The Effect Of Ligament Stiffness On The Stability Of The Human Spine

C. Botha and J. Spoelstra *

Abstract—**We consider the influence of ligaments on spinal stability. The human spine has an initial curvature which should be taken into account when dealing with spinal stability. A formula for the change in subtended distance between two curves will be derived. This formula will be integrated with the variational method in order to obtain a relation for spinal stability. Stability requires two conditions from a mathematical point of view. From these two conditions a criterion for spinal stability in obtained.**

Keywords: spinal ligaments, mechanics of the spine, spinal stability

1 Background

The problem to be investigated in this study is the role and influence of the spinal ligaments on spinal stability. The importance of the spinal ligaments and the effect on the total spinal stability have not been emphasized sufficiently in the literature and is currently a big field of interest world wide.
The spinal column has both intrinsic and extrinsic stability:
Intrinsic stability results from the opposing forces of (a) ligaments restraining vertebral motion, and (b) pressure within the *nucleus pulposus* tending to push the vertebrae apart (Nixon and Brown, 1986:100).
Extrinsic stability results largely from trunk musculature and intra-abdominal pressure, which is in turn maintained by abdominal wall musculature (Nixon and Brown, 1986:100).
A study was done on the muscles acting on the L4/L5 joint of the lumbar spine (Potvin and Brown, 2005:973-980). Bergmark (1989) was the first to fully define and examine the mechanical stability of a muscular system which can be considered stable when the potential energy, V(a function of several variables), of the entire system is at a relative minimum. A stable system must always be able to return to its original state of equilibrium in response to perturbations around this original state.
In obtaining these goals, one of the problems to be solved is the change in subtended distance between two curves.

*I want to thank dr. A.A.F. van de Ven from the TU/e (Technical University of Eindhoven, Netherlands) for his help on the variational method. North West University, Potchefstroom, Mathematics & Applied Mathematics, Potchefstroom 2531 South Africa Tel/Fax: 018-299 4391/2570 Email: corne.botha@nwu.ac.za

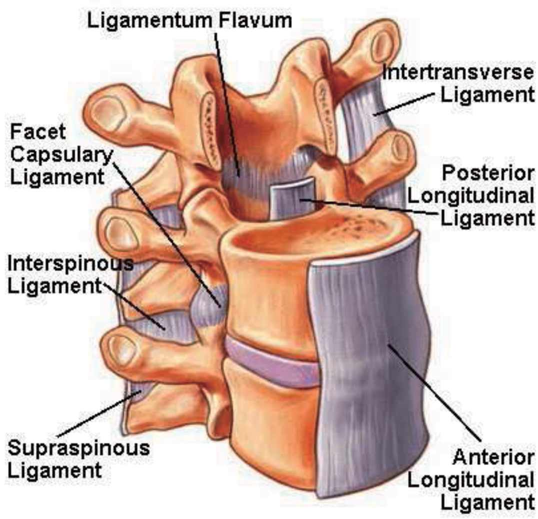

Figure 1: Spinal ligaments

From previous research we obtained formulae to determine the change in subtended distances between two lines/curves, but the requirement was that the initial line/curve on which the force is applied should be straight, and after deformation it is curved. In our study we have an initially curved rod. For this specific situation we could not find any information or formulae for determining the change in subtended distance between two curves (starting with an initially curved rod. In the next section we derive a formula for this special case.
The classical Euler buckling theory will also be used.

2 A formula to calculate the change in subtended distance between two curves

In this section we derive a formula to determine the change in distance between two curves, with the assumption and condition that the lengths of the two curves remain the same and equal, before and after deformation. The initial curve (before deformation), is described by the function $u_0(z)$. The deformed state of the curve is described by the function $u(z)$. (See Figure 2)

ISBN: 978-988-17012-9-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

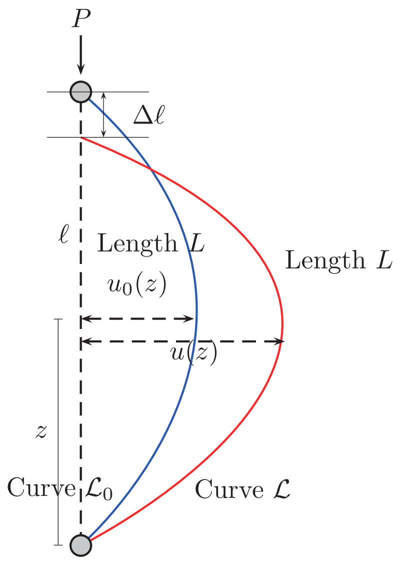

Figure 2: The buckling of an initially curved rod.

