Study of Solutions of a Nonlinear Fractional Partial Differential Equation

M.M. Rodrigues *

Abstract— **In this paper, we determine the invariants of scaling transformations for a nonlinear partial differential equation of fractional order. For its invariant solutions an ordinary differential equation of fractional order with the new independent variable $\eta = xt^{-\alpha/\beta}$ is derived. Using Banach fixed point theorem, the existence and uniqueness of solutions is studied for this kind of fractional differential equations.**

Keywords: Partial differential equation of fractional order, invariant solutions, Banach fixed point theorem.

1 Introduction

Consider the following fractional-order equation

$$\frac{\partial^\alpha u(t,x)}{\partial t^\alpha} + u\frac{\partial u(t,x)}{\partial x} = \frac{\partial^\beta u(t,x)}{\partial x^\beta}, \quad x>0, t>0\ , \quad (1)$$

with

$$(D^\alpha_{0+}u)(t,x) = \frac{\partial^\alpha u(t,x)}{\partial t^\alpha} = \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial t^n}\int_0^t (t-s)^{n-\alpha-1}u(s,x)ds, \quad (2)$$

and $n-1<\alpha<n\in\mathbb{N}$.

Here, both fractional derivatives present in (1) are defined in the Riemann-Liouville sense.

Our aim is to obtain the partial scale-invariant solutions for (1). In our investigations we use the method of the Lie group analysis.

The fractional calculus is one of the most accurate tools to refine the description of natural phenomena. Fractional differential equations have attracted in the recent years a considerable interest due to their frequent appearance in various fields and their more accurate models of systems under consideration provided by fractional derivatives. Some partial differential equations of fractional order of type like one-dimensional time-fractional diffusion-wave equation were successfully used for modelling relevant physical processes (see, for example, Caputo [2], Giona and Roman [3] , Hilfer [4], Mainardi [8], Metzler et al. [9], Pipkin [11], Podlubny [12]).

The Lie group analysis of this equation has been discussed by Buckwar and Luchko [1]. The scale-invariant solutions for the one-dimensional time-fractional diffusion-wave equation (with the fractional derivative in the Riemann-Liouville sense) and for the more general time- and space-fractional partial differential equation (with the Riemann- Liouville space-fractional derivative of order $\beta \leq 2$ instead of the second order space derivative in the one-dimensional time-fractional diffusion-wave equation) have been presented by Buckwar and Luchko [1] and Luchko and Gorenflo [7], respectively.

Recently, S. Yakubovich and M.M. Rodrigues deals with fractional generalization of the Laplace equation and of the fractionary telegraph equation for rectangular domains which is associated with the Riemann-Liouville fractional derivatives in [14], [15], respectively.

The paper is organized as follows. Section 2 is devoted to the necessary definitions concerning the Lie group method, though not in the most general form. In the next section, we will apply the similarity method to the partial differential equation of fractional order (1). At first we determine a symmetry group of scaling transformations for this equation. The last section is dedicated to study of the existence and uniqueness of solutions for this kind of fractional differential equations.

2 Preliminaries

We start by present some definitions concerning the Lie group method, though not in the most general form. Let us consider the abstract equation

$$F(u)=0, \quad u=u(t,x). \quad (3)$$

DEFINITION: A one-parameter family of scaling transformations, denoted by T_λ, is a transformation of $(x,t,u)-$space of the form

$$\bar{t}=\lambda^b t, \quad \bar{x}=\lambda^a x, \quad \bar{u}=\lambda^c u \quad (4)$$

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where a, b, and c are constants and λ is a real parameter restricted to an open interval I containing $\lambda = 1$.

