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Control and Stability on the Euclidean Group $\mathsf{SE}(2)$

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Abstract— **This paper considers control affine left-invariant systems evolving on matrix Lie groups. Any left-invariant optimal control problem (with quadratic cost) can be lifted, via the celebrated Maximum Principle, to a Hamiltonian system on the dual of the Lie algebra of the underlying state space G. The (minus) Lie-Poisson structure on the dual space $\mathfrak{g}^*$ is used to describe the (normal) extremal curves. The fully actuated case on the Euclidean group $\mathsf{SE}(2)$ is considered and the reduced Hamilton equations associated with an extremal curve are derived in a simple and elegant manner. Finally, the nature of the equilibrium states is fully investigated by the energy-Casimir method.**

Keywords: ***left-invariant control system, Pontryagin maximum principle, extremal curve, Lie-Poisson structure, Lyapunov stability, the energy-Casimir method***

1 Introduction

A wide range of dynamical systems from fields as diverse as classical and quantum mechanics, elasticity, electrical networks, and molecular chemistry can be modelled by invariant (control) systems on matrix Lie groups. A short list of invariant optimal control problems contains the ball-plate problem, various versions of the Euler elastic problem, the Dubins' problem as well as the (more general) sub-Riemannian geodesic problem. These problems (and many other) can be found, for instance, in the monographs by Jurdjevic [6], Bloch [2] or Agrachev and Sachkov [1]. See also [5], [18].

Substantial work on applied nonlinear control has drawn attention to (left-) invariant control systems with control affine dynamics, evolving on matrix Lie groups of low dimension. Such systems arise, for instance, in the airplane landing problem, the motion planning for wheeled robots, and the control of underactuated underwater vehicles (see e.g. [19], [12], [14], [9] and the references therein).

A left-invariant optimal control problem consists in minimizing some (practical) cost functional over the trajectories of a given left-invariant control system, subject to appropriate boundary conditions. The application of the Maximum Principle shifts the emphasis to the language of symplectic and Poisson geometries and to the associated Hamiltonian formalism. The Maximum Principle states that the optimal solutions are projections of the extremal curves onto the base manifold. (For invariant control systems the base manifold is a Lie group G.) The extremal curves are solutions of certain Hamiltonian systems on the cotangent bundle $T^*\mathsf{G}$. The cotangent bundle $T^*\mathsf{G}$ can be realized as the direct product $\mathsf{G}\times\mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual of the Lie algebra $\mathfrak{g}$ of G. As a result, each original (left-invariant) Hamiltonian induces a reduced Hamiltonian on the dual space (which comes equipped with a natural Poisson structure).

An arbitrary control affine left-invariant system on the Euclidean group $\mathsf{SE}(2)$ has the form

$$\dot g = g\,(A + u_1B_1 + \cdots + u_\ell B_\ell)\,, \quad g\in\mathsf{SE}(2),\ u\in\mathbb{R}^\ell$$

where $A, B_1, \ldots, B_\ell \in \mathfrak{se}(2)$, $1\le\ell\le 3$. (The elements $B_1,\ldots,B_\ell$ are assumed to be linearly independent.) There are essentially four kinds of such systems : single-input systems with drift, two-input systems with or without drift, and three-input systems. (The single-input drift-free systems represent a degenerate case of little interest.) The Euclidean elastic problem on $\mathbb{E}^2$ is associated with control systems of the first type (see [6], [5], [18]) whereas problems related to the motion of the wheeled mobile robot lead to optimal control problems associated with (drift-free) two-input systems (see [19], [12]; for a simplified model, the so-called unicycle, see e.g. [8].

In this paper, we consider an optimal control problem associated with a three-input control-affine system on the Euclidean group $\mathsf{SE}(2)$. The problem is lifted, via the Pontryagin Maximum Principle, to a Hamiltonian system on the dual of the Lie algebra $\mathfrak{se}(2)$. Now, the (minus) Lie-Poisson structure on $\mathfrak{se}(2)^*$ (identified here with $\mathbb{R}^3$) can be used to derive, in a general and elegant manner, the equations for extrema (cf. [6], [1], [8]) (see also [16], [17] for similar computations on the rotation group $\mathsf{SO}(3)$). The energy-Casimir method is used to fully investigate the nature of the equilibrium states (cf. [13], [14]).

