Inclusions Between the Spaces of Strongly Almost Convergent Sequences Defined by An Orlicz Function in A Seminormed Space

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Abstract—**The concept of strong almost convergence was introduced by Maddox in 1978 [*Math. Proc. Camb. Philos. Soc.*, 83 (1978), 61-64] which has various applications. In this paper we introduce some new sequence spaces which arise from the notions of strong almost convergence and an Orlicz function in a seminormed space. A new concept of uniform statistical convergence in a seminormed space has also been introduced.**

Index Terms—**sequence space, Banach limit, strong almost convergence, Orlicz function.**

I. Introduction

The first attempts to found a theory of sequence spaces and infinite matrices in the last two decades of the nineteenth century were motivated by problems in Fourier series, power series and systems of equations with infinitely many variables. The theory of sequence spaces has wide applications to several other branches of functional analysis, e.g., the theory of functions, summability theory, the theory of locally convex spaces. nuclear spaces and matrix transformations, and is indeed well developed to have a logic of its own. One of the typical problems concerning sequence spaces is the inclusion problem (Abelian Theorems) i.e. given sequence spaces S and T determine whether S is contained in T. The main purpose of this paper is to establish certain inclusion relations between the spaces of strongly almost convergent sequences defined by an Orlicz function in a seminormed space.

Let ℓ_∞, c and c_0 be the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively, normed by $\|x\|_\infty = \sup_k |x_k|$. A sequence $x \in \ell_\infty$ is said to be almost convergent if all Banach limits of x coincide (see Banach [1]). Lorentz [12] proved that x is almost convergent to a number l if and only if

$$t_{kn} = (k+1)^{-1} \sum_{i=0}^{k} x_{i+n} \to l$$

as $k \to \infty$ uniformly in n. Let $\hat{c}$ denote the space of all almost convergent sequences. Several authors including Lorentz [12], Duran [4] and King [8] have studied almost convergent sequences. Maddox [14] has defined x to be strongly almost convergent to a number l if

$$t_{kn}(|x-l|) = (k+1)^{-1} \sum_{i=0}^{k} |x_{i+n} - l| \to 0$$

as $k \to \infty$ uniformly in n. We denote the space of all strongly almost convergent sequences by $[\hat{c}]$ and the space of all sequences which are strongly almost convergent to zero by $[\hat{c}_0]$. It is easy to sec that $[\hat{c}_0] \subset [\hat{c}] \subset \hat{c} \subset \ell_\infty$.

Das and Sahoo [3] extended the space $[\hat{c}]$ to the space $[w_1]$, where $[w_1]$ is the space defined in [3] as follows:

$$[w_1] = \left\{ x : (m+1)^{-1} \sum_{k=0}^{m} t_{kn}(|x-l|) \to 0 \right.$$
$$\left. \text{as } m \to \infty \text{ uniformly in } n, \text{ for some } l \right\}.$$

It is obvious that $[\hat{c}] \subset [w_1]$ and $[\hat{c}] - \lim x = [w_1] - \lim x = l$.

Lindenstrauss and Tzafriri [11] used the idea of an Orlicz function M to construct the sequence space ℓ_M of all sequences of scalars (x_k) such that $\sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty$ for some $\rho > 0$. The space ℓ_M equipped with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}$$

is a BK space [7, p. 300] usually called an Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(x) = x^p$, $1 \le p < \infty$. We recall [7], [11] that an Orlicz function M is a function from $[0,\infty)$ to $[0,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for all $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. Note that an Orlicz function is always unbounded.

An Orlicz function M is said to satisfy the Δ_2-condition for all values of u if there exists a constant $K > 0$ such that $M(2u) \le KM(u)$, $u \ge 0$. It is easy to see that always $K > 2$ [10]. A simple example of an Orlicz function which satisfies the Δ_2-condition for all values of u is given by $M(u) = a|u|^\alpha (\alpha > 1)$, since $M(2u) = a2^\alpha |u|^\alpha = 2^\alpha M(u)$. The Orlicz function $M(u) = e^{|u|} - |u| - 1$ does not satisfy the Δ_2-condition.

The Δ_2-condition is equivalent to the inequality $M(lu) \le K(l)M(u)$ which holds for all values of u, where l can be any number greater than unity.