DEFINITION: The equation (3) is invariant under the one-parameter family T_λ of scaling transformations (4) iff T_λ takes any solution u of (3) to a solution $\bar{u}$ of the same equation:

$$\bar{u} = T_\lambda u \text{ and } F(\bar{u}) = 0. \tag{5}$$

DEFINITION: A real-valued function $\eta(t, x, u)$ is called an invariant of the one-parameter family T_λ, if it is unaffected by the transformations, in other words:

$$\eta(T_\lambda(t, x, u)) = \eta(t, x, u) \text{ for all } \lambda \in I. \tag{6}$$

On the half space $(t, x, u) : x > 0, t > 0$, the invariants of the family of scaling transformations (4) are provided by the functions (see [10])

$$\eta_1(t, x, u) = xt^{-a/b}, \eta_2(t, x, u) = t^{-c/b}u. \tag{7}$$

If the equation (3) is a second order partial differential equation of the form

$$G(x, t, u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}) = 0, \tag{8}$$

and this equation is invariant under T_λ, given by (4), then the transformation

$$u(t, x) = t^{-c/b}v(z), z = xt^{-a/b} \tag{9}$$

reduces the equation (8) to a second order ordinary differential equation of the form

$$g(z, v, v', v'') = 0. \tag{10}$$

For a proof of this fact we refer in the case of general Lie group methods to [10], for the special case of similarity methods to [6], and in some cases it can be easily checked directly.

3 Invariants of scaling transformations for a partial differential equation of fractional order

Here, we apply the method described in the previous section to the partial differential equation of fractional order (1).

We start to obtain the invariants of scaling transformations and using these invariants the partial differential equation of fractional order (1) is reduced to an ordinary differential equation.

THEOREM: The invariants of scaling transformations under which the equation (1) is invariant are given by the expressions

$$\eta_1(t, x, u) = xt^{-\alpha/\beta}, \eta_2(t, x, u) = t^{-\gamma}u, \tag{11}$$

with γ being an arbitrary real constant.

PROOF: At first, we determine a symmetry group of scaling transformations for this equation, and we have $(\bar{x} = \lambda^a x, \bar{t} = \lambda t, \bar{u} = \lambda^c u)$. In the case $\beta = n$ with $n \in \mathbb{N}$, we get

$$\frac{\partial^\beta \bar{u}(\bar{t}, \bar{x})}{\partial x^\beta} = \lambda^{c+a\beta} \frac{\partial^\beta \bar{u}(\bar{t}, \bar{x})}{\partial \bar{x}^\beta}. \tag{12}$$

For $n - 1 < \beta < n, n \in \mathbb{N}$, using variable substitution we have

$$\begin{aligned}
\frac{\partial^\beta \bar{u}(\bar{t}, \bar{x})}{\partial x^\beta} &= \frac{1}{\Gamma(n-\beta)} \frac{\partial^n}{\partial x^n} \int_0^x (x-s)^{n-s-1} u(\bar{t}, \bar{s}) ds \\
&= \frac{\lambda^{c+na}}{\Gamma(n-\beta)} \frac{\partial^n}{\partial \bar{x}^n} \int_0^{\bar{x}\lambda^{-a}} (\bar{x}\lambda^{-a} - s)^{n-\beta-1} \\
&\quad \times \bar{u}(\bar{t}, \lambda^a s) ds \\
&= \frac{\lambda^{c+na}}{\Gamma(n-\beta)} \frac{\partial^n}{\partial \bar{x}^n} \int_0^{\bar{x}} (\bar{x} - \tau)^{n-\alpha-1} \bar{u}(\bar{t}, \tau) d\tau \\
&= \lambda^{c+a\beta} \frac{\partial^\beta \bar{u}(\bar{t}, \bar{x})}{\partial \bar{x}^\beta}.
\end{aligned} \tag{13}$$

Analogously, we have

$$\frac{\partial^\alpha \bar{u}(\bar{t}, \bar{x})}{\partial t^\alpha} = \lambda^{c+\alpha} \frac{\partial^\alpha \bar{u}(\bar{t}, \bar{x})}{\partial \bar{t}^\alpha}$$

and

$$\frac{\partial \bar{u}(\bar{t}, \bar{x})}{\partial x} = \lambda^{c+a} \frac{\partial \bar{u}(\bar{t}, \bar{x})}{\partial \bar{x}}.$$