The paper is organized as follows. Section 2 con-

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tains mathematical preliminaries including invariant control systems, elements of Hamilton-Poisson formalism, a (coordinate-free) statement of the Maximum Principle as well as Lyapunov stability (and the energy-Casimir method); also, a particularly useful result due to P.S. Krishnaprasad is recalled. Section 3 deals with the particular case of the Euclidean group $\mathsf{SE}(2)$, the Lie-Poisson structure on $\mathfrak{se}(2)^*$ and a "typical" left-invariant optimal control on $\mathsf{SE}(2)$. Finally, section 4 contains a complete study of the stability nature of the equilibrium states of the extremal equations.

2 Preliminaries

2.1 Invariant Control Systems

Invariant control systems on Lie groups were first considered in 1972 by Brockett [3] and by Jurdjevic and Sussmann [7]. A *left-invariant control system* is a (smooth) control system evolving on some (real) Lie group, whose dynamics is invariant under left translations. For the sake of convenience, we shall assume that the state space of the system is a matrix Lie group and that there are no constraints on the controls. Such a control system (evolving on G) is described as follows (cf. [6], [1], [15])

$$\dot{g} = g\,\Xi(\mathbf{1}, g), \quad g \in \mathsf{G},\ u \in \mathbb{R}^\ell \tag{1}$$

where the parametrisation map $\Xi(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{g}$ is a (smooth) embedding. (Here $\mathbf{1} \in \mathsf{G}$ denotes the identity matrix and $\mathfrak{g}$ denotes the Lie algebra associated with G.) An admissible control is a map $u(\cdot) : [0, T] \to \mathbb{R}^\ell$ that is bounded and measurable. ("Measurable" means "almost everywhere limit of piecewise constant maps".) A *trajectory* for an admissible control $u(\cdot) : [0, T] \to \mathbb{R}^\ell$ is an absolutely continuous curve $g(\cdot) : [0, T] \to \mathsf{G}$ such that $\dot{g}(t) = g(t)\,\Xi(\mathbf{1}, u(t))$ for almost every $t \in [0, T]$. The Carathéodory existence and uniqueness theorem of ordinary differential equations implies the local existence and global uniqueness of trajectories. A controlled trajectory is a pair $(g(\cdot), u(\cdot))$, where $u(\cdot)$ is an admissible control and $g(\cdot)$ is the trajectory corresponding to $u(\cdot)$.

The attainable set from $g \in \mathsf{G}$ is the set $\mathcal{A}(g)$ of all terminal points $g(T)$ of all trajectories $g(\cdot) : [0, T] \to \mathsf{G}$ starting at g. It follows that $\mathcal{A}(g) = g\,\mathcal{A}(\mathbf{1})$. Thus, $\mathcal{A}(g) = \mathsf{G}$ if and only if $\mathcal{A}(\mathbf{1}) = \mathsf{G}$. Control systems which satisfy $\mathcal{A}(\mathbf{1}) = \mathsf{G}$ are called *controllable*. Let $\Gamma \subseteq \mathfrak{g}$ be the image of the parametrisation map $\Xi(\mathbf{1}, \cdot)$, and let $\mathsf{Lie}(\Gamma)$ denote the Lie subalgebra of $\mathfrak{g}$ generated by Γ. It is well known that a necessary condition for the control system (1) to be controllable is that G be connected and that $\mathsf{Lie}(\Gamma) = \mathfrak{g}$. If the group G is compact, then the condition is also sufficient.

For many practical control applications, (left-invariant) control systems contain a drift term and are affine in controls, i.e., are of the form

$$\dot{g} = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell), \quad g \in \mathsf{G},\ u \in \mathbb{R}^\ell \tag{2}$$

where $A, B_1, \ldots, B_\ell \in \mathfrak{g}$.

2.2 Invariant Optimal Control Problems

Consider a left-invariant control system (1) evolving on some matrix Lie group $\mathsf{G} \leq \mathsf{GL}(n, \mathbb{R})$ of dimension m. In addition, it is assumed that there is a prescribed (smooth) *cost function* $L : \mathbb{R}^\ell \to \mathbb{R}_{>0}$ (which is also called a Lagrangian). Let g_0 and g_1 be arbitrary but fixed points of G. We shall be interested in finding a controlled trajectory $(g(\cdot), u(\cdot))$ which satisfies

$$g(0) = g_0, \quad g(T) = g_1 \tag{3}$$

and which in addition *minimizes* the total cost functional $\mathcal{J} = \int_0^T L(u(t))\,dt$ among all trajectories of (1) which satisfy the same boundary conditions (3). The terminal time $T > 0$ can be either fixed or it can be free.