It is easy to see that $M_1 + M_2$ is an Orlicz function when M_1 and M_2 are Orlicz functions, and that the function M^v (v is a positive integer), the composition of an Orlicz

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function M with itself v times, is also an Orlicz function. If an Orlicz function M satisfies the Δ_2-condition, then so does the composite Orlicz function M^v.

The following inequalities [13, p. 190] are needed throughout the paper.

Let $p = (p_i)$ be a bounded sequence of positive real numbers. If $H = \sup_i p_i$, then for any complex a_i and b_i,

$$|a_i + b_i|^{p_i} \le C(|a_i|^{p_i} + |b_i|^{p_i}), \tag{1}$$

where $C = \max(1, 2^{H\ 1})$. Also for any complex λ,

$$|\lambda|^{p_i} \le \max(1, |\lambda|^H). \tag{2}$$

Throughout the paper, X denotes a seminormed space with seminorm q, $p = (p_i)$ is a sequence of positive real numbers, M is an Orlicz function and $w(X)$ denotes the space of all X-valued sequences.

We now introduce the following X-valued sequence spaces using an Orlicz function M.

$$[\hat{c}(M,p,q)] = \Big\{x \in w(X) : (k+1)^{-1} \sum_{i=0}^{k} \left[M\left(q\left(\frac{x_{i+n} - l}{\rho}\right)\right)\right]^{p_i} \to 0 \text{ as } k \to \infty, \text{ uniformly in } n, \text{ for some } l \text{ and } \rho > 0\Big\},$$

$$[w_1(M,p,q)] = \Big\{x \in w(X) : (m+1)^{-1} \sum_{k=0}^{m} (k+1)^{-1} \sum_{i=0}^{k} \left[M\left(q\left(\frac{x_{i+n} - l}{\rho}\right)\right)\right]^{p_i} \to 0 \text{ as } m \to \infty, \text{ uniformly in } n, \text{ for some } l \text{ and } \rho > 0\Big\},$$

$$[w_\infty(M,p,q)] = \{x \in w(X) : \sup_{m,n} (m+1)^{-1} \sum_{k=0}^{m} (k+1)^{-1} \sum_{i=0}^{k} \left[M\left(q\left(\frac{x_{i+n}}{\rho}\right)\right)\right]^{p_i} < \infty \text{ for some } \rho > 0\}$$

If we put $l = 0$ in $[w_1(M,p,q)]$, then we obtain $[w_0(M,p,q)]$.

If we take $X = \mathbb{C}$, $q(x) = |x|$, $M(x) = x$ and $p_i = 1$ for all i, then $[w_1(M,p,q)] = [w_1]$ and $[w_0(M,p,q)] = [w_0]$.

We denote $[w_1(M,p,q)]$, $[w_0(M,p,q)]$ and $[w_\infty(M,p,q)]$ as $[w_1(q)]$, $[w_0(q)]$ and $[w_\infty(q)]$ when $M(x) = x$ and $p_i = 1$ for all i.

II. Inclusion Theorems

In this section we examine some algebraic properties of the sequence spaces defined above and investigate some inclusion relations between these spaces.

In order to discuss the properties of the sequence spaces, we assume that (p_i) is bounded.

Theorem 2.1. $[\hat{c}(M,p,q)]$, $[w_1(M,p,q)]$, $[w_0(M,p,q)]$ and $[w_\infty(M,p,q)]$ are linear spaces over the complex field $\mathbb{C}$.

The proof is a routine verification by using standard techniques and hence is omitted.

Theorem 2.2. Let M, M_1, M_2 be Orlicz functions. Then

(i) if there is a positive constant β such that $M(t) \le \beta t$ for all $t \ge 0$, then $[z(M_1,p,q)] \subseteq [z(M_0 M_1,p,q)]$,

(ii) $[Z(M_1,p,q)] \cap [Z(M_2,p,q)] \subseteq [Z(M_1 + M_2,p,q)]$, where $Z = w_0, w_1, w_\infty$.

Proof. We prove the theorem for $Z = w_0$ and the other cases will follow similarly.

