Proceedings of the World Congress on Engineering 2011 Vol I
WCE 2011, July 6 - 8, 2011, London, U.K.

Reduced Differential Transform Method for Solving Klein Gordon Equations

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***Abstract*— Reduced differential transform method (RDTM) is implemented for solving the linear and nonlinear Klein Gordon equations. The approximate analytical solution of the equation is calculated in the form of a series with easily computable components. Comparing the methodology with some other known techniques shows that the present approach is effective and powerful. Three test modeling problems from mathematical physics are discussed to illustrate the effectiveness and the performance of the proposed method.**

***Index Terms*— Reduced differential transform method, Variational iteration method, Klein Gordon equations.**

I. Introduction

One of the most important of all partial differential equations occurring in applied mathematics is that associated with the name of Klein–Gordon. The Klein–Gordon equation plays an important role in mathematical physics such as plasma physics, solid state physics, fluid dynamics and chemical kinetics [1-3].

We consider the Klein–Gordon equation

$$u_{tt} - u_{xx} + u + Nu(x,t) = f(x,t) \qquad (1.1)$$

subject to initial conditions

$$u(x,0) = g(x),\ u_t(x,0) = h(x) \qquad (1.2)$$

where u is a function of x and t, $Nu(x,t)$ is a nonlinear function, and $f(x,t)$ is a known analytic function. Many numerical methods were developed for this type of nonlinear partial differential equations such as the Adomian Decomposition Method (ADM) [4-7], the EXP function method [8], the Homotopy Perturbation Method (HPM) [9], the Homotopy Analysis Method (HAM) [10] and Variational Iteration Method (VIM) [11-14].

In this paper, we solve some Klein–Gordon equations by the reduced differential transform method [15-18] which is presented to overcome the demerit of complex calculation of differential transform method (DTM) [19]. The main advantage of the method is the fact that it provides its user with an analytical approximation, in many cases an exact solution, in a rapidly convergent sequence with elegantly computed terms.

The structure of this paper is organized as follows. In section 2, we begin with some basic definitions and the use of the proposed method. In section 3, we apply the reduced differential transform method to solve three test examples in order to show its ability and efficiency.

Manuscript received March 16, 2011.

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II. Traditional Differential Transform Method

A. One Dimensional Differential Transform Method

The differential transform of the function $w(x)$ is defined as follows:

$$W(k) = \frac{1}{k!}\left[\frac{d^k}{dx^k} w(x)\right]_{x=0} \qquad (2.1)$$

where $w(x)$ is the original function and $W(k)$ is the transformed function. Here $\frac{d^k}{dx^k}$ means the k the derivative with respect to x.

The differential inverse transform of $W(k)$ is defined as

$$w(x) = \sum_{k=0}^{\infty} W(k) x^k . \qquad (2.2)$$

Combining (2.1) and (2.2) we obtain

$$w(x) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{d^k}{dx^k} w(x)\right]_{x=0} x^k . \qquad (2.3)$$

From above definitions it is easy to see that the concept of differential transform is derived from Taylor series expansion. With the aid of (2.1) and (2.2) the basic mathematical operations are readily be obtained and given in Table 1.

Table 1 One-dimensional differential transformation

Functional Form	Transformed Form
$u(x) \pm v(x)$	$U(k) \pm V(k)$
$cu(x)$	$cU(k)$
$\frac{d^m u(x)}{dx^m}$	$\frac{(k+m)!}{k!} U(k+m)$
$u(x)v(x)$	$\sum_{r=0}^{k} U(r)V(k-r)$

B. Two Dimensional Differential Transform Method

Similarly, the two dimensional differential transform of the function $w(x,t)$ can be defined as follows:

$$W(k,h) = \frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^k \partial t^h} w(x,t)\right]_{(0,0)} \qquad (2.4)$$

where $w(x,t)$ is the original function and $W(k,h)$ is the transformed function. The differential inverse transform of $W(k,h)$ is

$$w(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} W(k,h) x^k t^h \ . \ 2.5)$$

ISBN: 978-988-18210-6-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Then combining equation (2.4) and (2.5) we write

$$w(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^k \partial t^h}w(x,t)\right]_{(0,0)} x^k t^h \quad (2.6)$$

Therefore we can obtain basic mathematical operations of two-dimensional differential transform as follows in Table 2.