The cotangent bundle $T^*\mathsf{G}$ can be trivialized (from the left) such that $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual space of the Lie algebra $\mathfrak{g}$. Explicitly, $\xi \in T_g^*\mathsf{G}$ is identified with $(g, p) \in \mathsf{G} \times \mathfrak{g}^*$ via $p = dL_g^*(\xi)$. (Here, dL_g^* denotes the dual of the tangent map $dL_g = (L_g)_{*,\mathbf{1}} : \mathfrak{g} \to T_g\mathsf{G}$.) That is, $\xi(gA) = p(A)$ for $g \in \mathsf{G}$, $A \in \mathfrak{g}$. Each element (matrix) $A \in \mathfrak{g}$ defines a (smooth) function H_A on the cotangent bundle $T^*\mathsf{G}$ defined by $H_A(\xi) = \xi(gA)$ for $\xi \in T_g^*\mathsf{G}$. Viewed as a function on $\mathsf{G} \times \mathfrak{g}^*$, H_A is left-invariant, which is equivalent to saying that H_A is a function on $\mathfrak{g}^*$.

The canonical symplectic form ω on $T^*\mathsf{G}$ sets up a correspondence between (smooth) functions H on $T^*\mathsf{G}$ and vector fields $\vec{H}$ on $T^*\mathsf{G}$ given by $\omega_\xi\left(\vec{H}(\xi), V\right) = dH(\xi) \cdot V$ for $V \in T_\xi(T^*\mathsf{G})$. The Poisson bracket of two functions F, G on $T^*\mathsf{G}$ is defined by $\{F, G\}(\xi) = \omega_\xi\left(\vec{F}(\xi), \vec{G}(\xi)\right)$ for $\xi \in T^*\mathsf{G}$. If (ϕ_t) is the flow of the Hamiltonian vector field $\vec{H}$, then $H \circ \phi_t = H$ (conservation of energy) and $\frac{d}{dt}(F \circ \phi_t) = \{F, H\} \circ \phi_t = \{F \circ \phi_t, H\}$. For short, for any $F \in C^\infty(T^*\mathsf{G})$,

$$\dot{F} = \{F, H\}. \tag{4}$$

The dual space $\mathfrak{g}^*$ has a natural *Poisson structure*, called the "minus Lie-Poisson structure" and given by

$$\{F, G\}_-(p) = -p\,([dF(p), dG(p)])$$

for $p \in \mathfrak{g}^*$ and $F, G \in C^\infty(\mathfrak{g}^*)$. (Note that $dF(p)$ is a linear function on $\mathfrak{g}^*$ and hence is an element of $\mathfrak{g}$.) The (minus) Lie-Poisson bracket can be derived from the canonical Poisson structure on the cotangent bundle $T^*\mathsf{G}$ by a process called Poisson reduction (cf. [10], [8]). The Poisson manifold $(\mathfrak{g}, \{\cdot,\cdot\})$ is denoted by $\mathfrak{g}^*_-$. Each left-invariant Hamiltonian on the cotangent bundle $T^*\mathsf{G}$ is

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identified with its reduction on the dual space $\mathfrak{g}^*_-$. In the left-invariant realization of $T^*\mathsf{G}$, the equations of motion for the left-invariant Hamiltonian H are

$$\begin{aligned}\dot{g} &= g\,dH(p)\\ \dot{p} &= \mathrm{ad}^*_{dH(p)}p\end{aligned}$$

where ad^* denotes the coadjoint representation of $\mathfrak{g}$ (cf. [10], [6]). Note that for non-commutative Lie groups, the representation $T^*\mathsf{G} = \mathsf{G}\times\mathfrak{g}^*$ invariably leads to non-canonical coordinates.

If $(E_k)_{1\le k\le m}$ is a basis for the Lie algebra $\mathfrak{g}$, the structure constants $\left(c^k_{ij}\right)$ are defined by $[E_i, E_j] = \sum_{k=1}^m c^k_{ij}E_k$. Any element $p\in\mathfrak{g}^*$ can be expressed uniquely as $p = \sum_{k=1}^m p_k E^*_k$, where $(E^*_k)_{1\le k\le m}$ is the basis of $\mathfrak{g}^*$ dual to $(E_k)_{1\le k\le m}$. Then the (minus) Lie-Poisson bracket becomes

$$\{F, G\}_-(p) = -\sum_{i,j,k=1}^m c^k_{ij}p_k\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial p_j}.$$

A *Casimir function* of (the Poisson structure of) $\mathfrak{g}^*_-$ is a (smooth) function C on $\mathfrak{g}^*$ such that $\{C, F\}_- = 0$ for all $F\in C^\infty(\mathfrak{g}^*)$. The Casimir functions have the remarkable property that they are integrals of motion for any Hamiltonian system (i.e., they are constant along the flow of any Hamiltonian vector field) on $\mathfrak{g}^*_-$.

