Fixed Points of Quasi-Contractive Type Operators in Normed Spaces by a Three-step Iteration Process

Safeer Hussain Khan

Abstract— **We prove a strong convergence theorem to approximate fixed points of quasi-contractive operators recently introduced by Berinde by using a three-step iteration process recently given by Suantai .We prove our result in the setting of a normed space. As a corollary to this theorem we have a convergence theorem of Xu-Noor. The latter will then generalize and improve upon, among others, the corresponding result of Berinde and the theorems generalized therein.**

Index Terms— **Three-step iteration process, Quasi-contractive type operator, Fixed point, Strong convergence.**

I. INTRODUCTION

Throughout this paper, $\mathbf{N}$ denotes the set of all positive integers. Let C be a nonempty convex subset of a normed space E and $T : C \to C$ be a mapping. Let $\{a_n\},\{b_n\},\{c_n\},\{\alpha_n\}$ and $\{\beta_n\}$ be appropriately chosen sequences in $[0,1]$.

The Mann iteration process [7] is defined by the sequence $\{x_n\}$:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-a_n)x_n + a_n Tx_n, n \in \mathbf{N}. \end{cases} \quad (1)$$

The sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-a_n)x_n + a_n Ty_n, \\ y_n = (1-b_n)x_n + b_n Tx_n, \ n \in \mathbf{N} \end{cases} \quad (2)$$

is known as the Ishikawa iteration process [5].

We know that Ishikawa iteration process is a two-step process. In 2000, Noor [8] introduced a three-step iteration process to approximate solutions of variational inclusions in Hilbert spaces. Glowinski and Le Tallec [4] applied a three-step iteration process for finding the approximate solution of the elastoviscoplasticity problem, eigen value problem and liquid crystal theory.

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In 2002, Xu and Noor [11] introduced the following extension of the above Ishikawa iteration process:

$$\begin{cases} x_1 = x \in C, \\ z_n = (1-a_n)x_n + a_n Tx_n \\ y_n = (1-b_n)x_n + b_n Tz_n \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n, \ n \in \mathbf{N} \end{cases} \quad (3)$$

They used it to approximate fixed points in a uniformly convex Banach space.

Recently, Suantai [10] introduced the following iteration process:

$$\begin{cases} x_1 = x \in C, \\ z_n = (1-a_n)x_n + a_n Tx_n \\ y_n = b_n Tz_n + c_n Tx_n + (1-b_n-c_n)x_n \\ x_{n+1} = \alpha_n Ty_n + \beta_n Tz_n + (1-\alpha_n-\beta_n)x_n, \ n \in \mathbf{N} \end{cases} \quad (4)$$

He used it for weak and strong convergence of fixed points in a uniformly convex Banach space. It can be viewed as an extension of the iteration processes given by Noor [8],Glowinski and Le Tallec [4], Xu and Noor [11], Ishikawa [5] and Mann [7].

On the other hand Berinde [1] introduced a new class of quasi-contractive type operators and proved a strong convergence theorem for the Ishikawa iteration process (2) to approximate fixed points in a normed space. To appreciate this class of operators, we have to go through some definitions in a metric space (X,d) .

A mapping $T : X \to X$ is called an a-contraction if

$$d(Tx,Ty) \le a d(x,y) \text{ for all } x,y \in X, \quad (5)$$

where $0 < a < 1$.

The map T is called Kannan mapping [6] if there exists $b \in (0,\frac{1}{2})$ such that

$$d(Tx,Ty) \le b[d(x,Tx) + d(y,Ty)] \quad (6)$$

for all $x, y \in X$.

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A similar definition is due to Chatterjea [3]: there exists $c \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \le c[d(x, Ty) + d(y, Tx)] \qquad (7)$$

for all $x, y \in X$.

Combining the above three definitions, Zamfirescu [12] proved the following important result.

Theorem 1. *Let* (X, d) *be a complete metric space and* $T : X \to X$ *a mapping for which there exist real numbers* a, b *and* c *satisfying* $0 < a < 1$, $b \in (0, \frac{1}{2}), c \in (0, \frac{1}{2})$ *such that for each pair* $x, y \in X$, *at least one of the following conditions holds:*

(z_1) $d(Tx, Ty) \le ad(x, y)$ *for all* $x, y \in X$

(z_2) $d(Tx, Ty) \le b[d(x, Tx) + d(y, Ty)]$ *for all* $x, y \in X$

(z_3) $d(Tx, Ty) \le c[d(x, Ty) + d(y, Tx)]$ *for all* $x, y \in X$.

Then T *has a uique fixed point* p *and the Picard iteration* $\{x_n\}$ *defined by*

$$x_{n+1} = Tx_n, \quad n \in \mathbf{N}$$

converges to p *for any arbitrary but fixed* $x_1 \in X$.