Table 2. Two dimensional differential transformation

Functional Form	Transformed Form
$u(x,t)\pm v(x,t)$	$U(k,h)\pm V(k,h)$
$cu(x,t)$	$cU(k,h)$
$\frac{\partial u(x,t)}{\partial x}$	$(k+1)U(k+1,h)$
$\frac{\partial u(x,t)}{\partial t}$	$(h+1)U(k,h+1)$
$\frac{\partial u^{r+s}(x,t)}{\partial^r x \partial t^s}$	$\frac{(k+r)!}{k!}\frac{(h+s)!}{s!}U(k+r,h+s)$
$u(x,t)v(x,t)$	$\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)V(k-r,s)$

Now we can state our main results in the next section.

III. Reduced Differential Transform for Klein–Gordon Equations

The basic definitions and operations of reduced differential transform method [15-17] are introduced as follows:

Definition 1

If function $u(x,t)$ is analytic and differentiated continuously with respect to time t and space x in the domain of interest, then let

$$U_k(x)=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0} \quad (3.1)$$

where the t-dimensional spectrum function $U_k(x)$ is the transformed function. In this paper, the lowercase $u(x,t)$ represent the original function while the uppercase $U_k(x)$ stand for the transformed function.

Definition 2

The reduced differential transform of a sequence $\{U_k(x)\}_{k=0}^{\infty}$ is defined as follows:

$$u(x,t)=\sum_{k=0}^{\infty}U_k(x)t^k . \quad (3.2)$$

Then combining equation (3.1) and (3.2) we write

$$u(x,t)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0} t^k . \quad (3.3)$$

Some basic properties of the reduced differential transformation obtained from definitions (3.1) and (3.2) are summarized in Table 3.

The proofs of Table 3 and the basic definitions of reduced differential transform method are available in [18].

To illustrate, Consider the following Klein–Gordon equations (1.1):

$$L_t u(x,t)-L_x u(x,t)+u(x,t)+Nu(x,t)=f(x,t) \quad (3.4)$$

with initial conditions

$$u(x,0)=g(x),\ u_t(x,0)=h(x) \quad (3.5)$$

where $L_t=\frac{\partial^2}{\partial t^2}$, $L_x=\frac{\partial^2}{\partial x^2}$, $Nu(x,t)$ is a nonlinear term.

Assume that we can write $u(x,t)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}U_{n,m}t^n x^m$

Then, we get terms

$U_{0,0},U_{0,1}x,U_{0,2}x^2,...,U_{1,0}t,U_{1,1}tx,U_{1,2}tx^2,...,U_{2,0}t^2,U_{2,1}t^2x,...$

The first group can be written $t^0\sum_{m=0}^{\infty}U_{0,m}x^m$, the second group as $t^1\sum_{m=0}^{\infty}U_{1,m}x^m$, the third group as $t^2\sum_{m=0}^{\infty}U_{2,m}x^m$ etc.

Table 3. Basic operations of RDTM

Functional Form	**Transformed Form**
$u(x,t)$	$\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}$
$u(x,t)\pm v(x,t)$	$U_k(x)\pm V_k(x)$
$\alpha u(x,t)$	$\alpha U_k(x)$ (α is a constant)
$x^m t^n$	$x^m\delta(k-n)$
$x^m t^n u(x,t)$	$x^m U_{k-n}(x)$
$u(x,t)v(x,t)$	$\sum_{r=0}^{k}U_r(x)V_{k-r}(x)$
$\frac{\partial^r}{\partial t^r}u(x,t)$	$\frac{(k+r)!}{k!}U_{k+r}(x)$
$\frac{\partial}{\partial x}u(x,t)$	$\frac{\partial}{\partial x}U_k(x)$
$Nu(x,t)$	**Maple code** NF:=Nu(x,t):#Nonlinear function odr:=3:# Order u[t]:=sum(u[b]*t^b,b=0..odr): NF:=subs({Nu(x,t)=u[t]},NF): s:=expand(NF,t): dt:=unapply(s,t): for i from 0 to odr do n[i]:=((D@@i)(dt)(0)/i!): print(N[i],n[i]); # Transform Function od:

Thus

$$u(x,t)=\sum_{n=0}^{\infty}U_n(x)t^n$$

where $U_n(x)=\sum_{m=0}^{\infty}U_{n,m}x^m$ so that the double series turns to a single series.