Let $x \in [w_0(M_1,p,g)]$ so that

$$A_{mn} \equiv (m+1)^{-1} \sum_{k=0}^{m} (k+1)^{-1} \sum_{i=0}^{k} \left[M_1\left(q\left(\frac{x_{i+n}}{\rho}\right)\right)\right]^{p_i} \to 0 \text{ as } m \to \infty,$$

uniformly in n and for some $\rho > 0$. Since $M(t) \le \beta t$ for all $t \ge 0$,

$$(m+1)^{-1} \sum_{k=0}^{m} (k+1)^{-1} \sum_{i=0}^{k} [M(y_{i+n})]^{p_i} \le \max(1 + \beta^H)(m+1)^{-1} \sum_{k=0}^{m} (k+1)^{-1} \sum_{i=0}^{k} [y_{i+n}]^{p_i},$$

where $y_{i+n} = M_1\left(q\left(\frac{x_{i+n}}{\rho}\right)\right)$ and hence $x \in [w_0(MoM_1,p,q)]$.

(iii) The proof is immediate using (1). ■

Theorem 2.3. Let $h = \inf p_i > 0$, M be an Orlicz function and if $\lim_{u\to\infty} \frac{M(u/\rho)}{(u/\rho)} > 0$ for some $\rho > 0$, then $[Z(M,p,q)] \subseteq [Z(p,q)]$, where $Z = w_0, w_1$ and w_∞.

Proof. We prove the theorem for $Z = w_1$ and the other cases will follow similarly. If $\lim_{u\to\infty} \frac{M(u/\rho)}{(u/\rho)} > 0$ then there exists a number $\alpha > 0$ such that $M(u/\rho) \ge \alpha(u/\rho)$ for all $u > 0$ and some $\rho > 0$. Let $x \in [w_1(M,p,q)]$ so that $(m+1)^{-1} \sum_{k=0}^{m} (k+1)^{-1} \sum_{i=0}^{k} \left[M\left(q\left(\frac{x_{i+n} - l}{\rho}\right)\right)\right]^{p_i} \to 0$ as $m \to \infty$, uniformly in n, for some $l \in X$ and $\rho > 0$.

$$(m+1)^{-1} \sum_{k=0}^{m} (k+1)^{-1} \sum_{i=0}^{k} \left[M\left(q\left(\frac{x_{i+n} - l}{\rho}\right)\right)\right]^{p_i} \ge \max(\alpha^h, \alpha^H)(m+1)^{-1} \sum_{k=0}^{m} (k+1)^{-1} \sum_{i=0}^{k} \left[\left(q\left(\frac{x_{i+n} - l}{\rho}\right)\right)\right]^{p_i}.$$

Hence $x \in [w_1(p,q)]$. ■

Theorem 2.4. Let M be an Orlicz function which satisfies Δ_2-condition, q_1, q_2 be seminorms. Then

(i) $[w_0(M,p,q_1)] \cap [w_0(M,p,q_2)] \subseteq [w_0(M,p,q_1 + q_2)]$,

(ii) If there exists a constant $L > 1$ such that $q_2(x) \le L q_1(x)$ for all $x \in X$, then $[w_0(M,p,q_1)] \subseteq [w_0(M,p,q_2)]$.

Proof. The proof of (i) is straightforward using (1).

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(ii) Let $x \in [w_0(M,p,q_1)]$. Since M satisfies Δ_2-condition,

$$(m+1)^{-1}\sum_{k=0}^{m}(k+1)^{-1}\sum_{i=0}^{k}\left[M\left(q_2\left(\frac{x_{i+n}}{\rho}\right)\right)\right]^{p_i}$$
$$\leq (m+1)^{-1}\sum_{k=0}^{m}(k+1)^{-1}\sum_{i=0}^{k}\left[M\left(Lq_1\left(\frac{x_{i+n}}{\rho}\right)\right)\right]^{p_i}$$
$$\leq (m+1)^{-1}\sum_{k=0}^{m}(k+1)^{-1}\sum_{i=0}^{k}\left[K(L)M\left(q_1\left(\frac{x_{i+n}}{\rho}\right)\right)\right]^{p_i}$$
$$\leq \max(1,(K(L))^H)(m+1)^{-1}\sum_{k=0}^{m}(k+1)^{-1}\sum_{i=0}^{k}\left[M\left(q_1\left(\frac{x_{i+n}}{\rho}\right)\right)\right]^{p_i}$$
$$\to 0 \text{ as } m\to\infty, \text{ uniformly in } n.$$

Hence $x \in [w_0(M,p,q_2)]$. ■

Theorem 2.5. For any Orlicz function M, $[\hat{c}(M,p,q)] \subseteq [w_1(M,p,g)]$.