An operator T satisfying the contractive conditions $(z_1), (z_2)$ and (z_3) in the above theorem is called Zamfirescu operator. The class of Zamfirescu operators is one of the most studied classes of quasi contractive type operators. In this class, Mann and Ishikawa iteration processes are known to converge to a unique fixed point of T.

In 2005, Berinde [1] introduced a new class of quasi-contractive type operators on a normed space X satisfying

$$\|Tx - Ty\| \le \delta\|x - y\| + L\|Tx - x\| \qquad (8)$$

for any $x, y \in X$, $0 < \delta < 1$ and $L \ge 0$.

Note that the contractive condition (5) makes T a continuous function on X while this is not the case with the contractive conditions (6)-(8) .

Berinde [1] proved that the class of operators given by (8) is wider than the class of Zamfirescu operators and used the Ishikawa iteration process (2) to approximate fixed points of this class of operators in a normed space. Actually, his main theorem is the following:

Theorem 2. *Let* C *be a nonempty closed bounded convex subset of a normed space* E. *Let* $T : C \to C$ *be an operator satisfying* (8). *Let* $\{x_n\}$ *be defined by the Ishikawa iterative process* (2). *If* $F(T) \ne \phi$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$, *then* $\{x_n\}$ *converges strongly to a fixed point of* T.

Our purpose in this paper is to prove a strong convergence therorem using the iteration process (4) for quasi-contractive type operators as given in (8) to approximate fixed points in normed spaces. On similar lines, we will also get a strong convergence theorem via the iteration process (3). It will improve and unify a number of results including Berinde's results [1, 2].

II. Main results

We now prove our main theorem as follows.

Theorem 3. *Let* C *be a nonempty closed convex subset of a normed space* E. *Let* $T : C \to C$ *be an operator satisfying* (8) *and* $F(T) \ne \phi$. *Let* $\{x_n\}$ *be defined by the iteration process* (4). *If* $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $[0,1]$, $a_n + b_n \in [0,1]$ *and* $\alpha_n + \beta_n \in [0,1]$ *such that* $\sum_{n=1}^{\infty}(\alpha_n + \beta_n) = \infty$, *then* $\{x_n\}$ *converges strongly to a fixed point of* T.

Proof Assume that $F(T) \ne \phi$. Let $w \in F(T)$. Then

$$\begin{aligned} \|x_{n+1} - w\| &= \left\| \begin{matrix} \alpha_n Ty_n + \beta_n Tz_n \\ + (1 - \alpha_n - \beta_n)x_n - w \end{matrix} \right\| \\ &\le \alpha_n \|Ty_n - w\| + \beta_n \|Tz_n - w\| \\ &\quad + (1 - \alpha_n - \beta_n)\|x_n - w\|. \qquad (9) \end{aligned}$$

Now for $x = w$ and $y = y_n$, (8) gives

$$\|Ty_n - w\| \le \delta\|y_n - w\| \qquad (10)$$

and with $x = w$ and $y = z_n$, we get

$$\|Tz_n - w\| \le \delta\|z_n - w\|. \qquad (11)$$

Also, the choice $x = w$ and $y = x_n$ provides

$$\|Tx_n - w\| \le \delta\|x_n - w\|. \qquad (12)$$

But

$$\begin{aligned} \|z_n - w\| &\le a_n\|Tx_n - w\| + (1 - a_n)\|x_n - w\| \\ &\le a_n\delta\|x_n - w\| + (1 - a_n)\|x_n - w\| \\ &\le (1 - a_n(1 - \delta))\|x_n - w\|. \qquad (13) \end{aligned}$$

Thus

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$$
\begin{aligned}
\|y_n - w\| &= \|b_n T z_n + c_n T x_n + (1-b_n-c_n)x_n - w\| \\
&\le b_n\|Tz_n - w\| + c_n\|Tx_n - w\| \\
&\quad + (1-b_n-c_n)\|x_n - w\| \\
&\le b_n\delta\|z_n - w\| + c_n\delta\|x_n - w\| \\
&\quad + (1-b_n-c_n)\|x_n - w\| \\
&\le b_n\delta(1-a_n(1-\delta))\|x_n - w\| \\
&\quad + c_n\delta\|x_n - w\| + (1-b_n-c_n)\|x_n - w\| \\
&= \begin{bmatrix} b_n\delta(1-a_n(1-\delta)) + c_n\delta \\ +(1-b_n-c_n) \end{bmatrix}\|x_n - w\|.
\end{aligned} \tag{14}
$$