2.3 The Maximum Principle

The *Pontryagin Maximum Principle* is a necessary condition for optimality expressed most naturally in the language of the geometry of the cotangent bundle $T^*\mathsf{G}$ of G (cf. [1], [6]). To an optimal control problem (with fixed terminal time)

$$\int_0^T L(u(t))\,dt\to\min \tag{5}$$

subject to (1) and (3), we associate, for each real number λ and each control parameter $u\in\mathbb{R}^\ell$, a Hamiltonian function on $T^*\mathsf{G} = \mathsf{G}\times\mathfrak{g}^*$:

$$\begin{aligned}H^\lambda_u(\xi) &= \lambda\,L(u) + \xi\,\left(g\,\Xi(\mathbf{1}, u)\right)\\ &= \lambda\,L(u) + p\,\left(\Xi(\mathbf{1}, u)\right),\quad \xi = (g, p)\in T^*\mathsf{G}.\end{aligned}$$

The Maximum Principle can be stated, in terms of the above Hamiltonians, as follows :

THE MAXIMUM PRINCIPLE. *Suppose the controlled trajectory* $(\bar{g}(\cdot), \bar{u}(\cdot))$ *defined over the interval* $[0, T]$ *is a solution for the optimal control problem (1)-(3)-(5). Then, there exists a curve* $\xi(\cdot) : [0, T]\to T^*\mathsf{G}$ *with* $\xi(t)\in T^*_{\bar{g}(t)}\mathsf{G}$, $t\in[0, T]$, *and a real number* $\lambda\le 0$, *such that the following conditions hold for almost every* $t\in[0, T]$ *:*

$$(\lambda, \xi(t))\not\equiv(0, 0) \tag{6}$$

$$\dot{\xi}(t) = \vec{H}^\lambda_{\bar{u}(t)}(\xi(t)) \tag{7}$$

$$H^\lambda_{\bar{u}(t)}(\xi(t)) = \max_u H^\lambda_u(\xi(t)) = \textit{constant}. \tag{8}$$

An optimal trajectory $\bar{g}(\cdot) : [0, T]\to\mathsf{G}$ is the projection of an integral curve $\xi(\cdot)$ of the (time-varying) Hamiltonian vector field $\vec{H}^\lambda_{\bar{u}(t)}$ defined for all $t\in[0, T]$. A trajectory-control pair $(\xi(\cdot), u(\cdot))$ defined on $[0, T]$ is said to be an *extremal pair* if $\xi(\cdot)$ is such that the conditions (6), (7) and (8) of the Maximum Principle hold. The projection $\xi(\cdot)$ of an extremal pair is called an extremal. An extremal curve is called normal if $\lambda = -1$ and abnormal if $\lambda = 0$. In this paper, we shall be concerned only with normal extremals.

If the maximum condition (8) eliminates the parameter u from the family of Hamiltonians (H_u), and as a result of this elimination, we obtain a smooth function H (without parameters) on $T^*\mathsf{G}$ (in fact, on $\mathfrak{g}^*_-$), then the whole (left-invariant) optimal control problem reduces to the study of trajectories of a fixed Hamiltonian vector field $\vec{H}$.

2.4 The Energy-Casimir Method

Given a (complete) vector field X on the (smooth) manifold M, with flow $(\phi_t)_{t\in\mathbb{R}}$, let $z_e\in\mathsf{M}$ be an equilibrium point (state) of X (i.e. $X(z_e) = 0$ or, equivalently, $\phi_t(z_e) = z_e$ for all $t\in\mathbb{R}$). Recall that z_e is *Lyapunov stable* (or nonlinearly stable) if for any open neighborhood V of z_e, there is an open neighborhood $V'\subset V$ of z_e such that $\phi_t(z)\in V$ for any $z\in V'$ and any $t > 0$.

The *energy-Casimir method* [4] is a generalization of the classical Lagrange-Dirichlet stability test. It gives sufficient conditions for Lyapunov stability of equilibrium states for certain types of Hamilton-Poisson dynamical systems (cf. [10], [11]). The method is restricted to certain types of systems, since its implementation relies on an abundant supply of Casimir functions.

