# Behavior of Axial Dual-spin Spacecraft

Vladimir Aslanov

*Abstract* – **This paper presents the study of behavior of the axial dual-spin spacecraft. The spacecraft is composed of two rigid bodies: an asymmetric platform and an axisymmetric rotor aligned with the platform principal axis. Such aircraft is often called gyrostat. The paper discusses three types of gyrostats: oblate, prolate and intermediate. The rotor can rotate freely relative of the platform and an internal angular momentum is equal to zero. We consider the dynamics of gyrostats in the absence of external torque. The dynamics is described by ordinary differential equations in the Andoyer-Deprit canonical variables. The stationary solutions are found and studied their stability. Also we obtain general exact analytical solutions in terms of elliptic functions. These results can be interpreted as the development of the classical Euler case for a solid, when added to one degree of freedom - the relative rotation of bodies. Results of the study can be useful for the analysis of dynamics of dual-spin spacecrafts and for studying the chaotic behavior of the spacecrafts.**

*Index Terms* – **Dual-spin spacecraft. Axial gyrostat. Andoyer-Deprit variables. Solutions in terms of elliptic functions**

## I. Introduction

THE dynamics of a rotating body is a classic topic of study in mechanics. In the eighteenth and nineteenth centuries, several aspects of the motion of a rotating rigid body were studied by famous mathematicians of all time as Euler, Cauchy, Jacobi, Poinsot, Lagrange and Kovalevskaya. However, the study of the dynamics of rotating bodies is still very important for numerous applications such as the dynamics of satellite-gyrostat and spacecraft. A dual-spin spacecraft consists of an inertially fixed or slowly spinning platform connected to a rotor that spins relatively fast to provide attitude stability. The classical gyrostat model has a balanced axisymmetric rotor coupled to a platform that may be unbalanced or asymmetric. Both bodies are rigid and are connected by a rigid shaft about which relative spin may occur, driven by either a constant-speed. Rumyantsev [2] developed Lyapunov's ideas arising from the theory of stability of the equilibrium figure of a rotating liquid contained within a gyrostat. The Lyapunov-Rumyantsev theorem is widely used in the design of artificial satellites and liquid-filled projectiles. In [2] introduced the Andoyer-Deprit canonical variables to establish the Hamiltonian structure of an asymmetric gyrostat in the gravitational field. Kinsey et al. [3] focused upon the capture dynamics of the precession phase lock, a phenomenon that could prevent the successful despin of a dual-spin spacecraft by developing a control strategy that employed closed-loop feedback control of the motor torque when the system was near resonance. Hall [4] proposed a procedure based upon the global analysis of the rotational dynamics. Hall and Rand [5] considered spinup dynamics of classical axial gyrostat composed of an asymmetric platform and an axisymmetric rotor. They obtained averaged equations of motion for the slowly varying relative rotation of the bodies (disturbed motion) and the analytical solutions in terms of Jacobi's elliptic functions for the projections of angular momentum in the case of constant relative rotation (undisturbed motion) . Aslanov [6] obtained explicit analytical time dependences of the Andoyer–Deprit variables corresponding to heteroclinic orbits for all the phase portrait forms of undisturbed motion of axial gyrostats. Although previous works provide insight into the behavior of the axial gyrostats, equations of motion have not been reduced to the system with one degree of freedom and were not found exact analytical solutions for the Andoyer-Deprit canonical variables for the undisturbed motion. Therefore, this paper presents the study of non-linear dynamic behavior of the classical axial gyrostats with zero external torque in the undisturbed. We consider three types of the gyrostats classical [7]: oblate, prolate and intermediate.

This paper is organized as follows. In Section 1, aim of this paper is formulated. In Section 2, the motion of the axial gyrostats as two rigid bodies connected by a rigid shaft is considered. The gyrostats dynamics is described by ordinary differential equations in the Andoyer-Deprit canonical. Section 3 gives the stationary position and their stability of the gyrostats. In Section 4, a bifurcation diagram and phase portraits are constructed for three types of gyrostats: oblate, prolate and intermediate. The main features of the phase space of the unperturbed system are defined. In Section 5, the general exact analytical solutions for the undisturbed motion of three types of the gyrostats are found in terms of Jacobi's elliptic functions and elementary functions.