Consider the curves to be composed of i chords each with the same length L_i, subtended by δ_i on the z-axis. (See Figure 3)
The lengths of the cords L_i remain the same after deformation.

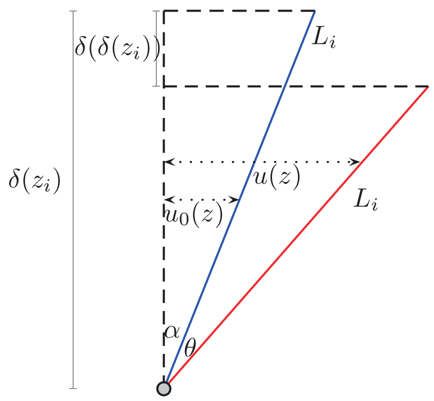

Figure 3: Composed curved rod.

From simple trigonometry we know that

$$\begin{aligned} \tan\alpha &= u_0'(z) \\ \tan(\alpha+\theta) &= u'(z) \\ \cos\alpha &= \frac{1}{1+[u_0'(z)]^2} \\ \cos(\alpha+\theta) &= \frac{1}{1+[u'(z-\delta(\delta z))]^2}. \end{aligned} \tag{1}$$

Another assumption we make is that the difference between $u'(z-\delta(\delta z))$ and $u'(z)$ is negligibly small. That means $u'(z-\delta(\delta z)) \approx u'(z)$. From this we obtain an equation to determine the difference in the lengths subtending the initial chord and the deformed chord:

$$\begin{aligned} \delta z - \delta(\delta z) &= L\cos(\alpha+\theta) \\ &= \frac{\delta z}{\cos\alpha}\cos(\alpha+\theta) \\ \delta(\delta z) &= \delta z\left[1 - \frac{\sqrt{1+[u_0'(z)]^2}}{\sqrt{1+[u'(z-\delta(\delta z))]^2}}\right] \end{aligned} \tag{2}$$

Equation 2 holds for each chord i. Adding up all the small $\delta(\delta z_i)$ we find an expression for $\Delta\ell$:

$$\begin{aligned} \Delta\ell &= \sum_{i=1}^{n}\delta(\delta z_i) \\ &= \sum_{i=1}^{n}\left[1-\sqrt{\frac{1+[u_0'(z_i)]^2}{1+[u'(z_i)]^2}}\right]\delta z_i \end{aligned} \tag{3}$$

Using the limiting process, we obtain

$$\int_0^l \left[1-\sqrt{\frac{1+[u_0'(z)]^2}{1+[u'(z)]^2}}\right]dz \tag{4}$$

3 Variational method

Having an expression for Δl, our next step is to look at the classical Euler buckling theory, but we modify the theory to hold for an initially curved rod, since the human spine has an initial curvature. We apply the minimal potential approach on an initially curved rod, considering the spine as a whole.

In Figure 2 P is the applied force (from the top); $u(z)$ is the displacement; l is the length of the rod (the length remains the same); z represents the distance from the initial point and $u_0(z)$ is the initial displacement of the curved rod.
The bending moment for a straight rod that is buckled, is given by

$$M(z) = Pu(z). \tag{5}$$

The constitutive equation is given by

$$M(z) = -EI\frac{d^2u}{dz^2} \tag{6}$$

where EI is the bending stiffness.
But the human back is not straight initially, and therefore we are considering an initially curved rod (see Figure 2). We consider the difference between the initial bending and the actual bending that occurred to obtain the actual bending. Therefore the actual bending is given by

$$u''(z) - u_0''(z) \tag{7}$$

ISBN: 978-988-17012-9-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

and now the constitutive equation becomes

$$M(z) = -EI(u''(z) - u_0''(z)). \tag{8}$$

We rewrite the constitutive equation as

$$u''(z) + \lambda^2 u(z) = u_0''(z) = Pu(z) \tag{9}$$

with

$$\lambda = \sqrt{\frac{P}{EI}} \tag{10}$$

where we have the conditions:

$$u(0) = u(l) = 0. \tag{11}$$

By definition, the potential energy in the deformed state of the material

$$U = W - A_u^{(b)} \tag{12}$$

where U is the total potential energy, W is the elastic potential energy, and $A_u^{(b)}$ refers to the work done by (given) external forces.

