Proceedings of the World Congress on Engineering 2010 Vol I
WCE 2010, June 30 - July 2, 2010, London, U.K.

On Generalized Stochastic Differential Equation and Black−Scholes Dynamic Process

Y.Zheng

***Abstract* — In the preaent paper, we examined the generalized stochastic differential equation $dS = \mu(S,t)\ dt + \sigma(S,t)\ dX$ under the assumption that both the drift and the volatility are functions of the price of an asset S at time *t* respectively. The generalized Black-Scholes equations are derived based on the special conditions of $(\mu(S,t) - rS)/\sigma(S,t) = \omega(t,r)$. The stochastic differential equation can be used to describe the generalized Black-Scholes option pricing dynamic process with unique continuous solution. Especially, we obtain the classical Black-Scholes equation when the drift and volatility are both homogeneous functions of asset price S in the form of $\mu(S,t) = \mu S$ and $\sigma(S,t) = \sigma S$ with $\omega(t,r) = (\mu - r)/\sigma$. The mathematical technique employed in this paper also has the significance in studying some other problems.**

***Index Terms* — Black-Scholes equation, generalized stochastic differential equation,option pricing.**

Manuscript received October 20, 2009. This work was supported in part by the Crescent City Doctoral Scholarship.

Y.Zheng is with the Department of Economics and Finance, University of New Orleans, 2000 Lakeshore Drive UNiversity of New Olreans #2384, New Orleans LA 70148, USA (e-mail: taotaot@hotmail.com)

I. INTRODUCTION

Derivative instruments are contracts that promise some payment or delivery in the future contingent on the evolution of the price of some other asset, which is called the underlying. Derivative markets provide an excellent setting for mathematical analysis because the price of the underlying asset can be modeled as a diffusion process, and in this case finding the correct price for the derivative contracts gives rise to interesting mathematical problems. The most popular example is provided by option on stocks, where the option is the derivative contract and the stock is the underlying [1-2].

The pricing of options is a central problem for financial investment in derivative markets. It is both theoretical and practical problem since the use of options thrives in the financial market. An option is, like a contract, an agreement between to parties on buying of an asset by one party, and selling it by the other. In option pricing theory, the Black—Scholes equation is one of the most effective model for pricing option. The equation assumes the existence of perfect capital markets and the security prices are log normally distributed or equivalently, the log—returns are normally distributed. To these, they add the assumptions that trading in all securities in continuous and that the distribution of the rates of return is stationary [3-11].

Black-Scholes assumed that the stochastic differential equation

$$dS = \mu S\ dt + \sigma S\ dX \tag{1}$$

ISBN: 978-988-17012-9-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

as dynamics of the asset price, as the starting point and derived their partial differential equation by an arbitrage argument

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{ss} + rSV_s - rV = 0 \qquad (2)$$

This is the fundamental Black-Scholes equation. Where μ is the expected increase rate of S, σ is volatility, r is the risk-free interest rate(μ , σ , r are constants), and X is a normally distributed random variable with zero mean and $(dX)^2 \approx dt$ as $dt \to 0$.

In this paper, we examined the generalized stochastic differential equation under the assumption that both the drift and the volatility are functions of the price of an asset at time t with a view to derive the condition for the stochastic differential equation can be used to describe the generalized Black-Scholes option pricing dynamic process.

II. MODIFIED BLACK-SCHOLES EQUATION MODELS

We assume that the price S of an asset at time t ($0 < t < T$) follows the stochastic differential equation of the general form

$$dS = \mu(S,t)\, dt + \sigma(S,t)\, dX \qquad (3)$$

Where $\mu = \mu(S,t)$ and $\sigma = \sigma(S,t)$ are functions of S and t respectively. The coefficient $\mu(\cdot,\cdot)$ is called the drift and $\sigma(\cdot,\cdot)$ is called the diffusion coefficient. In financial modeling, $\sigma(\cdot,\cdot)$ is called the volatility of S.

In following, we show that under appropriate conditions on $\mu = \mu(S,t)$ and $\sigma = \sigma(S,t)$, the stochastic differential equation (3) can be used to describe the Black-Scholes dynamic process with unique continuous solution and they will be strong Markov processes. We derive two modified Black-Scholes differential equation models of which one is characterized only by volatility coefficient $\sigma = \sigma(S,t)$ but another model characterized by the coefficients of both drift $\mu = \mu(S,t)$ and volatility $\sigma = \sigma(S,t)$ of S.