Hence, we can rewrite (1) in terms of $\bar{u}(\bar{t}, \bar{x})$ as

$$\begin{aligned}
\frac{\partial^\alpha \bar{u}(\bar{t}, \bar{x})}{\partial t^\alpha} + \bar{u}(\bar{t}, \bar{x}) \frac{\partial \bar{u}(\bar{t}, \bar{x})}{\partial x} - \frac{\partial^\beta \bar{u}(\bar{t}, \bar{x})}{\partial x^\beta} &= \lambda^{c+\alpha} \frac{\partial^\alpha u(\bar{t}, \bar{x})}{\partial \bar{t}\alpha} \\
&+ \lambda^{2c+a} \frac{\partial u(\bar{t}, \bar{x})}{\partial \bar{x}} \\
&- \lambda^{c+a\beta} \frac{\partial^\beta u(\bar{t}, \bar{x})}{\partial \bar{x}^\beta},
\end{aligned} \tag{14}$$

since $a = \frac{\alpha}{\beta}$ and $c = \alpha - \frac{\alpha}{\beta}$. Taking into account the previous relations and setting $\gamma = c$ in (11), we get the statement of our theorem.

By analogy with the partial differential equation (8), we use the transformation

$$u(t, x) = t^{\frac{\alpha}{\beta} - \alpha} v(\eta), \quad \eta = xt^{-\alpha/\beta} \tag{15}$$

to determine the scale-invariant solutions of the partial differential equation of fractional order (1). It is easy to see that, the partial derivative, u_x, is given in terms of derivatives of v by

$$u_x = t^{-\alpha} v'(\eta)$$

THEOREM: The transformation

$$u(t, x) = t^{\frac{\alpha}{\beta} - \alpha} v(\eta), \quad \eta = xt^{-\alpha/\beta}$$

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reduces the partial differential equation of fractional order (1) to the ordinary differential equation of fractional order of the form

$$\left(P^{1+\alpha/\beta-2\alpha,\alpha}_{\beta/\alpha}v\right)(\eta)+v(\eta)v'(\eta)=\eta^{-\beta}(D^{-\beta,\beta}_{1}v)(\eta), \tag{16}$$

with the left and right-hand sided Erdlyi-Kober fractional differential operators:

$$\left(P^{\tau,\alpha}_{\delta}g\right)(\eta)=\left(\prod_{j=0}^{n-1}\left(\tau+j-\frac{1}{\delta}\eta\frac{d}{d\eta}\right)\right)\left(K^{\tau+\alpha,n-\alpha}_{\delta}g\right)(\eta), \tag{17}$$

$\eta>0,\delta>0,\alpha>0,$

$$n=\begin{cases}[\alpha]+1, & \alpha\notin\mathbb{N}\\ \alpha, & \alpha\in\mathbb{N},\end{cases}$$

and where

$$\left(K^{\tau,\alpha}_{\delta}g\right)(\eta)=\frac{1}{\Gamma(\alpha)}\int_1^{\infty}(u-1)^{\alpha-1}u^{-(\tau+\alpha)}g(\eta u^{1/\delta})du,$$

for $\alpha>0$ and

$$\left(K^{\tau,\alpha}_{\delta}g\right)(\eta)=g(\eta)),\alpha=0$$

is the left-hand sided Erdlyi-Kober fractional integral operator,

$$\left(D^{\tau,\beta}_{\delta}g\right)(\eta)=\left(\prod_{j=1}^{n}\left(\tau+j+\frac{1}{\delta}\eta\frac{d}{d\eta}\right)\right)\left(I^{\tau+\beta,n-\beta}_{\delta}g\right)(\eta), \tag{18}$$

$\eta>0,\delta>0,\beta>0,$

$$n=\begin{cases}[\beta]+1, & \beta\notin\mathbb{N}\\ \beta, & \beta\in\mathbb{N},\end{cases}$$

where

$$\left(I^{\tau,\beta}_{\delta}g\right)(\eta)=\begin{cases}\frac{1}{\Gamma(\beta)}\int_0^1(1-u)^{\beta-1}u^{\tau}g(\eta u^{1/\delta})du, & \beta>0,\\ g(\eta) & \beta=0\end{cases}$$

is the right-hand sided Erdlyi-Kober fractional integral operator.