Then using of (9) through (14), we obtain

$$
\begin{aligned}
\|x_{n+1} - w\| &\le \alpha_n\|Ty_n - w\| + \beta_n\|Tz_n - w\| \\
&\quad + (1-\alpha_n-\beta_n)\|x_n - w\| \\
&\le \alpha_n\delta\|y_n - w\| + \beta_n\delta\|z_n - w\| \\
&\quad + (1-\alpha_n-\beta_n)\|x_n - w\| \\
&\le \left\{ \begin{array}{l} \alpha_n\delta\begin{bmatrix} b_n\delta(1-a_n(1-\delta)) \\ + c_n\delta + (1-b_n-c_n) \end{bmatrix} \\ + \beta_n\delta(1-a_n(1-\delta)) \\ + (1-\alpha_n-\beta_n) \end{array} \right\}\|x_n - w\|
\end{aligned}
$$

Rearranging the terms, we get

$$
\begin{aligned}
\|x_{n+1} - w\| &\le \left\{ \begin{array}{l} 1-(1-\delta)\alpha_n\begin{bmatrix} 1+a_nb_n \\ +(b_n+c_n)\delta \end{bmatrix} \\ -(1-\delta)\beta_n[1+a_n\delta] \end{array} \right\}\|x_n - w\| \\
&\le [1-(1-\delta)\alpha_n - (1-\delta)\beta_n]\|x_n - w\| \\
&= [1-(1-\delta)(\alpha_n+\beta_n)]\|x_n - w\|
\end{aligned}
$$

for all $n \in \mathbf{N}$.

By induction,

$$
\begin{aligned}
\|x_{n+1} - w\| &\le \prod_{k=1}^{n}[1-(1-\delta)(\alpha_k+\beta_k)]\|x_1 - w\| \\
&= \|x_1 - w\|\exp\left(\sum_{k=1}^{n} -(1-\delta)(\alpha_k+\beta_k)\right) \\
&= \|x_1 - w\|\exp\left(-(1-\delta)\sum_{k=1}^{n}(\alpha_k+\beta_k)\right)
\end{aligned}
$$

for all $n \in \mathbf{N}$.

Since $0<\delta<1, \alpha_n, \beta_n \in [0,1]$ and $\sum_{n=1}^{\infty}(\alpha_n+\beta_n) = \infty$, we get that

$$
\limsup_{n\to\infty}\|x_n - w\| \le \limsup_{n\to\infty}\|x_1 - w\| \times \exp\left(-(1-\delta)\sum_{k=1}^{n}(\alpha_k+\beta_k)\right) \le 0.
$$

Hence $\lim_{n\to\infty}\|x_n - w\| = 0.$ Consequently $x_n \to w \in F(T).$ This completes the proof.

We also have the following theorem which actually can be seen as a corollary to the above theorem.

Theorem 4. *Let C be a nonempty closed convex subset of a normed space E. Let $T : C \to C$ be an operator satisfying (8). Let $\{x_n\}$ be defined by the iterative process (3). If $F(T) \ne \phi$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof As (4) reduces to (3) by choosing $c_n = \beta_n = 0$, the proof follows on the lines similar to the above theorem.

Theorem 3 (as well as Theorem 4) now immediately gives Theorem 1 of [1] as follows:

Corollary 1. $([1], Theorem 1)$ *Let C be a nonempty closed convex subset of a normed space E. Let $T : C \to C$ be an operator satisfying (8). Let $\{x_n\}$ be defined through the iterative process (2). If $F(T) \ne \phi$ and $\sum_{n=1}^{\infty}a_n = \infty$, then $\{x_n\}$ converges strongly to a fixed point of T.*

We also have the following corollaries.

Corollary 2. $([2], Theorem 2)$ *Let E be a Banach space and C a non-empty closed convex subset of E. Let $T : C \to C$ be a Zamfirescu operator. Let $\{x_n\}$ be defined by the Ishikawa iteration process (2) with $\sum_{n=1}^{\infty}a_n = \infty$. Then $\{x_n\}$ converges strongly to the unique fixed point of T.*

Proof The operator T has a unique fixed point by Theorem 1 and hence the result follows from Theorem 3by putting $a_n = c_n = \beta_n = 0.$

Corollary 3. $([2], Theorem 1)$ *Let E be a Banach space and C a non-empty closed convex subset of E. Let $T : C \to C$ be a Zamfirescu operator. Define $\{x_n\}$ by the Mann iteration process (1) with $\sum_{n=1}^{\infty}a_n = \infty$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof Set $a_n = b_n = c_n = \beta_n = 0$ for all $n \in \mathbf{N}$, in Theorem 3.

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Remarks (1) *The Chatterjea's and the Kannan's contractive conditions* (6) *and* (7) *are both included in the class of Zamfirescu operators and so their convergence theorems for the modified Noor iteration process* (4) *are obtained in Theorem* 3 .

(2) *Theorem* 4 *of Rhoades* [9] *in the context of Mann iteration on a uniformly convex Banach space has been extended in Corollary* 2 *to the case of an Ishikawa iteration on arbitrary Banach space and more generally by Theorem* 3 *to the case of modified three-step iteration process in normed spaces.*

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