## II. Equations of motion

The gyrostat (P+R) consists of the balanced platform (P), axisymmetric rigid body and of the unbalanced rotor (R). The differential equations of the motion for the angular momentum variables of a rigid axial gyrostat with zero external torque may be written as [7]

$$\frac{dh_1}{dt} = \frac{I_2 - I_3}{I_2 I_3} h_2 h_3 , \tag{1}$$

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$$\frac{dh_2}{dt} = \left( \frac{I_3 - I_P}{I_3} h_1 - h_a \right) \frac{h_3}{I_P} \quad (2)$$

$$\frac{dh_3}{dt} = \left( \frac{I_P - I_2}{I_2} h_1 + h_a \right) \frac{h_2}{I_P} \quad (3)$$

where $e_i$ are principal axes of $P + R$ ($i = 1,2,3$), $h_a = I_S(\omega_S + \omega_1) = const$ is angular momentum of $R$ about $e_1$, $h_1 = I_1\omega_1 + I_S\omega_S$ is angular momentum of $P + R$ about $e_1$, $h_i = I_i\omega_i$ are angular momentums of $P + R$ about $e_i$ ($i = 2,3$), $I_i$ are moments of inertia of $P + R$ about $e_i$ ($i = 1,2,3$), $I_P = I_1 - I_S$ is moment of inertia of $P$ about $e_1$, $I_S$ is moment of inertia of $R$ about $e_1$, $t$ is time, $\omega_i$ are angular velocities of $P$ about $e_i$ ($i = 1,2,3$), $\omega_S$ is angular velocity of $R$ about $e_1$ relative to $P$.

Since there are external moments, angular momentum is conserved and first integral of the motion is

$$G = \sqrt{h_1^2 + h_2^2 + h_3^2} = const \quad (4)$$

This first integral can be to reduce the number of equations (1) - (3) by one. However it gives complicated equations of the motion. The equations of motion can be simplified by using the canonical Andoyer-Deprit variables [8]: $l, g, h, L, G, H$ . In our case the first integral (4) directly is included in the Andoyer-Deprit variables. Using the change of variables

$$h_1 = L, \quad h_2 = \sqrt{G^2 - L^2} \sin l, \quad h_3 = \sqrt{G^2 - L^2} \cos l \quad (5)$$

we obtain the equations of motion in terms of Andoyer–Deprit variables

$$\dot{l} = \frac{1}{I_P} \left[ L - h_a - \frac{1}{2} L \left( a + b + (b - a)\cos 2l \right) \right], \quad (6)$$

$$\dot{L} = \frac{1}{2I_P} (b - a) \left( G^2 - L^2 \right) \sin 2l \quad (7)$$

where $\dot{x} = dx / dt$ , $a = I_P / I_2$ , $b = I_P / I_3$ . The body axes have been chosen so that $I_2 > I_3$ (or equivalently $b > a$ ). The transformation of equations (6) – (7) to dimensionless form is obtained by scaling two momentum, time and axial torque as follows

$$s = \frac{L}{G}, \; d = \frac{h_a}{G}, \; \tau = t\frac{G}{I_p} \quad (8)$$

Derivatives with respect to $\tau$ are denoted by a derivative sign: $x' = dx / d\tau$ . Carrying out change of variables (8) leads to the to the equivalent set of uncanonical dimensionless equations

$$l' = \frac{\partial H}{\partial s} = s - d - \frac{s}{2}\left[ (a + b) + (b - a)\cos 2l \right], \quad (9)$$

$$s' = -\frac{\partial H}{\partial l} = \frac{1}{2}(b - a)\left(1 - s^2\right)\sin 2l \quad (10)$$

where $H$ is a dimensionless Hamiltonian by

$$H(l,s) = \frac{1 - s^2}{4}\left[ (a + b) + (b - a)\cos 2l \right] + \frac{s^2}{2} - sd = h = const \quad (11)$$