PROOF: Let consider $n-1<\alpha<n,\ n=1,2,\ldots$ and $\eta=xt^{-\alpha/\beta}$. The partial derivative of fractional order $\frac{\partial^\alpha u}{\partial t^\alpha}$ is given as

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha}&=\frac{\partial^n}{\partial t^n}\left(\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-\tau)^{n-\alpha-1}\tau^{\gamma}v(x\tau^{-\alpha/\beta})d\tau\right)\\&=\frac{\partial^n}{\partial t^n}\left(t^{n-\alpha+\gamma}(K^{1+\gamma,n-\alpha}_{\beta/\alpha}v)(\eta)\right)\end{aligned} \tag{19}$$

with left-hand sided Erdlyi-Kober fractional integral operator $(K^{1+\gamma,n-\alpha}_{\beta/\alpha}v)(\eta)$. Since $(K^{\tau,0}_{\delta}v)=v(\eta)$, the same relation holds for $\alpha=n=1,2,\ldots$. In view of relation $(\eta=xt^{-\alpha/\beta},\phi\in C^1(0,\infty))$

$$t\frac{\partial}{\partial t}\phi(\eta)=tx\left(-\frac{\alpha}{\beta}\right)t^{-\frac{\alpha}{\beta}-1}\phi'(\eta)=-\frac{\alpha}{\beta}\eta\frac{d}{d\eta}\phi(\eta), \tag{20}$$

we arrive at

$$\begin{aligned}&\frac{\partial^n}{\partial t^n}\left(t^{n-\alpha+\gamma}(K^{1+\gamma,n-\alpha}_{\beta/\alpha}v)(\eta)\right)\\=&\frac{\partial^{n-1}}{\partial t^{n-1}}\left(t^{n-\alpha+\gamma-1}(n-\alpha+\gamma-\frac{\alpha}{\beta}\eta\frac{d}{d\eta})(P^{1+\gamma,n-\alpha}_{\beta/\alpha}v)(\eta)\right)\\=&\ \cdots\\=&\ t^{\gamma-\alpha}\prod_{j=0}^{n-1}\left(1+\gamma-\alpha+j-\frac{\alpha}{\beta}\eta\frac{d}{d\eta}\right)(K^{1+\gamma-\alpha,n-\alpha}_{\beta/\alpha}v)(\eta)\\=&\ t^{\gamma-\alpha}\left(P^{1+\gamma-\alpha,\alpha}_{\beta/\alpha}v\right)(\eta).\end{aligned}$$

Finally, for the transformation (15) we have

$$\frac{\partial^\alpha}{\partial t^\alpha}=t^{\gamma-\alpha}\left(P^{1+\gamma-\alpha,\alpha}_{\beta/\alpha}v\right)(\eta). \tag{21}$$

By analogy, using the relation

$$\frac{\partial^\beta}{\partial x^\beta}=t^{\gamma}\frac{\partial^n}{\partial x^n}\left(x^{n-\beta}\left(I^{0,n-\beta}_{1}v\right)(\eta)\right),\quad \eta=xt^{-\alpha/\beta} \tag{22}$$

for $n-1<\beta\le n,\ n=1,2,\ldots$ and

$$x\frac{\partial}{\partial x}\phi(\eta)=xt^{-\frac{\alpha}{\beta}}\phi'(\eta)=\eta\frac{d}{d\eta}\phi(\eta),\quad \eta=xt^{-\alpha/\beta}, \tag{23}$$

we obtain

$$\frac{\partial^\beta}{\partial x^\beta}=t^{\gamma}x^{-\beta}\left(D^{-\beta,\beta}_{1}v\right)(\eta). \tag{24}$$

Now, substituting (21) and (24) into (1) we obtain

$$t^{\gamma-2\alpha}\left(P^{1+\gamma-\alpha,\alpha}_{\beta/\alpha}v\right)(\eta)+t^{\gamma-2\alpha}v(\eta)v'(\eta)=t^{\gamma-\alpha}x^{-\beta}\left(D^{-\beta,\beta}_{1}v\right)(\eta), \tag{25}$$

with $\eta>0$. This last equation is equivalent to equation (17) in which the variables t and x do not occur.

4 Existence and Uniqueness of Solutions

This section will be developed to the existence and uniqueness of solutions for equations

$$(D^{\alpha}_{t_0^+}u)(t,x)=f\left(u(t,x),(D_{x_0^+}u)(t,x),D^{\beta}_{x_0^+}u(t,x)\right), \tag{26}$$

$t\in[t_0,T_0],x\in[x_0,X_0],0<\alpha<1,0<\beta<1$ employing the Banach fixed point theorem.