$$W = \frac{1}{2} EI \int_0^l (u''(z) - u_0''(z))^2 \, dz \tag{13}$$

and

$$\begin{aligned} A_u^{(b)} &= P\Delta l \\ &= P \int_0^l \left[1 - \sqrt{\frac{1 + (u_0'(z))^2}{1 + (u'(z))^2}} \, dz \right] \end{aligned}$$

Using Taylor series expansion and looking at the maximum value for $|u_0'(z)|$ and therefore $\max|\tan\theta|$, assuming $\theta < 10^\circ$ we can neglect all the fourth order and higher order terms. That gives us

$$\begin{aligned} A_u^{(b)} &\approx P \left[\ell - \int_0^l \left[1 + \frac{1}{2}((u_0'(z))^2 - (u'(z))^2) \right] \right] dz \\ &= -\frac{P}{2} \int_0^l [(u_0'(z))^2 - (u'(z))^2] \, dz \end{aligned} \tag{14}$$

Substituting (13) and (14) back into (12) we get an expression for the potential energy

$$\begin{aligned} U &= \frac{1}{2} EI \int_0^l (u''(z) - u_0''(z))^2 \, dz \\ &\quad - \frac{1}{2} P \int_0^l [(u'(z))^2 - (u_0'(z))^2] \, dz \\ &= U\{u(z)\}. \end{aligned} \tag{15}$$

For verification of the constitutive equation (9) we obtained, we now want to look at the effect that a small disturbance δ will have on the potential function $U(z)$. For this we consider the variation of U and u

$$\delta U = U(u(z) + \delta u(z)) - U(u(z)) \tag{16}$$

where $|\delta u(z)|$ is small, and $\delta u(z)$ satisfies the kinematic boundary conditions

$$\delta u(0) = \delta u(l) = 0. \tag{17}$$

By substituting (15) in (16) we obtain, after some calculation,

$$\begin{aligned} \delta U(z) &= EI \int_0^l (u''(z) - u_0''(z)) \frac{d^2}{dz^2} [\delta(u(z))] \, dz \\ &\quad - P \int_0^l \left[u'(z) \frac{d\delta u(z)}{dz} \right] dz. \end{aligned} \tag{18}$$

Using integration by parts with the constraints

$$u''(l) - u_0''(l) = u''(0) - u_0''(0) = 0. \tag{19}$$

and the kinematic boundary conditions from (17) we obtain

$$\delta U = \int_0^\ell \Big[EI(u^{iv}(z) - u_0''(z) + Pu''(z)) \Big] \delta(u(z)) \, dz \tag{20}$$

But δU should be zero for all $\delta u(z)$, because we are looking for an equilibrium state $u(z)$ which implies $\delta U = 0$ This gives us the following equation and constraints

$$EIu^{(4)}(z) + Pu''(z) = EIu_0^{(4)}(z)$$

subject to

$$\begin{aligned} &EI[u''(0) - u_0''(0)] = EI[u''(l) - u_0''(l)] = 0, \\ &u(0) = u_0(0) = u(l) = u_0(l) = 0. \end{aligned}$$

Integrating the equation twice yields

$$EIu''(z) + Pu(z) = EIu_0''(z) + A_1 z + A_2$$

but from our constraints we have that

$$A_1 = A_2 = 0.$$

Therefore

$$u''(z) + \lambda^2 u(z) = u_0''(z) \tag{21}$$

ISBN: 978-988-17012-9-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

where

$$\lambda = \sqrt{\frac{P}{EI}}.$$

Equation (21) is similar to (9), therefore we have a nonhomogeneous linear equation.
In order to solve this differential equation we need an estimation for the initial form of the spine. Looking at the literature we see a representation of the human spine in Figure 4.