Proof. If $(k+1)^{-1}\sum_{i=0}^{k}\left[M\left(q\left(\frac{x_{i+n}-l}{\rho}\right)\right)\right]^{p_i} \to 0$ as $k\to\infty$ uniformly in n, for some l and $\rho>0$, then its arithmetic mean also converges to 0 as $m\to\infty$ uniformly in n. ■

Although it seems likely that $[\hat{c}(M,p,q)]$ is strictly contained in $[w_1(M,p,q)]$, we have been unable to prove it. It is therefore an open question.

Theorem 2.6. If M is an Orlicz function which satisfies Δ_2-condition and X is seminormed algebra, then $[\hat{c}_0(M,p,q)]$ is an ideal in $\ell_\infty(q)$, where $\ell_\infty(q) = \{x \in w(X) : \sup_k q(x_k) < \infty\}$.

Proof. Let $x \in [\hat{c}_0(M,p,q)]$ and $y \in l_\infty(q)$. We show that $xy \in [\hat{c}_0(M,p,q)]$. Since $y \in l_\infty(q)$, there exists a constant $T>1$ such that $q(y_k) < T$ for all k. Since M satisfies Δ_2-condition, there exists a constant $K>1$ such that

$$(k+1)^{-1}\sum_{i=0}^{k}\left[M\left(q\left(\frac{x_{i+n}y_{i+n}}{\rho}\right)\right)\right]^{p_i}$$
$$\leq (k+1)^{-1}\sum_{i=0}^{k}\left[M\left(q\left(\frac{x_{i+n}}{\rho}\right)q(y_{i+n})\right)\right]^{p_i}$$
$$\leq (k+1)^{-1}\sum_{i=0}^{k}\left[M\left(Tq\left(\frac{x_{i+n}}{\rho}\right)\right)\right]^{p_i}$$
$$\leq \max(1,(K(L))^H)(k+1)^{-1}\sum_{i=0}^{k}\left[M\left(q\left(\frac{x_{i+n}}{\rho}\right)\right)\right]^{p_i}$$
$$\to 0 \text{ as } k\to\infty, \text{ uniformly in } n, \text{ for some } \rho>0.$$

Thus $xy \in [\hat{c}_0(M,p,q)]$ and the proof is complete. ■

III. The Space of Multipliers of $[w_\infty(M,p,q)]$

Suppose (X,q) is seminormed algebra. We define $S([w_\infty(M,p,g)])$, the space of multipliers of $[w_\infty(M_\infty,p,q)]$, as

$$S([w_\infty(M,p,q)]) = \{a \in w(X) : (a_k x_k) \in [w_\infty(M,p,q)] \text{ for all } x=(x_k)\in[w_\infty(M,p,q)]\}.$$

Theorem 3.1. If M satisfies the Δ_2-condition, then $l_\infty(q) \subseteq S([w_\infty(M,p,q)])$.

Proof. Let $a=(a_k)\in l_\infty(q)$, $T=\sup_k q(a_k)$ and $x \in [w_\infty(M,p,q)]$. Then $\sup_{m,n}(m+1)^{-1}\sum_{k=0}^{m}(k+1)^{-1}\sum_{i=0}^{k}\left[M\left(q\left(\frac{x_{i+n}}{\rho}\right)\right)\right]^{p_i} < \infty$ for some $\rho>0$. Since M satisfies the Δ_2-condition, there exists a constant $K_1>1$ such that