Note that, the equation (1) is a particular case of the latter equation.

We begin introducing some notations and results for further consideration.

Let $I=[a,b]$ $(a<b,a,b\in\mathbb{R})$ and $m\in\mathbb{N}_0$. Denoting by C^m a usual space of functions v which are m times continuously differentiable on I with the norm

$$\|v\|_{C^m}=\sum_{k=0}^{m}\|v^{(k)}\|_C=\sum_{k=0}^{m}\max_{x\in\Omega}|v^{(k)}(x)|,$$

$m\in\mathbb{N}_0$.

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In particular, for $m = 0$, $C^0(I) \equiv C(I)$ is the space of continuous functions v on I with the norm $\|v\|_C = \max\limits_{x\in I} |v(x)|$.

For $0 \le \gamma < 1$, denoting by $C^\gamma(I)$ a weighted space of functions v for $x \in (a,b]$ such that $(x-a)^{1+\gamma}v(x) \in C[a,b]$ and

$$\|v\|_{C^\gamma} = \|(x-a)^{1+\gamma}v(x)\|_C.$$

Now, we present some definitions and facts from the theory of fractional differential operators that will be useful in what follows.

DEFINITION: (see [13]) By $AC^n([a,b]), n \in \mathbb{N}$, one denotes the class of functions $v(x)$, which are continuously differentiable on the segment $[a,b]$ up to the order $n-1$ and $v^{(n-1)}(x)$ is absolutely continuous on $[a,b]$.

DEFINITION: (see [13]) By $I^\gamma_{a^+}(L_1)$ denotes the class of functions v represented by a left-sided fractional integral of a summable function, that is, $v = I^\gamma_{a^+}\varphi$, $\varphi \in L_1(a,b)$.

This class of functions is described below.

THEOREM: (see [13]) A function $v(x) \in I^\gamma_{a^+}(L_1)$, $\gamma > 0$ if and only if $(I^{n-\gamma}_{a^+}v)(x) \in AC^n([a,b])$, $n = [\gamma]+1$ and $(I^{n-\gamma}_{a^+}v)^{(k)}(a) = 0$, $k = 0,1,\ldots,n-1$.

THEOREM: (see [13]) Let $\gamma \ge 0$ and $v(x) \in AC^n([a,b])$, $n = [\gamma]+1$. Then $D^\gamma_{a^+}v$ exists almost everywhere and may be represented in the form

$$\begin{aligned} D^\gamma_{a^+}v &= \textstyle\sum_{k=0}^{n-1} \frac{v^{(k)}(a)}{\Gamma(1+k-\gamma)}(x-a)^{k-\gamma} \\ &\quad + \frac{1}{\Gamma(n-\gamma)}\int_a^x \frac{v^{(n)}(t)}{(x-t)^{\gamma-n+1}}dt. \end{aligned} \tag{27}$$

THEOREM: (see [5]) Let $\gamma \ge 0$ and $n = [\gamma]+1$. If $v(x) \in AC^n([a,b])$, then the Caputo fractional derivative ${}^C D^\gamma_{a^+}v$ exists almost everywhere on $[a,b]$, and if $\gamma \notin \mathbb{N}_0$, ${}^C D^\gamma_{a^+}v$ is represented by

$${}^C D^\gamma_{a^+}v = \frac{1}{\Gamma(n-\gamma)}\int_a^x \frac{v^{(n)}(t)}{(x-t)^{\gamma-n+1}}dt := (I^{n-\gamma}_{a^+}D^n v)(x), \tag{28}$$

where $D = d/dx$.

REMARK: If $\gamma \notin \mathbb{N}_0$ and $n = [\gamma]+1$, then

$$\left|I^{n-\gamma}_{a^+}D^n v)(x)\right| \le \frac{\|v^{(n)}\|_C}{|\Gamma(n-\gamma)|(n-\gamma+1)}(x-a)^{n-\gamma}. \tag{29}$$