Let $(\mathsf{M}, \{\cdot,\cdot\}, H)$ be a (finite-dimensional) Hamilton-Poisson dynamical system and $z_e\in\mathsf{M}$ an equilibrium state of the Hamiltonian vector field $\vec{H}$. (In fact, we shall be concerned only with the Poisson space $\mathfrak{g}^*_-$.) The method proceeds in the following algorithmic way :
Step 1. Find a constant of motion for the system (usually the energy H).
Step 2. Find a family $\mathcal{C}$ of constants of motion.
Step 3. Relate the equilibrium state z_e of the system to a constant of motion $C\in\mathcal{C}$ (usually a Casimir function) by requiring that $H + C$ have a critical point at z_e.
Step 4. Check that the second variation $\delta^2(H + C)$ at z_e is positive (or negative) definite.
Then the equilibrium state z_e of the system is Lyapunov stable.

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2.5 A Class of Optimal Control Problems

Consider now a left-invariant optimal control problem (2)-(3)-(5) with quadratic cost of the form

$$L(u_1, \ldots, u_\ell) = \frac{1}{2}\left(c_1 u_1^2 + \cdots + c_\ell u_\ell^2\right)$$

where $c_1, \ldots, c_\ell$ are (positive) constants. The terminal time $T > 0$ is fixed in advance. The maximum condition (8) of the Maximum Principle implies that (for $\lambda = -1$) the optimal controls $\bar{u}(\cdot)$ satisfy

$$-\frac{\partial L}{\partial u_i} + \frac{\partial}{\partial u_i}\left(p\left(A + u_1 B_1 + \cdots + u_\ell B_\ell\right)\right) = 0$$

or

$$-c_i u_i + p(B_i) = 0, \quad i = 1, \ldots, \ell.$$

The following result holds.

Proposition 1 (Krishnaprasad [8])**.** *For the optimal control problem (2)-(3)-(5), every normal extremal is given by*

$$\bar{u}_i(t) = \frac{1}{c_i}\, p(t)(B_i), \quad i = 1, \ldots, \ell$$

where $p(\cdot) : [0, T] \to \mathfrak{g}^*$ *is an integral curve of the vector field* $\vec{H}$ *on* $\mathfrak{g}^*_-$ *corresponding to the reduced Hamiltonian*

$$H(p) = p(A) + \frac{1}{2}\left(\frac{1}{c_1} p(B_1)^2 + \cdots + \frac{1}{c_\ell} p(B_\ell)^2\right).$$

Furthermore, in coordinates on $\mathfrak{g}^*_-$*, the (components of the) integral curves satisfy*

$$\dot{p}_i = -\sum_{j,k=1}^{m} c_{ij}^k p_k \frac{\partial H}{\partial p_j}, \quad i = 1, \ldots, m. \tag{9}$$

3 Optimal Control on the Euclidean Group $\mathsf{SE}(2)$

The Euclidean group

$$\mathsf{SE}(2) = \left\{ \begin{bmatrix} 1 & 0 \\ \mathbf{v} & R \end{bmatrix} : \mathbf{v} \in \mathbb{R}^{2\times 1} \quad \text{and} \quad R \in \mathsf{SO}(2) \right\}$$

is a (real) three-dimensional connected matrix Lie group. The associated Lie algebra is given by

$$\mathfrak{se}(2) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{bmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

Let

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

be the standard basis of $\mathfrak{se}(2)$ with the following table for the bracket operation

$[\cdot,\cdot]$	E_1	E_2	E_3
E_1	0	0	$-E_2$
E_2	0	0	E_1
E_3	E_2	$-E_1$	0

If we identify $\mathfrak{se}(2)$ with $\mathbb{R}^3$ by the isomorphism

$$\begin{bmatrix} 0 & 0 & 0 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{bmatrix} \in \mathfrak{se}(2) \mapsto \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3,$$

the expression for the Lie bracket becomes

$$\begin{aligned} \mathbf{x} \oslash \mathbf{y} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} (\mathbf{x} \wedge \mathbf{y}) \\ &= (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, 0). \end{aligned}$$

(Here, the symbol $\wedge$ denotes the usual vector product in $\mathbb{R}^3$.) Hence, we identify $\mathfrak{se}(2)$ with (the Lie algebra) $\mathbb{R}^3_{\oslash}$.