$$(m+1)^{-1}\sum_{k=0}^{m}(k+1)^{-1}\sum_{i=0}^{k}\left[M\left(q\left(\frac{a_{i+n}x_{i+n}}{\rho}\right)\right)\right]^{p_i}$$
$$\leq (m+1)^{-1}\sum_{k=0}^{m}(k+1)^{-1}\sum_{i=0}^{k}\left[M\left(q(a_{i+n})q\left(\frac{x_{i+n}}{\rho}\right)q(y_{i+n})\right)\right]^{p_i}$$
$$\leq (m+1)^{-1}\sum_{k=0}^{m}(k+1)^{-1}\sum_{i=0}^{k}\left[M\left((1+[T])q\left(\frac{x_{i+n}}{\rho}\right)\right)\right]^{p_i}$$
$$\leq (K_1(1+[T]))^H(m+1)^{-1}\sum_{k=0}^{m}(k+1)^{-1}\sum_{i=0}^{k}\left[M\left(q\left(\frac{x_{i+n}}{\rho}\right)\right)\right]^{p_i}$$

where $[T]$ denotes the integer part of T.

Hence $a \in S([w_\infty(M,p,q)])$. ■

IV. Composite Space $[w_1(M^v,p,q)]$ Using Composite Orlicz Function M^v

Taking Orlicz function M^v instead of M in the space $[w_1(M,p,q)]$, we define the composite space $[w_1(M^v,p,q)]$ as follows:

Definition 4.1. For a fixed natural number v, we define

$$[w_1(M^v,p,q)] = \Big\{x\in w(X) : (m+1)^{-1}\sum_{k=0}^{m}(k+1)^{-1}\sum_{i=0}^{k}\left[M^v\left(q\left(\frac{x_{i+n}-l}{\rho}\right)\right)\right]^{p_i} \to 0 \text{ as } m\to\infty, \text{ uniformly in } n, \text{ for some } l\in X \text{ and } \rho>0\Big\}.$$

Theorem 4.2. For any Orlicz function M and $v\in\mathbb{N}$,

(i) $[w_1(M^v,p,q)] \subseteq [w_1(p,q)]$ if there exists a constant $\alpha\geq 1$ such that $M(t)\geq \alpha t$ for all $t\geq 0$.

(ii) Suppose there exists a constant β, $0<\beta\leq 1$ such that $M(t)\leq\beta t$ for all $t\geq 0$ and let $n,v\in\mathbb{N}$ be

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such that $n < v$, then $[w_1(p,q)] \subseteq [w_1(M^n,p,q)] \subseteq [w_1(M^v,p,q)]$.

Proof. (i) Since $M(t) \geq \alpha t$ for all $t \geq 0$ and M is non-decreasing and convex, we have $M^v(t) \geq \alpha^v t$ for each $v \in \mathbb{N}$. Let $x \in [w_1(M^v,p,q)]$. Using (2), we have

$$(m+1)^{-1}\sum_{k=0}^{m}(k+1)^{-1}\sum_{i=0}^{k}[q(x_{i+n}-l)]^{p_i} \leq \max(1,\rho^H)\max(1,\alpha^{-vH})(m+1)^{-1} \sum_{k=0}^{m}(k+1)^{-1}\sum_{i=0}^{k}\left[M^v\left(q\left(\frac{x_{i+n}-l}{\rho}\right)\right)\right]^{p_i}$$

and hence $x \in [w_1(p,q)]$.

(ii) Since $M(t) \leq \beta t$ for all $t \geq 0$ and M is non-decreasing and convex, we have $M^n(t) \leq \beta^n t$ for each $n \in \mathbb{N}$. The first inclusion is easily proved by using (2). To prove the second inclusion, suppose that $v - n = s$ and let $x \in [w_1(M^n,p,q)]$. Again, using (2), we have

$$(m+1)^{-1}\sum_{k=0}^{m}(k+1)^{-1}\sum_{i=0}^{k}\left[M^v\left(q\left(\frac{x_{i+n}-l}{\rho}\right)\right)\right]^{p_i} \leq \max(1,\beta^{sH})(m+1)^{-1}\sum_{k=0}^{m}(k+1)^{-1} \sum_{i=0}^{k}\left[M^n\left(q\left(\frac{x_{i+n}-l}{\rho}\right)\right)\right]^{p_i} \to 0 \text{ as } m\to\infty, \text{ uniformly in } n$$

and hence $x \in [w_1(M^v,p,q)]$. ■

Example 4.3. $M_1(t) = e^t - 1 \geq t$ and $M_2(t) = \dfrac{t^2}{1+t}$ for all $t \geq 0$ satisfy the conditions given in Theorem 4.2(i),(ii) respectively.