Solving the expression (11) with respect to the $\cos 2l$ we obtain an equation of the phase trajectory

$$\cos 2l = \frac{(a + b - 2)s^2 + 4ds + 4h - a - b}{\left(1 - s^2\right)(b - a)} \quad (12)$$

## III. STATIONARY SOLUTIONS

We define stationary solutions of equations (9) and (10). Equating to zero these equations leads to four stationary solutions. The first and second stationary solutions are described by, respectively

$$\cos(2l_*) = 1, \quad s_* = \frac{d}{1 - b}, \quad (13)$$

$$\cos(2l_*) = -1, \quad s_* = \frac{d}{1 - a} \quad (14)$$

The third and fourth stationary solutions correspond to the cases when axis of rotation gyrostat $e_1$ coincides with the angular momentum, or takes the opposite direction

$$\cos(2l_*) = \frac{2 - a - b - 2d}{b - a}, \quad s_* = 1 \quad (15)$$

$$\cos(2l_*) = \frac{2 - a - b + 2d}{b - a}, \quad s_* = -1 \quad (16)$$

We will perform the standard procedure of linearization (9) and (10) in the vicinity of a stationary position $\Delta l = l_* - l, \Delta s = s_* - s$ , then a characteristic equation can be written as

$$\begin{vmatrix} \dfrac{\partial^2 H}{\partial s \partial l} - \lambda & \dfrac{\partial^2 H}{\partial s^2} \\ -\dfrac{\partial^2 H}{\partial l^2} & -\dfrac{\partial^2 H}{\partial l \partial s} - \lambda \end{vmatrix} = 0 \quad (17)$$

This characteristic equation for first stationary solution (13) becomes

$$\lambda^2 - (b - a)(1 - b)\left(1 - s_*^2\right) = 0$$

The equilibrium position (13) is obviously stable if

$$b > 1, \; (I_P > I_3) \quad (18)$$

and unstable if

$$b < 1, \; (I_P < I_3) \quad (19)$$

For the second stationary solution (14), the characteristic equation (17) can be written as

$$\lambda^2 - (b - a)(a - 1)\left(1 - s_*^2\right) = 0$$

then the second stationary solution (19) will be stable if

$$a < 1, \; (I_P < I_2) \quad (20)$$

and unstable for

$$a > 1, \; (I_P > I_2) \quad (21)$$

Thus, the equilibrium position $l_* = n\pi, n \in \mathbb{Z}$ is stable, if the moment of inertia of the platform $I_P$ greater than the smaller moments of inertia of gyrostat $I_2$ and unstable, if $I_P$ less than $I_2$. The equilibrium position $l_* = \pi/2 + \pi n,\ n \in \mathbb{Z}$is stable if the moment of inertia of the platform is less than the larger of the transverse moments of inertia gyrostat and unstable if more than that moment of inertia.

For the third and fourth stationary solutions (15) and (16), the characteristic equation (17) can be written as

$$\lambda^2 - (b-a)^2\left(1 - \cos^2 2l_*\right) = 0$$

This equation has only real roots, so the third and fourth stationary solutions (15) and (16) are unstable.

## IV. Bifurcation Diagram

An axial gyrostat is oblate if $I_p > I_2$ or equivalently $b > a > 1$; it is prolate if $I_p < I_3$ or if $a < b < 1$; and it is intermediate if $I_3 < I_p < I_2$ or if $b > 1 > a$. We have three areas on the bifurcation diagram, which correspond to stationary solutions of (13) - (16) and the conditions that determine stability or instability of (18) - (21) as shown in Fig. 1:

1) oblate gyrostat

centers: $l_c = n\pi, n \in \mathbb{Z};\ s_c = d/(1-b)$ (22)

saddles: $l_s = \pi/2 + \pi n,\ n \in \mathbb{Z};\ s_s = d/(1-a)$ (23)