Recalling equation (26), since $u(t,x)$ belongs to classes $I^\alpha_{t_0^+}(L_1)$, $I^\beta_{x_0^+}(L_1)$ by $t \in [t_0,T_0]$ and $x \in [x_0,X_0]$, respectively, we take operator $I^\alpha_{t_0^+}$ from both sides of (26) to obtain

$$(I^\alpha_{t_0^+}D^\alpha_{t_0^+}u)(t,x) = I^\alpha_{t_0^+}\left(f\left(u(t,x), D_{x_0^+}u(t,x), D^\beta_{x_0^+}u(t,x)\right)\right), \tag{30}$$

with f be a real continuous function. Using identity $\left(I^\alpha_{t_0^+}D^\alpha_{t_0^+}u\right)(t,x) = u(t,x)$ it becomes

$$u(t,x) = I^\alpha_{t_0^+}\left(f\left(u(t,x), D_{x_0^+}u(t,x), D^\beta_{x_0^+}u(t,x)\right)\right) \tag{31}$$

Now, we will establish an auxiliary result.

LEMMA: The fractional integration operator $I^\gamma_{t_0^+}$ of order γ with $\gamma \in \mathbb{R}^+$ forms a map from $C[t_0,T_0]$ to itself for each $t \in [x_0,X_0]$, and we have the estimate

$$\left\|I^\gamma_{t_0^+}u\right\|_{C[t_0,T_0]} \le \frac{(T_0-t_0)^{\gamma+1}}{\Gamma(\gamma+1)}\|u\|_{C[t_0,T_0]}.$$

PROOF: First we prove that, if $u(t,x) \in C[t_0,T_0]$ then $(I^\gamma_{t_0^+}u)(t,x) \in C[t_0,T_0]$. In fact, for any $t \in [t_0,T_0]$ and $\Delta t > 0$, $t+\Delta t \le T_0$ we have

$$\begin{aligned} &\left|(I^\gamma_{t_0^+}u)(t+\Delta t,x) - (I^\gamma_{t_0^+}u)(t,x)\right| \\ &= \frac{1}{\Gamma(\gamma)}\left|\int_{t_0}^{t+\Delta t}(t+\Delta t-z)^{\gamma-1}u(z,x)dz \right. \\ &\quad \left. - \int_{t_0}^{t}(t-z)^{\gamma-1}u(z,x)dz\right| \\ &\le \frac{1}{\Gamma(\gamma)}\left[\left|\int_{t_0}^{t}u(z,x)\left((t+\Delta t-z)^{\gamma-1}-(t-z)^{\gamma-1}\right)dz\right| \right. \\ &\quad \left. + \left|\int_{t}^{t+\Delta t}(t+\Delta t-z)^{\gamma-1}u(z,x)dz\right|\right] \\ &\le \frac{\|u(t,x)\|_{C[t_0,T_0]}}{(\gamma)\Gamma(\gamma)}\left[((t+\Delta t-t_0)^\gamma-(t-t_0)^\gamma)+2(\Delta t)^\gamma\right] \\ &\le \frac{\|u(t,x)\|_{C[t_0,T_0]}}{\Gamma(\gamma+1)}\left[((t+\Delta t-t_0)^\gamma-(t-t_0)^\gamma)+2(\Delta t)^\gamma\right]. \end{aligned} \tag{32}$$

Therefore, when $\Delta t \to 0^+$ we have

$$\left|(I^\gamma_{t_0^+}u)(t+\Delta t,x) - (I^\gamma_{t_0^+}u)(t,x)\right| \to 0.$$

Similarly it is valid when $\Delta t \to 0^-$. Hence, $I^\gamma_{t_0^+}u \in C[t_0,T_0]$. Consequently,

$$\begin{aligned} \left\|I^\gamma_{t_0^+}u\right\|_{C[t_0,T_0]} &= \max_{t\in[t_0,T_0]}\left|\frac{1}{\Gamma(\gamma)}\int_{t_0}^{t}(t-z)^{\gamma-1}u(z,x)dz\right| \\ &\le \frac{\|u\|_{C[t_0,T_0]}}{\Gamma(\gamma)}\int_{t_0}^{t}(t-z)^{\gamma-1}dz \\ &\le \frac{(T_0-t_0)^{\gamma+1}}{(\gamma)\Gamma(\gamma)}\|u\|_{C[t_0,T_0]} \\ &\le \frac{(T_0-t_0)^{\gamma+1}}{\Gamma(\gamma+1)}\|u\|_{C[t_0,T_0]}. \end{aligned} \tag{33}$$