Let the nonlinear term $Nu(x,t)$,write

$$Nu(x,t)=\sum_{n=0}^{\infty}N_n(U_1(x),...,U_n(x))t^n=\sum_{n=0}^{\infty}N_n(x)t^n .$$

Calculation of $N_n(x)$ was given in the Table 3. The approximate solution using the t partial solution is given by:

$$u(x,t)=\Phi+L_t^{-1}f(x,t)+L_t^{-1}L_xu(x,t)-L_t^{-1}Nu(x,t)-L_t^{-1}u(x,t) \quad (3.6)$$

where

$$\Phi=u(0,x)+tu_t(0,x)=g(x)+th(x)$$

and $L_t^{-1}=\int_0^t\int_o^t(.)dtdt$. Substituting for $u(x,t),Nu(x,t)$ we have

$$\sum_{n=0}^{\infty}U_n(x)t^n=g(x)+th(x)+\int_0^t\int_0^t\sum_{n=0}^{\infty}F_n(x)t^ndtdt$$
$$+\int_0^t\int_0^t\frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty}U_n(x)t^ndtdt-\int_0^t\int_0^t\sum_{n=0}^{\infty}N_n(x)t^ndtdt$$
$$-\int_0^t\int_0^t\sum_{n=0}^{\infty}U_n(x)t^ndtdt$$

We now carry out the above integrations to write

$$\sum_{n=0}^{\infty}U_n(x)t^n=g(x)+th(x)+\sum_{n=0}^{\infty}\frac{t^{n+2}}{(n+1)(n+2)}F_n(x)$$
$$+\sum_{n=0}^{\infty}\frac{t^{n+2}}{(n+1)(n+2)}\frac{\partial^2}{\partial x^2}U_n(x)-\sum_{n=0}^{\infty}\frac{t^{n+2}}{(n+1)(n+2)}N_n(x)$$
$$-\sum_{n=0}^{\infty}\frac{t^{n+2}}{(n+1)(n+2)}U_n(x)$$

Let $n\to n-2$ on the right side. Then

$$\sum_{n=0}^{\infty}U_n(x)t^n=g(x)+th(x)+\sum_{n=2}^{\infty}\frac{t^n}{(n-1)n}F_{n-2}(x)$$
$$+\sum_{n=2}^{\infty}\frac{t^n}{(n-1)n}\frac{\partial^2}{\partial x^2}U_{n-2}(x)-\sum_{n=2}^{\infty}\frac{t^n}{(n-1)n}N_{n-2}(x) \quad (3.7)$$
$$-\sum_{n=2}^{\infty}\frac{t^n}{(n-1)n}U_{n-2}(x)$$

Finally, equation coefficients of like powers of t, we derive the recursion formula for the coefficients (according to the RDTM and Table 3)

$$U_0(x)=g(x),U_1(x)=h(x) \quad (3.8)$$

and

$$\frac{(n+2)!}{n!}U_{n+2}(x)=F_n(x)+\frac{\partial^2}{\partial x^2}U_n(x)-N_n(x)-U_n(x) \quad (3.9)$$

where $U_n(x),F_n(x)$ and $N_n(x)$ are the transformations of the functions $u(x,t),f(x,t)$ and $Nu(x,t)$ respectively. Substituting (3.8) into (3.9) and by a straight forward iterative calculations, we get the following $U_n(x)$ values. Then the inverse transformation of the set of values $\{U_n(x)\}_{n=0}^{p}$ give approximation solution as,

$$\tilde{u}_p(x,t)=\sum_{n=0}^{p}U_n(x)t^n$$

where p is order of approximation solution.

Therefore, the exact solution of problem is given by

$$u(x,t)=\lim_{p\to\infty}\tilde{u}_p(x,t) .$$

IV. Applications

To show the efficiency of the new method described in the previous part, we present some examples.