2) prolate gyrostat

saddles: $l_s = n\pi, n \in \mathbb{Z};\ s_s = d/(1-b)$ (24)

centers $l_c = \pi/2 + \pi n,\ n \in \mathbb{Z};\ s_c = d/(1-a)$ (25)

3) intermediate gyrostat

centers: $l_c = n\pi, n \in \mathbb{Z};\ s_c = d/(1-b)$ (26)

centers: $l_c = \pi/2 + \pi n,\ n \in \mathbb{Z};\ s_c = d/(1-a)$ (27)

saddles: $l_s = \pm\dfrac{1}{2}\arccos\dfrac{2-a-b+2d}{b-a};\ s_s = -1$ (28)

saddles: $l_s = \pm\dfrac{1}{2}\arccos\dfrac{2-a-b-2d}{b-a};\ s_s = 1$ (29)

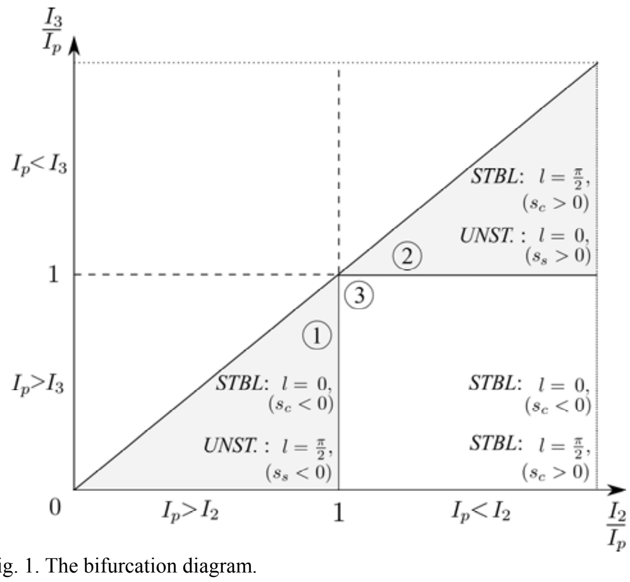


Fig. 1. The bifurcation diagram.

Examples of phase trajectories for the oblate gyrostat and the prolate gyrostat are shown in $s, l$ coordinates in Figs. 2-3.

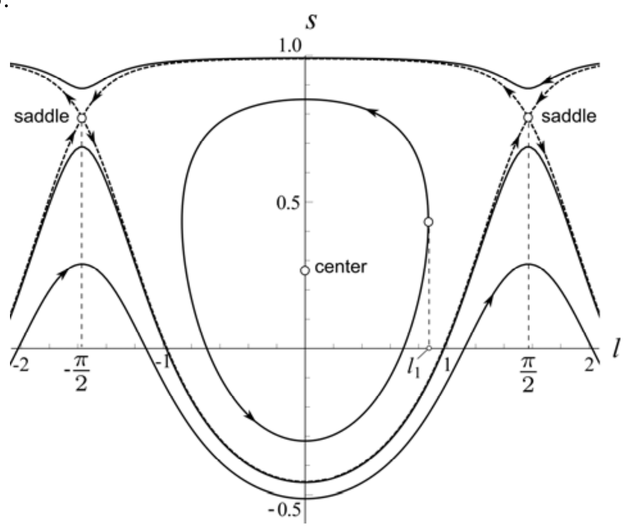


Fig. 2. Phase trajectory for the oblate gyrostat: $I_2 = 2.1\,kg\,m^2$, $I_3 = 1.6\,kg\,m^2$, $I_p = 2.5\,kg\,m^2$, $d = -0.15$, $(s_c = 0.267,\ s_s = 0.788)$

