0(\Delta(z))}. \quad (35)$$

Note that the function

$$U(\Delta(z)+\theta_1) = h(\Delta(z)+\theta_1) + \theta_1(p+qz)r_1(\Delta(z))\frac{1-h(\Delta(z))}{\Delta(z)+\theta_1} \quad (36)$$

has a meaningful interpretation. It is the Laplace transform of the « blocking time » i.e. the elapsed time from the instant at which a customer begins his service until the time at which the server is available for beginning the service of each other customer.

A similar interpretation may be given to the function

$$U_{01}(\Delta(z)+\theta_1) = a_{01}(\Delta(z)+\theta_1) + \theta_1(p+qz)r_1(\Delta(z))\frac{1-a_{01}(\Delta(z))}{\Delta(z)+\theta_1}. \quad (37)$$

Now denote by

$$P_m = \lim_{t\to\infty} P\{N(t)=m\} = Q_{00}(m) + P_{01}(m-1,\infty) + P_{10}(m,\infty)$$

the stationary probability that there are m customers in the system. Passing to the generating function, we have

$$Q(z) = Q_{00}(z) + Q_{10}(z) + zQ_{01}(z,\infty).$$

Using the obtained relations we have

$$\begin{aligned} Q(z) = &\left\{1 + \frac{\theta_0 - \theta_0 r_0(\Delta(z))}{\Delta(z)} + \left(z + \theta_1(p+qz)\frac{1-r_1(\Delta(z))}{\Delta(z)}\right)\right. \\ &\left.\times\left(\lambda + \frac{\nu}{z}\right)\frac{1-h(\Delta(z)+\theta_1)}{\Delta(z)+\theta_1}\right\}Q_{00}(z) + \gamma(z)\times \\ &\times\left\{\frac{1-a_{10}(\Delta(z))}{\Delta(z)} + (z+\theta_1(p+qz))\times\right. \\ &\left.\times\frac{1-r_1(\Delta(z))}{\Delta(z)}\frac{1-a_{01}(\Delta(z)+\theta_1)}{\Delta(z)+\theta_1}\right\} - \\ &-\frac{\nu}{z}\frac{1-h(\Delta(z)+\theta_1)}{\Delta(z)+\theta_1}\left(z+\theta_1(p+qz)\frac{1-r_1(\Delta(z))}{\Delta(z)}\right)P_{00}(0). \end{aligned} \quad (38)$$

By using normalization condition $Q(1)=1$, we obtain

$$P_{00}(0) = \frac{[1+\theta_0 + (1+\theta_1 r_1)(\lambda+\nu)h_1]Q_{00}(1) - 1}{\nu h_1(1+\theta_1 r_1)}$$

(39)

From (35), we have

$$Q_{00}(1) = \frac{\delta(1+U_{01}(\theta_1)) + \{\delta + \nu U(\theta_1) - \nu(\delta-\lambda)U'(\theta_1)\}P_{00}(0)}{\nu U(\theta_1) - (\lambda+\nu)(\delta-\lambda)U'(\theta_1) + \theta_0(\delta-\lambda)r_0}. \quad (40)$$

From (39) and (40), we obtain finally

$$P_{00}(0) = \frac{\delta[1+\theta_0(1+\theta_1 r_1)(\lambda+\nu)h_1][1+U_{01}(\theta_1)]}{\nu h_1(1+\theta_1 r_1)A - [\delta + \nu U(\theta_1) - \nu(\delta-\lambda)U(\theta_1)]},$$

where $r_i = -r_i'(0)$ and

$$A = \nu U(\theta_1) - (\lambda+\nu)(\delta-\lambda)U'(\theta_1) + \theta_0(\delta-\lambda)r_0.$$

In order to found the functions $a_{10}(s)$ and $a_{01}(s)$, we may use an argumentation similar to that of section 3. More precisely, the conditional limiting distribution of the residual service time provided that the server is busy and available is equal to

$$\lim_{t\to\infty} P(\xi(t) < x / E(t)=0, C(t)=1) = \frac{\int_0^x [1-H(x)]}{\int_0^\infty [1-H(x)]}$$

Taking the Laplace transform, we have that

$$a_{10}(s) = sf_{10}(1,s) = \frac{1-h(s)}{sh_1}Q_{10}(0,\infty),$$

Similarly, the Laplace transform of the limiting distribution of the residual renewal time provided that the server is down

$$\lim_{t\to\infty} P(\xi(t) < x / E(t)=1, C(t)=0)$$

is equal to

$$\frac{\theta_0}{\theta_0+\lambda+\nu}Q_{00}(1)\frac{1-r_0(s)}{s^2 r_0} + h(\theta_1)(p+qz)\frac{1-r_1(s)}{s^2 r_1}.$$

Then we have

$$a_{01}(s) = \frac{Q_{01}(0,\infty)}{s}\left\{\frac{\theta_0}{\theta_0+\lambda+\nu}Q_{00}(1)\frac{1-r_0(s)}{r_0} + h(\theta_1)(p+qz)Q_{01}(1,\infty)\frac{1-r_1(s)}{r_1}\right\}.$$

Consequently, the function $Q(z)$ given by (38) is now entirely determined.

V. Conclusion

We have provided the study of the above defined retrial queue with negative arrivals and unreliable server. It will be interesting to provide a more detailed study by considering sample path properties as in [6,7].

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