Consider an option whose value is $V(S,t)$, then from Itô 's Lemma

$$\begin{aligned} dV &= \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 \\ &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}[\mu(S,t)\, dt + \sigma(S,t)\, dX] \\ &\quad + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}[\mu(S,t)\, dt + \sigma(S,t)\, dX]^2 \\ &= [\frac{\partial V}{\partial t} + \mu(S,t)\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2(S,t)\frac{\partial^2 V}{\partial S^2}\ ]dt \\ &\quad + \sigma(S,t)\frac{\partial V}{\partial S}\ dX \end{aligned} \qquad (4)$$

Model (I):

We construct a portfolio Ψ of one option and a number $-\Phi$ of assets, the value of the portfolio is:

$$\Psi = V - \Phi S$$

The change of the portfolio over dt is

$$\begin{aligned} d\Psi &= dV - \Phi dS \\ &= [\frac{\partial V}{\partial t} + \mu(S,t)\frac{\partial V}{\partial S} \\ &\quad + \frac{1}{2}\sigma^2(S,t)\frac{\partial^2 V}{\partial S^2} - \Phi\mu(S,t)]dt \\ &\quad + \sigma(S,t)[\frac{\partial V}{\partial S} - \Phi]dX \end{aligned} \qquad (5)$$

In order to cancel the random term of $d\Psi$, we choose $\Phi = \frac{\partial V}{\partial S}$, to obtain

$$d\Psi = [\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2(S,t)\frac{\partial^2 V}{\partial S^2}]dt \qquad (6)$$

ISBN: 978-988-17012-9-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

This implies that the portfolio is risk-less over time dt and the portfolio must instantaneously earn the same rate of return as other risk-free securities. The return on an amount Ψ invested in a risk-less asset over dt is $r\Psi dt$. Then

$$d\Psi = r\Psi dt = [rV - r\frac{\partial V}{\partial S}S]dt \qquad (7)$$

After rearrange Eqs.(6) and (7), we obtain a modified Black-Scholes parabolic partial differential equation for call option price $V(S,t)$ of the form :

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2(S,t)\frac{\partial^2 V}{\partial S^2} - rV = 0 \qquad (8)$$

It is seen that the model is no relation with drift coefficient and it is only affected by volatility coefficient $\sigma(S,t)$. Specially, if we let $\sigma(s,t) = \sigma S$, then Eq.(8) reduces the classical Black-Scholes equation

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - rV = 0$$

Model (II):

Consider an option whose value is $V(S,t)$, formula (2.2) indicates that

$$dV = [\frac{\partial V}{\partial t} + \mu(S,t)\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2(S,t)\frac{\partial^2 V}{\partial S^2}]dt + \sigma(S,t)\frac{\partial V}{\partial S}\,dX$$

We rewrite the above formula as:

$$dV(S,t) = h(S,t)dt + g(S,t)dX \qquad (9)$$

Consider construct a portfolio $\Psi = V_1 - \lambda V_2 + C$ of two options with maturity T_1 and T_2 respectively. In terms of Itô 's Lemma, the change of the portfolio over time internal dt is

$$d\Psi = (h_1 - \lambda h_2)dt + (g_1 - \lambda g_2)dX + rCdt \qquad (10)$$

In order to cancel the stochastic term of the risk free portfolio, we choose the parameter $\lambda = g_1 / g_2$, in terms of $C = \Psi - (V_1 - \lambda V_2)$, obtain

$$d\Psi = (h_1 - \frac{g_1}{g_2}h_2)dt + r[\Psi - (V_1 - \frac{g_1}{g_2}V_2)]dt$$

It follows from $d\Psi = r\Psi dt$, that

$$(h_1 - \frac{g_1}{g_2}h_2) + r(-V_1 + \frac{g_1}{g_2}V_2) = 0$$

This yields

$$(h_1 - rV_1)/g_1 = (h_2 - rV_2)/g_2 \qquad (11)$$

It is seen from Eq.(11) that its left-hand is independent of T_2 while the right-hand side is independent of T_1. Therefore, the both sides are independent of both T_1 and T_2. So that the quantity $(h - rV)/g$ is only depends on t and S. Let this value is $\omega(t,r)$, i.e.