V. Inclusion relation with uniform statistical convergence

The idea of statistical convergence was introduced by Fast [5] and studied by various authors (e.g., [2], [6], [9], [15], [21]). Although statistical convergence was introduced over nearly the last fifty years, it has become an active area of research in recent years. This concept has been applied in various areas such as approximation theory [18], turnpike theory [16], [17], [20] etc.

Definition 5.1 ([6]). The number sequence $x = (x_k)$ is said to be statistically convergent to l if for each $\epsilon > 0$,

$$\lim_{n\to\infty} n^{-1}|\{k \leq n : |x_k - l| \geq \epsilon\}| = 0,$$

where the vertical bars denote the cardinality of the set which they enclose. The set of all statistically convergent sequences is denoted by S.

The concept, of uniform statistical convergence was introduced by Pehlivan and Fisher [19] as follows:

Definition 5.2 ([19]). The number sequence x is uniformly statistically convergent to 0 provided that for each $\epsilon > 0$,

$$\lim_{k\to\infty}(k+1)^{-1}\max_{n\geq 0}|\{0 \leq i \leq k : |x_{i+n}| \geq \epsilon\}| = 0\,.$$

The set of all uniformly statistically null sequences is denoted by S_{u_0}.

We now introduce the following definition:

Definition 5.3. A sequence $x = (x_k)$ in X is said to be uniformly statistically convergent to $l \in X$ if for each $\epsilon > 0$,

$$\lim_{k\to\infty}(k+1)^{-1}\max_{n\geq 0}|\{0 \leq i \leq k : q(x_{i+n} - l) \geq \epsilon\}| = 0\,.$$

We shall denote the set of all uniformly statistically convergent sequences by $S_u(q)$.

Theorem 5.4. For any Orlicz function M, $S_u(q) \cap l_\infty(q) \subseteq [w_1(M,p,q)] \cap l_\infty(q)$.

Proof. Let $x \in S_u(q) \cap l_\infty(q)$. Let $y_{i+n} = x_{i+n} - l$. Since $x \in l_\infty(q)$, there exists $K > 0$ such that $M\left(q\left(\frac{y_{i+n}}{\rho}\right)\right) \leq K$ for every $\rho > 0$ and for all y_{i+n}. Then for given $\epsilon > 0$,

$$\begin{aligned}
&(m+1)^{-1}\sum_{k=0}^{m}(k+1)^{-1}\sum_{i=0}^{k}\left[M\left(q\left(\frac{y_{i+n}}{\rho}\right)\right)\right]^{p_i} \\
&= (m+1)^{-1}\sum_{k=0}^{m}(k+1)^{-1} \sum_{i=0,q(y_{i+n})\geq\epsilon}^{k}\left[M\left(q\left(\frac{y_{i+n}}{\rho}\right)\right)\right]^{p_i} \\
&\quad + (m+1)^{-1}\sum_{k=0}^{m}(k+1)^{-1} \sum_{i=0,q(y_{i+n})<\epsilon}^{k}\left[M\left(q\left(\frac{y_{i+n}}{\rho}\right)\right)\right]^{p_i} \\
&\leq K^H(m+1)^{-1}\sum_{k=0}^{m}(k+1)^{-1} \max_{n\geq 0}|\{0 \leq i \leq k : q(y_{i+n}) \geq \epsilon\}| \\
&\quad + \max\left(1, \left(M\left(\frac{\epsilon}{\rho}\right)\right)^H\right).
\end{aligned}$$

We now select N_ϵ such that

$$(k+1)^{-1}|\{0 \leq i \leq k : q(y_{i+n})| \geq \epsilon\}| < \frac{\epsilon}{K^H}$$

for each n and $k > N_\epsilon$. Now for $k > N_\epsilon$,

$$(M+1)^{-1}\sum_{k=0}^{m}(k+1)^{-1}\sum_{i=0}^{k}\left[M\left(q\left(\frac{y_{i+n}}{\rho}\right)\right)\right]^{p_i} < K^H\frac{\epsilon}{K^H} + \max\left(1, \left(M\left(\frac{\epsilon}{\rho}\right)\right)^H\right).$$

Hence $x \in [w_1(M,p,q)] \cap l_\infty(q)$. ■

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