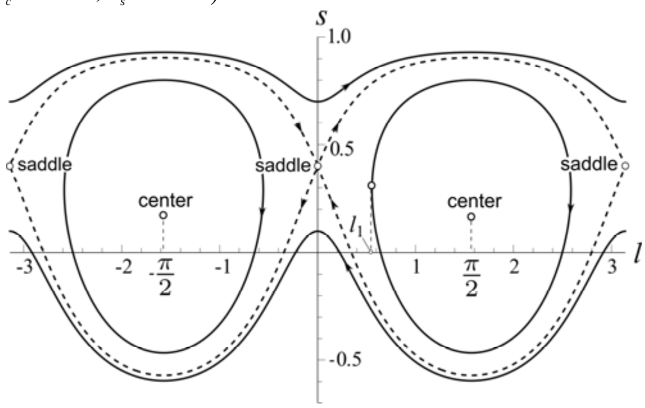


Fig. 3. Phase trajectory for the prolate gyrostat: $I_2 = 2.0\,kg\,m^2$, $I_3 = 1.6\,kg\,m^2$, $I_p = 1.4\,kg\,m^2$, $d = 0.05$ $(s_s = 0.4,\ s_c = 0.167)$.

In Fig. 4 (for the area 3 on bifurcation diagram), there are two types of separatrix, one of which contains saddles (28) and another saddles (29). In the phase space bounded by these separatrices, there is continuous motion with sequential change in the sign of dimensionless momentum $s$.

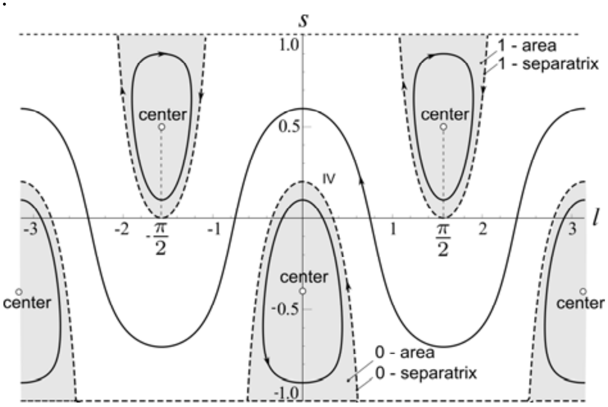


Fig. 4. Phase trajectory for the intermediate gyrostat: $I_2 = 2.0\,kg\,m^2$,

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$I_3 = 1.6\,kg\,m^2$, $I_p = 1.8\,kg\,m^2$, $d = 0.05$,
$\left(s_c = -0.4\left(l_c = 0\right),\ s_c = 0.5\left(l_c = -\pi/2,\ \pi/2\right)\right)$

## V. Integration by Quadrature the Equations of Undisturbed Motion

### A. Separation of variables

By deleting the coordinate l from the equation (10) and making use of equation (12), we obtain the new form

$$s' = \pm\frac{1}{2}\left(\left[\left(1-s^2\right)(b-a)\right]^2 - \left[(a+b-2)s^2 + 4ds + 4h - a - b\right]^2\right) = \pm\sqrt{F(s)} \quad (30)$$

where

$$F(s) = -4 f_a(s) f_b(s) \quad (31)$$

$$f_\gamma(s) = \frac{1}{2}(1-\gamma)s^2 - ds + \frac{\gamma}{2} - h,\ (\gamma = a,b) \quad (32)$$

Separating the variables in the equation (35) and integrating it we obtain

$$\tau = \pm\int\frac{ds}{\sqrt{F(s)}} + const \quad (33)$$

In a general case, this integral is an elliptic integral. Transform the integral to the Legendre normal form [10] depends on the type and location of the roots of the fourth-degree polynomial (31) as the product of two polynomials of second degree (32). We represent the roots of the quadratic equations

$$f_\gamma(s) = 0\ (\gamma = a,b)$$

as

$$s_{1,2}^{\gamma} = \frac{d \pm \sqrt{D_\gamma}}{1-\gamma},\ D_\gamma = d^2 + (2h-\gamma)(1-\gamma) \quad (34)$$