A. Example 1

We first consider the homogeneous Klein–Gordon equation [11]

$$u_{tt}-u_{xx}-u=0 \quad (4.1)$$

with initial conditions:

$$u(x,0)=1+\sin(x),\ u_t(x,0)=0 \quad (4.2)$$

where $u=u(x,t)$ is a function of the variables x and t.

Then, by using the basic properties of the reduced differential transformation, we can find the transformed form of equation (4.1) as

$$\frac{(k+2)!}{k!}U_{k+2}(x)=\frac{\partial^2}{\partial x^2}U_k(x)+U_k(x) . \quad (4.3)$$

Using the initial conditions (4.2), we have

$$U_0(x)=1+\sin(x), U_1(x)=0 \quad (4.4)$$

Now, substituting (4.4) into (4.3), we obtain the following $U_k(x)$ values successively

$$U_2(x)=\frac{1}{2}, U_3(x)=0, U_4(x)=\frac{1}{24},$$
$$U_5(x)=0, U_6(x)=\frac{1}{720}$$
$$U_k(x)=\begin{cases}0, & \text{for k is odd}\\ \frac{1}{k!}, & \text{for k is even}\end{cases}$$

Finally the differential inverse transform of $U_k(x)$ gives

$$u(x,t)=\sum_{k=0}^{\infty}U_k(x)t^k=\sin(x)+\sum_{k=0,2,4,...}^{\infty}\frac{1}{k!}t^k \quad (4.5)$$

Hence the closed form of (3.5) is

$$u(x,t)=\sin(x)+\cosh(t)$$

which is the exact solutions of (4.1)–(4.2).

B. Example 2

We next consider the inhomogeneous nonlinear Klein-Gordon equation [21]

$$u_{tt}-u_{xx}+u^2=-x\cos(t)+x^2\cos^2(t) \quad (4.6)$$

with the initial conditions:

$$u(x,0)=x,\ u_t(x,0)=0 \quad (4.7)$$

Taking differential transform of (4.6) and the initial conditions (4.7) respectively, we obtain

$$\frac{(k+2)!}{k!}U_{k+2}(x)-\frac{\partial^2}{\partial x^2}U_k(x)+N_k(x)=F_k(x) \quad (4.8)$$

where $N_k(x)$ and $F_k(x)$ are transformed form of $u^2(x,t)$ and $-x\cos(t)+x^2\cos^2(t)$.

ISBN: 978-988-18210-6-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

For the easy to follow of the reader, we can give the first few nonlinear term are

$$N_0 = U_0^2(x) \qquad F_0 = -x + x^2$$
$$N_1 = 2U_0(x)U_1(x) \qquad F_1 = 0$$
$$N_2 = 2U_0(x)U_2(x) + U_1^2(x) \qquad F_2 = \frac{x}{2} - x^2$$
$$N_3 = 2U_0(x)U_3(x) + 2U_1(x)U_2(x) \qquad F_3 = 0$$
$$\vdots \qquad \vdots$$

The transformed initial conditions

$$U_0(x) = x,\ U_1(x) = 0 \tag{4.9}$$

Then substituting (4.9) into (4.8) we have

$$U_2(x) = -\frac{x}{2}, U_3(x) = 0, U_4(x) = \frac{x}{24},$$
$$U_5(x) = 0, U_6(x) = -\frac{x}{720}$$

and

$$U_k(x) = \begin{cases} 0, & \text{for k is odd} \\ \dfrac{(-1)^{k/2} x}{k!}, & \text{for k is even} \end{cases}.$$

Finally the differential inverse transform of $U_k(x)$ gives

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^k = x \sum_{k=0}^{\infty} \frac{(-1)^{k/2}}{k!} t^k = x\cos(t)$$

which is the exact solution [20].