THEOREM: Let f be a real continuous function satisfy-

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ing

$$\begin{aligned}|f(u_1, D_{x_0^+}u_1, D^{\beta}_{x_0^+}u_1) - f(u_2, D_{x_0^+}u_2, D^{\beta}_{x_0^+}u_2)| \\ \leq A\,(\|u_1 - u_2\| \\ + \|D_{x_0^+}(u_1 - u_2)\| + \|D^{\beta}_{x_0^+}(u_1 - u_2)\|\Big) . \qquad (34)\end{aligned}$$

Integral equation (26) has a unique solution whenever $0 < \xi < 1$.

PROOF: We denote by X the Banach space

$$X_0 = \{u : u(.,x) \in C([x_0, X_0]), u(t,.) \in C([t_0, T_0])\},$$

$$X_1 = \{u : u(.,x) \in C^1([x_0, X_0]), u(t,.) \in C([t_0, T_0])\},$$

$$X = \{u : u(.,x) \in C^2([x_0, X_0]), u(t,.) \in C([t_0, T_0])\},$$

and by Y the Banach space

$$Y = \{u : u(.,x) \in C^{\lambda}([x_0, X_0]), u(t,.) \in C([t_0, T_0])\}.$$

Next, we put $T : X \to Y$,

$$(Tu)(t,x) = I^{\alpha}_{t_0^+}\left(f\left(u(t,x), D_{x_0^+}u(t,x), D^{\beta}_{x_0^+}u(t,x)\right)\right).$$

Hence, we rewrite equation (31) in the form

$$u(t,x) = (Tu)(t,x).$$

Calling definitions of $C^{\gamma}-$norm, C^2-norm and $C-$norm and taking into account (27), (28) and (29) and the hypotheses of the theorem, we have

$$\begin{aligned}
&\|Tu_1 - Tu_2\|_Y \\
= \;& \Big\| I^{\alpha}_{t_0^+}\left(f\left(u_1, D_{x_0^+}u_1, D^{\beta}_{x_0^+}u_1\right)\right) \\
& - I^{\alpha}_{t_0^+}\left(f\left(u_2, D_{x_0^+}u_2, D^{\beta}_{x_0^+}u_2\right)\right)\Big\|_Y \\
& \leq A\frac{(T_0 - t_0)^{\alpha+1}}{\Gamma(\alpha+1)}\Big[\|u_1 - u_2\|_Y + \|D_{x_0^+}(u_1 - u_2)\|_Y \\
& + \|D^{\beta}_{x_0^+}(u_1 - u_2)\|_Y\Big] \\
= \;& A\frac{(T_0 - t_0)^{\alpha+1}}{\Gamma(\alpha+1)}\Big[(X_0 - x_0)^{1+\lambda}\|u_1 - u_2\|_X \\
& + (X_0 - x_0)^{1+\lambda}\|u_1 - u_2\|_X + \\
& \Bigg(\frac{(X_0 - x_0)^{1+\lambda-\beta}}{|\Gamma(1-\beta)|} \\
& + \frac{(X_0 - x_0)^{2+\lambda-\beta}}{\Gamma(2-\beta)} + \frac{(X_0 - x_0)^{3+\lambda-\beta}}{\Gamma(2-\beta)(3-\beta)}\Bigg)\|u_1 - u_2\|_X\Big] \\
= \;& A\frac{(T_0 - t_0)^{\alpha+1}}{\Gamma(\alpha+1)}M\|u_1 - u_2\|_X \\
= \;& \xi\|u_1 - u_2\|_X,
\end{aligned}$$

where we let

$$\xi \;=\; A\frac{(T_0 - t_0)^{\alpha+1}}{\Gamma(\alpha+1)}M$$

and

$$M = \max\Big\{(X_0 - x_0)^{1+\lambda}, \tfrac{(X_0-x_0)^{1+\lambda-\beta}}{|\Gamma(1-\beta)|} + \tfrac{(X_0-x_0)^{2+\lambda-\beta}}{\Gamma(2-\beta)} + \tfrac{(X_0-x_0)^{3+\lambda-\beta}}{\Gamma(2-\beta)(3-\beta)}\Big\}.$$

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