### B. Analytical solutions for the oblate gyrostat

The type of the roots (34) of the polynomial (31) depend on the value of the constant $h$. For different types of the motion of the oblate gyrostat $(b > a > 1)$ $h$ corresponds to the following condition

$$h_c > h_L > h_s > h_R \quad (35)$$

where $h_L$ and $h_R$ are correspond respectively to libration and rotation. The constant $h$ in the center (22) - $h_c$ and in the saddle (23) - $h_s$ is

$$h_c = \frac{1}{2}\left(b - \frac{d^2}{1-b}\right),\ h_s = \frac{1}{2}\left(a - \frac{d^2}{1-a}\right) \quad (36)$$

We have libration's solution if an arbitrary constant $h = h_L$ satisfy condition (35), and then the phase trajectory belongs to the closed area (Fig. 2), which includes the center (22). The roots of the polynomial (31) with (34) are given by

$$s_{1,2} = s_{1,2}^b = \frac{d \mp \sqrt{D_b}}{1-b},\ D_b = d^2 + (2h_L - b)(1-b) > 0$$

$$s_{3,4} = s_{1,2}^a = s_s \pm i s_k,\quad s_k = \frac{\sqrt{-D_a}}{1-a},$$

$$D_a = d^2 + (2h_L - a)(1-a) < 0$$

Two real roots $s_1 > s_2$ and two complex conjugate roots $s_{3,4} = s_s \pm i s_k$ take place because the integral (38) can be written as

$$\lambda\tau = \int_{s_2}^{s}\frac{ds}{\sqrt{(s_1-s)(s-s_2)(s-s_3)(s-s_4)}} \quad (37)$$

Change of variable [9]

$$\left(\tan\frac{\varphi}{2}\right)^2 = \frac{\cos\theta_1}{\cos\theta_2}\frac{s_1-s}{s-s_2} \quad (38)$$

converts the integral (37) to the Legendre normal form

$$\omega\tau = \int_{\pi}^{\varphi}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$$

where

$\tan\theta_1 = \frac{s_1 - s_s}{s_k}$, $\tan\theta_2 = \frac{s_2 - s_s}{s_k}$ ($\theta_1$, $\theta_2$ are acute angles),

$$\omega = \frac{\lambda}{\mu},\ k = \frac{\sin\theta_1 - \sin\theta_2}{2},\ \mu = -\frac{\left(\cos\theta_1\cos\theta_2\right)^{1/2}}{s_k}$$

We proceed to study the rotation when $h = h_R$ in the condition (35). The four real roots of the equation $F(s) = 0$ take place: two roots $(s_2 < s < s_1)$ correspond to the upper phase trajectories and two roots $(s_4 < s < s_3)$ the lower phase trajectories as shown in Fig.2.

$$s_{3,2} = s_{1,2}^a = \frac{d \pm \sqrt{D_a}}{1-a},\ D_a = d^2 + (2h_R - a)(1-a) > 0$$

$$s_{4,1} = s_{1,2}^b = \frac{d \pm \sqrt{D_b}}{1-b},\quad D_b = d^2 + (2h_R - b)(1-b) > 0$$

Since $D_a < D_b$ then the real roots are as follows

$$-1 < s_4 < s_3 < s_2 < s_1 < 1 \quad (39)$$

In this case the integral (38) has the form

$$\lambda\tau = \int_{s_i}^{s}\frac{ds}{\sqrt{(s_1-s)(s-s_2)(s-s_3)(s-s_4)}} \quad (40)$$

where index of the lower limit of the integral $i = 2$ for the upper phase trajectories and $i = 4$ for the lower phase trajectories. By a change of variables [9] the integral (40) can be reduced to the Legendre normal integral

$$\omega\tau = \int_0^{\varphi}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} \quad (41)$$