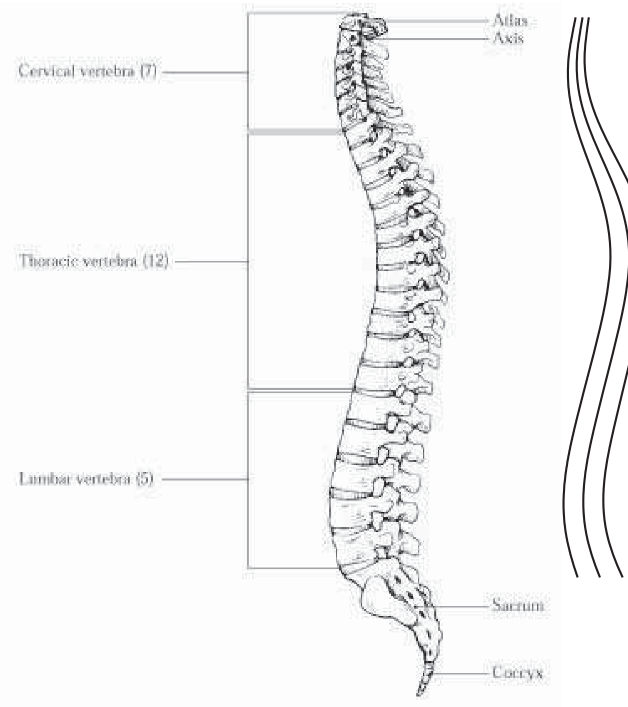

Figure 4: A function describing the initial form of the human spine

Hereby we can describe the form of the spine by fitting a curve with optimization. The anterior (front), posterior and central line from a mathematical fitting is given in Figure 4 below. The equation describing the anterior, posterior and central line of the curved spine, is given by

$$\begin{aligned} x &= 4 + 0.065y + 11.7\sin(\frac{6\pi y}{1320}) - 9.8\sin(2\frac{6\pi y}{1320}) \\ &+ -3\sin(3\frac{6\pi y}{1320}) \end{aligned} \tag{22}$$

where y, measured on a scale with the vertical distance form the top to the bottom of the curve given by the function, is 340 units, as seen on the right hand side of Figure 4.
Keeping (22) in mind we choose our initial displacement function to be

$$u_0(z) = \epsilon \sin\frac{2\pi z}{l} \quad \Longrightarrow u_0''(z) = -\epsilon\left(\frac{2\pi}{l}\right)^2 \sin\frac{\pi z}{l} \tag{23}$$

which gives us

$$u''(z) + \lambda^2 u(z) = -\epsilon\left(\frac{2\pi}{l}\right)^2 \sin\frac{2\pi z}{l} \tag{24}$$

subject to

$$u(0) = u(l) = 0.$$

Solving this nonhomogeneous linear equation, we choose the particular solution to be

$$\begin{aligned} u_p(z) &= c\sin\frac{2\pi z}{l} \\ u_p'(z) &= c\left(\frac{2\pi}{l}\right)\cos\frac{2\pi z}{l} \\ u_p''(z) &= -c\left(\frac{2\pi}{l}\right)\sin\frac{2\pi z}{l} \end{aligned}$$

Next we determine the constant c by substituting the particular solution back into the original problem:

$$-\left(\frac{2\pi}{l}\right)^2 c\sin\frac{2\pi z}{l} + \lambda^2 c\sin\frac{2\pi z}{l} = -\epsilon\left(\frac{2\pi}{l}\right)^2 \sin\frac{2\pi z}{l}$$

$$\left[-\left(\frac{2\pi}{l}\right)^2 + \lambda^2\right]c = -\epsilon\left(\frac{2\pi}{l}\right)^2$$

Thus we have

$$c = \frac{-\epsilon\left(\frac{2\pi}{l}\right)^2}{\left(\frac{2\pi}{l}\right)^2\left[-1 + \left(\frac{l}{2\pi}\right)^2\lambda^2\right]} = \frac{\epsilon}{1 - \left(\frac{l\lambda}{2\pi}\right)^2}$$

which gives a displacement function for the particular solution

$$u_p(z) = \frac{\epsilon}{1 - \left(\frac{l\lambda}{2\pi}\right)^2}\sin\frac{2\pi z}{l} \tag{25}$$

For the general homogeneous equation we have

$$u_h(z) = A\cos\lambda z + B\sin\lambda z$$

where A and B are constants to be determined. This gives us the displacement function for the nonhomogeneous linear equation

$$\begin{aligned} u(z) &= u_p(z) + u_h(z) \\ &= \frac{\epsilon}{1 - \left(\frac{\ell\lambda}{2\pi}\right)^2}\sin\frac{2\pi z}{l} + A\cos\lambda z + B\sin\lambda z \end{aligned} \tag{26}$$

ISBN: 978-988-17012-9-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

subject to

$$u(0) = u(l) = 0$$

and therefore it simplifies to

$$u(z) = \frac{\epsilon}{1 - \left(\frac{l\lambda}{2\pi}\right)^2} \sin \frac{2\pi z}{l} \tag{27}$$

Let

$$\frac{\ell\lambda}{2\pi} = \hat{\lambda},$$

then we have

$$u(z) = \frac{\epsilon}{1 - \left(\frac{l\lambda}{2\pi}\right)^2} \sin \frac{2\pi z}{l}$$

Hence, we found a formula for the displacement function $u(z)$.