C. Example 3

We now consider the nonlinear Klein-Gordon equation [22]

$$u_{tt} - u_{xx} + \frac{3}{4}u - \frac{3}{2}u^3 = 0 \tag{4.10}$$

with initial conditions

$$u(x,0) = -\text{sech}(x),\ u_t(x,0) = \frac{1}{2}\text{sech}(x)\tanh(x) \tag{4.11}$$

The exact solution of this problem is $u(x,t) = -\text{sech}\left(x + \frac{t}{2}\right)$. If we want to solve this equation by means of RDTM, using Table 3, we can find the transformed form of equation (4.10) as

$$\frac{(k+2)!}{k!} U_{k+2}(x) = \frac{\partial^2}{\partial x^2} U_k(x) - \frac{3}{4} U_k(x) + N_k(x) \tag{4.12}$$

where $N_k(x)$ is transformed form of $-\frac{3}{2}u^2(x,t)$ and the transformed initial conditions

$$U_0(x) = -\text{sech}(x), U_1(x) = \frac{1}{2}\text{sech}(x)\tanh(x) \tag{4.13}$$

Substituting (4.13) into (4.12), we obtain the following $U_k(x)$ values successively.

Then, the inverse transformation of the set of values $\{U_k(x)\}_{k=0}^{6}$ gives six-term approximation solution as

Therefore, the exact solution of problem is given by

$$u(x,y) = \lim_{n\to\infty} \tilde{u}_n(x,y).$$

This solution is convergent to the exact solution [22] and the same as approximate solution of the variational iteration method [11]. (see Figure 1)

$$\begin{aligned} \tilde{u}_6(x,t) = \sum_{k=0}^{6} U_k(x)t^k &= -\frac{1}{\cosh(x)} + \frac{\sinh(x)}{2\cosh(x)}t \\ &- \frac{(\cosh^2(x)-2)}{8\cosh^3(x)}t^2 + \frac{(\cosh^2(x)-6)\sinh(x)}{48\cosh^4(x)}t^3 \\ &- \frac{(\cosh^4(x)-20\cosh^2(x)+24)}{384\cosh^5(x)}t^4 \\ &+ \frac{(\cosh^4(x)-60\cosh^2(x)+120)\sinh(x)}{3840\cosh^6(x)}t^5 \\ &- \frac{(\cosh^6(x)-182\cosh^4(x)+840\cosh^2(x)-720)}{46080\cosh^7(x)}t^6 \end{aligned} \tag{4.14}$$

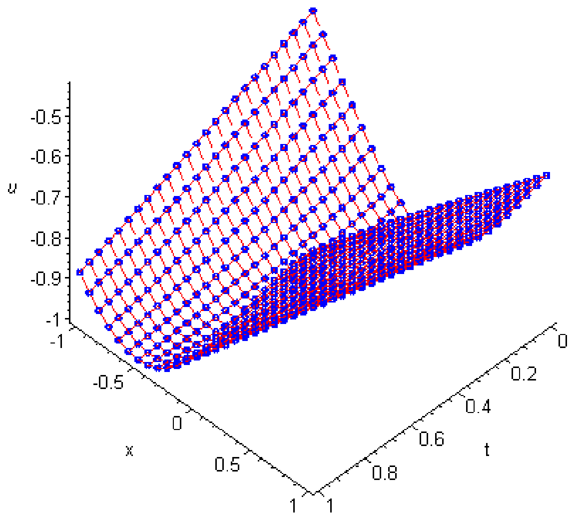

Figure 1. The comparison of the RDTM approximation and the exact solution.

Figure 1 shows the comparison of the RDTM approximation solution of order six and the exact solution $u(x,t) = -\text{sech}\left(x + \frac{t}{2}\right)$, the solid line represents the solution by the reduced differential transform method, while the circle represents the exact solution. From the figure 1, it is clearly seen that the RDTM approximation and the exact solution are in good agreement.

V. Conclusions

Analytical solutions enable researchers to study the effect of different variables or parameters on the function under study easily. Its small size of computation in comparison with the computational size required in other numerical methods, and its rapid convergence show that the method is reliable and introduces a significant improvement in solving Klein-Gordon equations over existing methods. From this study concluded that, it can be the reduced differential transform method outlined in the previous section finds quite practical approximate analytical results with less computational work.

ISBN: 978-988-18210-6-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

ACKNOWLEDGMENT

This study was supported by the Coordinatorship of Selcuk University's Scientific Research Projects (BAP).

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