Then the general solutions can be written for the upper area $(s_2 < s < s_1)$

$$s = \frac{s_2 s_{31} - s_3 s_{21} sn^2(\omega\tau,k)}{s_{31} - s_{21} sn^2(\omega\tau,k)} \quad (42)$$

and for the low area $(s_4 < s < s_3)$

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$$s = \frac{s_4 s_{31} + s_1 s_{43} sn^2(\omega\tau, k)}{s_{31} + s_{41} sn^2(\omega\tau, k)} \quad (43)$$

where $sn(\omega\tau, k)$ is elliptic sine

$$\omega = \frac{\lambda}{\mu},$$

$$k^2 = \frac{(s_3 - s_4)(s_2 - s_1)}{(s_3 - s_1)(s_2 - s_4)},\ \mu = 2(s_{31}s_{24})^{-1/2},\ s_{ij} = s_j - s_i$$

### C. Analytical solutions for the prolate gyrostat

The saddles and the centers are located at the points (24), (25) for the prolate gyrostat $(a < b < 1)$ and the constant $h$ satisfies the following condition for different types of motion

$$0 < h_c < h_L < h_s < h_R \quad (44)$$

where

$$h_c = \frac{1}{2}\left(a - \frac{d^2}{1-a}\right), h_s = \frac{1}{2}\left(b - \frac{d^2}{1-b}\right) \quad (45)$$

There is a libration, when arbitrary constant $h = h_L$ satisfies to condition (44). The roots (34) of polynomial (31) are written by

$$s_{1,2} = s_{1,2}^a = \frac{d \pm \sqrt{D_a}}{1-a},\ D_a = d^2 + (2h_L - a)(1-a) > 0$$

$$s_{3,4} = s_{1,2}^b = s_s \pm i s_k,\ s_k = \frac{\sqrt{-D_b}}{1-b},$$

$$D_b = d^2 + (2h_L - b)(1-b) < 0$$

From this is clear that desired solutions coincide with the solutions (38).

In the case of rotation of the prolate gyrostat there are four real roots (34)

$$s_{1,4} = s_{1,2}^a = \frac{d \pm \sqrt{D_a}}{1-a}, D_a = d^2 + (2h_R - a)(1-a) > 0$$

$$s_{2,3} = s_{1,2}^b = \frac{d \pm \sqrt{D_b}}{1-b},\ D_b = d^2 + (2h_R - b)(1-b) > 0$$

The location of these real roots

$$-1 < s_4 < s_3 < s_2 < s_1 < 1$$

coincide with the location of the roots (39). Therefore, in this case the general solutions are the solutions (42) and (43).

### D. Analytical solutions for the intermediate gyrostat

The moments of inertia of the intermediate gyrostat determined by the following relation $I_3 < I_p < I_2$, $(b < 1 < a)$. In this case we have two groups of areas of librations, when the phase trajectories are closed: 0-areas, which includes centers (26), and 1-areas containing centers (27). These areas correspond to values of the arbitrary constant of the Hamiltonian $h_{L0}$ и $h_{L1}$. As shown in Fig. 4 the phase portrait has a single area of rotations and opened trajectories, in which $h = h_R$. The constant $h$ for the different types of motion corresponds to the following condition:

$$0 < h_{c1} < h_{L1} < h_{s1} < h_R < h_{s0} < h_{L0} < h_{c0} \quad (46)$$

where $h_{c0}$ and $h_{s0}$ correspond respectively to the centers (26) and the saddles (28)

$$h_{c0} = \frac{1}{2}\left(b - \frac{d^2}{1-b}\right),\ h_{s0} = \frac{1}{2} + d \quad (47)$$

$h_{c1}$ and $h_{s1}$ correspond to the centers (27) and the saddles (29)

$$h_{c1} = \frac{1}{2}\left(a - \frac{d^2}{1-a}\right), h_{s1} = \frac{1}{2} - d \quad (48)$$

For the librations in the 0-areas ($h = h_{L0}$), which includes centers (26), we have the following roots of the polynomial (31)

$$s_{1,4} = s_{1,2}^a = \frac{d \pm \sqrt{D_{a0}}}{1-a} \quad (49)$$

where $D_{a0} = d^2 + (2h_{L0} - a)(1-a) > 0$, and

$$s_{3,2} = s_{1,2}^b = \frac{d \pm \sqrt{D_{b0}}}{1-b} \quad (50)$$

where $D_{b0} = d^2 + (2h_{L0} - b)(1-b) > 0$.