$$\frac{h - rV}{g} = \omega(t,r)$$

Then

$$h = rV + \omega(t,r)g$$

In terms of Eqs.(4) and (9), we obtain finally the second modified general Black-Scholes parabolic partial differential equation for call option price $V(S,t)$ of the

ISBN: 978-988-17012-9-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

form :

$$\frac{\partial V}{\partial t}+\mu(S,t)\frac{\partial V}{\partial S}+\frac{1}{2}\sigma^2(S,t)\frac{\partial^2 V}{\partial S^2}$$
$$=rV+\omega(t,r)\sigma(S,t)\frac{\partial V}{\partial S}$$

i.e.,

$$\frac{\partial V}{\partial t}+[\mu(S,t)-\omega(t,r)\sigma(S,t)]\frac{\partial V}{\partial S}$$
$$+\frac{1}{2}\sigma^2(S,t)\frac{\partial^2 V}{\partial S^2}-rV=0 \qquad (12)$$

Eq.(12) indicates that the second model affected by both the coefficients of both drift $\mu(S,t)$ and volatility $\sigma(S,t)$ of S.

It is seen that if Eqs.(6) and (12) be used to describe the same Black-Scholes dynamic processes, then the coefficients $\mu(S,t)$ and $\sigma(S,t)$ must satisfied the following conditions.

$$(\mu(S,t)-rS)/\sigma(S,t)=\omega(t,r)$$

Especially, we can obtain the classical Black-Scholes equation when $\mu(S,t)=\mu S$ and $\sigma(S,t)=\sigma S$ with $\omega(t,r)=(\mu-r)/\sigma$.

III. Conclusions

In the preaent paper, we examined the generalized stochastic differential equation $dS=\mu(S,t)\,dt+\sigma(S,t)\,dX$ under the assumption that both the drift and the volatility are functions of the price of an asset at time t respectively. It is shown that under conditions $(\mu(S,t)-rS)/\sigma(S,t)=\omega(t,r)$, the stochastic differential equation can be used to describe the generalized Black-Scholes option pricing dynamic process with unique continuous solution. Especially, we can obtain the classical Black-Scholes equation when $\mu(S,t)=\mu S$ and $\sigma(S,t)=\sigma S$ with $\omega(t,r)=(\mu-r)/\sigma$.

References

[1] M. Baxter and A. Rennie, Financial Calculus: In introduction to derivative pricing, Cambridge University Press, Cambridge, England, 1996.

[2] R. J. Jr. Rendleman, Applied derivatives: options, futures, and swaps, Blackwell, Malden, MA, 2002..

[3] L. Arnold, Stochastic differential equations: theory and applications, John Wile & Sons, New York, 1974.

[4] F. Black and M. Scholes, The pricing of options and corporate liabilities, Journal of Political Economy, Vol.81, 1973, pp. 637-659.

[5] D. Duffie, Dynamic asset pricing theory, seconded, Princeton University Press, Princeton, N.J., 1996.

[6] Joseph Stampfli and Victor Goodman, The Mathematics of Finance: Modeling and Hedging, ,2001,Thomson Learning.

[7] L. Jodar, P. Sevilla-Peris, J.C.Cortes, R.Sala, A new direct method for solving the Black-Scholes equation, Applied Mathematics Letters, Vol.18, 2005, pp. 29-32.

[8] Marianito R. Rodrigo, Rogemar S. Mamon, An alternative approach to solving the Black-Scholes equation with time-varying parameters, Applied Mathematics Letters, Vol . 19, 2006, pp. 398-402.

[9] E. Ahmed and H.A. Abdusalam, On modified Black-Scholes equation, Chaos Solitons & Fractals, Vol.22, 2004, pp.583-587.

[10] H. Gong, A. Thavaneswaran, Recursive estimation for continuous time stochastic volatility models, Applied Mathematics Letters,Vol. 22, 2009, pp. 1770-1774.

[11] Martin Bohner, Yao Zheng, On analytical solutions of the Black–Scholes equation, Applied Mathematics Letters, Vol. 22, 2009, pp.309-313.