4 Stability analysis

Our objective is to have a potential energy function in order to analyze the stability as stated earlier. We used the first condition for stability, that the first derivative of the potential function should be zero (mechanical equilibrium), to derive the displacement function in (27). The initial displacement function as given by (23).
Next we look at the second condition for spinal stability, using the method of variation. We want to consider the effect that a small disturbance δ will have on the potential function $U(z)$. For this we consider the variation of U and u, but now we develop our potential function U up to second order in δu:

$$\begin{aligned} U(u+\delta u) &= U(u(z)) + EI \int_0^\ell (u''(z) - u_0''(z))\delta u''(z)\, dz \\ &\quad -P \int_0^\ell u'(z)\delta u'(z)\, dz \\ &\quad +\frac{1}{2}EI \int_0^\ell [\delta u''(z)]^2 - \frac{1}{2}P \int_0^\ell [\delta u'(z)]^2\, dz \\ &= U + \delta U + \delta^2 U \end{aligned} \tag{28}$$

where $|\delta u(z)|$ is small, and $\delta u(z)$ satisfies the kinematic boundary conditions

$$\delta u(0) = \delta u(l) = 0. \tag{29}$$

The equilibrium state is stable if $\delta^2 U > 0$ for all admissible $\delta u(z)$, with $\delta u(z) \neq 0$. A complete representation for $\delta u(z)$ is thus

$$\delta u(z) = \sum_{k=1}^{\infty} a_k \sin \frac{2k\pi z}{\ell} \tag{30}$$

where a_k is arbitrary.
Then

$$\begin{aligned} \delta^2 U(u(z)) &= \frac{1}{2} EI \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \int_0^\ell \Bigg[\left(\frac{2k\pi}{\ell}\right)^2 \left(\frac{2m\pi}{\ell}\right)^2 \\ &\quad \sin\left(\frac{2k\pi}{\ell}\right) \sin\left(\frac{2m\pi}{\ell}\right) \\ &\quad - \frac{P}{EI}\left(\frac{2k\pi}{\ell}\right)\left(\frac{2m\pi}{\ell}\right) \\ &\quad \cos\left(\frac{2k\pi}{\ell}\right) \cos\left(\frac{2m\pi}{\ell}\right) \Bigg] dz \end{aligned} \tag{31}$$

Choose $a_1 = 1, a_k = 0, k \geq 2$ which is the most unfavorable situation. This yields

$$\begin{aligned} \delta^2 U(u(z)) &= \frac{1}{2} \int_0^\ell \Bigg[\left(\frac{2\pi}{\ell}\right)^4 \sin^2 \frac{2\pi z}{\ell} - \\ &\quad \left(\frac{2\pi}{\ell}\right)^2 \left(\frac{P}{EI}\right) \cos^2 \frac{2\pi z}{\ell} \Bigg] dz \\ &= \frac{1}{2}\left(\frac{2\pi}{\ell}\right)^2 \left(\frac{\ell}{2}\right) \left[\left(\frac{2\pi}{\ell}\right)^2 EI - P \right] \end{aligned} \tag{32}$$

Hence, for stability of the equilibrium state we have

$$\delta^2 U(u(z)) > 0 \iff \left(\frac{2\pi}{\ell}\right)^2 EI - P > 0 \tag{33}$$

The conclusion from (33) is:

- for

  $$P < \left(\frac{2\pi}{\ell}\right)^2 EI$$

  the equilibrium state of $u(z)$ is stable

- for

  $$P > \left(\frac{2\pi}{\ell}\right)^2 EI$$

  the equilibrium state of $u(z)$ is unstable

ISBN: 978-988-17012-9-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

5 Conclusions and future work

We obtained a sufficient result to draw conclusions from for the stability of the human spine. The relation between the force working on the spine (P) and the bending stiffness (EI) (see equation 33) can be tested by using data from the literature. This will be our next objective.

References

[1] Bergmark, A. 1989. Stability of the lumbar spine. A study in mechanical engineering. *Acta orthopaedica Scandinavica. Supplementum*, 230:1–54.

[2] Nixon, J. and Brown, H.M. 1986. Intervertebral disc mechanics: a review. *Journal of the Royal Society of Medicine*, 79:100–104.

[3] Potvin, J.R. and Brown, H.M. 2005. An equation to calculate individual muscle contributions to joint stability. *Journal of Biomechanics*, 38:973–980.

ISBN: 978-988-17012-9-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)