For the 1-areas ($h = h_{L1}$), which includes centers (27), the roots of the polynomial (31) are

$$s_{4,3} = s_{1,2}^a = \frac{d \pm \sqrt{D_{a1}}}{1-a} \quad (51)$$

$D_{a1} = d^2 + (2h_{L1} - a)(1-a) > 0$,

$$s_{2,1} = s_{1,2}^b = \frac{d \pm \sqrt{D_{b1}}}{1-b} \quad (52)$$

$D_{b1} = d^2 + (2h_{L1} - b)(1-b) > 0$.

The numbering of the roots of (49) - (52) corresponds to the following sequence

$$s_4 < -1 < s_3 < s_2 < 1 < s_1 \quad (53)$$

Physical motion is realized in the range $s \in (s_3, s_2)$. In this case the integral (33) becomes

$$\lambda\tau = \int_{s_3}^{s} \frac{ds}{\sqrt{(s-s_1)(s-s_2)(s-s_3)(s-s_4)}} \quad (54)$$

where $\lambda = \sqrt{(A_\Sigma - C)(C - B_\Sigma)/(A_\Sigma B_\Sigma)}$.

The elliptic integral (54) reduces to the Legendre normal form (41) with the following change of variables [9]

$$s = \frac{s_3 s_{42} - s_4 s_{32}\sin^2\varphi}{s_{42} - s_{32}\sin^2\varphi}$$

where

$$\omega = \frac{\lambda}{\mu},\ k^2 = \frac{(s_1 - s_4)(s_2 - s_3)}{(s_1 - s_3)(s_2 - s_4)},$$

$$\mu = 2(s_{31}s_{42})^{-1/2}, s_{ij} = s_j - s_i$$

Then the general solutions can be written as

$$s = \frac{s_3 s_{42} - s_4 s_{32} sn^2(\omega\tau, k)}{s_{42} - s_{32} sn^2(\omega\tau, k)} \quad (55)$$

We consider the area of rotation (Fig. 5), bounded by 0- and 1-separatrices. Range of variation of arbitrary constant $h_R \in (h_{s1}, h_{s0})$ or, according to (47) and (48)

$$h_R \in \left(\frac{1}{2} - d, \frac{1}{2} + d\right)$$

Then the four roots (34) have the form

$$s_{4,3} = s_{1,2}^a = \frac{d \pm \sqrt{D_a}}{1-a} \quad (56)$$

where $D_a = d^2 + (2h_R - a)(1-a) > 0$

$$s_{2,1} = s_{1,2}^b = \frac{d \pm \sqrt{D_b}}{1-b} \quad (57)$$

where $D_b = d^2 + (2h_R - b)(1-b) > 0$.

Physical motion is realized in the range $s \in (s_3, s_2)$. The location of the roots (56) and (57) corresponds to (53), therefore the solution (55) describes also the rotation of the intermediate gyrostat.

## VI. Conclusion

We have shown that the equations of motion for the axial gyrostats can be reduced to two first-order ordinary differential equations for the Andoyer-Deprit canonical variables. The stationary solutions are found and studied their stability. Also we obtain the general exact analytical solutions in terms of elliptic functions. Note that an analytical description of the motion along the separatrix is easily obtained. It's enough to substitute $sn(u,1) = \tanh(u)$ in the founded solutions for the libration or the rotation. These results can be interpreted as the development of the classical Euler case for a solid, when added one degree of freedom - the relative rotation of bodies. Results of the study can be useful for the analysis of dynamics of dual-spin spacecraft and for studying the chaotic behavior of the spacecrafts.

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