We identify (the dual space) $\mathfrak{se}(2)^*$ with matrices of the form

$$\begin{bmatrix} 0 & p_1 & p_2 \\ 0 & 0 & \frac{1}{2}p_3 \\ 0 & -\frac{1}{2}p_3 & 0 \end{bmatrix}$$

via the nondegenerate pairing given by the trace of the product. Thus, $se(2)^*$ is isomorphic to $\mathbb{R}^3$ via

$$\begin{bmatrix} 0 & p_1 & p_2 \\ 0 & 0 & \frac{1}{2}p_3 \\ 0 & -\frac{1}{2}p_3 & 0 \end{bmatrix} \in \mathfrak{se}(2)^* \mapsto \mathbf{p} = (p_1, p_2, p_3) \in \mathbb{R}^3$$

so that, in these coordinates, the pairing between $\mathfrak{se}(2)^*$ and $\mathfrak{se}(2)$ becomes

$$\langle \mathbf{p}, \mathbf{x} \rangle = p_1 x_1 + p_2 x_2 + p_3 x_3$$

(i.e. the usual scalar product in $\mathbb{R}^3$). Then each extremal curve $p(\cdot)$ in $\mathfrak{se}(2)^*$ is identified with a curve $P(\cdot)$ in $\mathfrak{se}(2)$ via the formula $\langle P(t), X \rangle = p(t)(X)$ for all $X \in \mathfrak{se}(2)$. Thus

$$P(t) = \begin{bmatrix} 0 & 0 & 0 \\ P_1(t) & 0 & -P_3(t) \\ P_2(t) & P_3(t) & 0 \end{bmatrix} \tag{10}$$

where $P_i(t) = \langle P(t), E_i \rangle = p(t)(E_i), \quad i = 1, 2, 3.$

The (minus) Lie-Poisson bracket on $\mathfrak{se}(2)^*$ is given by

$$\begin{aligned} \{F, G\}_-(p) &= -\sum_{i,j,k=1}^{3} c_{ij}^k p_k \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial p_j} \\ &= -\begin{vmatrix} p_1 & p_2 & 0 \\ \frac{\partial F}{\partial p_1} & \frac{\partial F}{\partial p_2} & \frac{\partial F}{\partial 0} \\ \frac{\partial G}{\partial p_1} & \frac{\partial G}{\partial p_2} & \frac{\partial G}{\partial p_3} \end{vmatrix} \\ &= -(p_1, p_2, 0) \bullet (\nabla F \times \nabla G). \end{aligned}$$

(Here, the covector $p = p_1 E_1^* + p_2 E_2^* + p_3 E_3^* \in \mathfrak{se}(2)^*$ is identified with the vector $\mathbf{P} = (P_1, P_2, P_3) \in \mathbb{R}^3$.) The equation of motion (4) becomes

$$\begin{aligned} \dot{F} &= \{F, H\}_- \\ &= -(p_1, p_2, 0) \bullet (\nabla F \times \nabla H) \\ &= \nabla F \bullet ((p_1, p_2, 0) \times \nabla H) \end{aligned}$$

and so

$$\begin{aligned} \dot{\mathbf{P}} &= (p_1, p_2, 0) \times \nabla H \\ &= \begin{bmatrix} 0 & 0 & P_2 \\ 0 & 0 & -P_1 \\ -P_2 & P_1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \\ \frac{\partial H}{\partial p_3} \end{bmatrix}. \end{aligned}$$

Hence, we get the following (scalar) equations of motion

$$\dot{P}_1 = \frac{\partial H}{\partial p_3} P_2 \tag{11}$$

$$\dot{P}_2 = -\frac{\partial H}{\partial p_3} P_1 \tag{12}$$

$$\dot{P}_3 = \frac{\partial H}{\partial p_2} P_1 - \frac{\partial H}{\partial p_1} P_2. \tag{13}$$

The function $C = P_1^2 + P_2^2$ is a Casimir function.

We consider the following left-invariant optimal control problem

$$\dot{g} = g\left(u_1 E_1 + u_2 E_2 + u_3 E_3\right), \quad u \in \mathbb{R}^3 \tag{14}$$

$$g(0) = g_0, \quad g(T) = g_1 \quad (g_0, g_1 \in \mathsf{SE}(2)) \tag{15}$$

$$\frac{1}{2}\int_0^T \left(c_1 u_1^2(t) + c_2 u_2^2(t) + c_3 u_3^2(t)\right) dt \to \min. \tag{16}$$

This problem is related to the Riemannian problem on the group of (rigid) motions of a plane (cf. [1], [6]). Note that the underlying control system is controllable.

Proposition 2. *Given the left-invariant optimal control problem (14)-(15)-(16), the extremal control* $\bar{u} = (\bar{u}_1, \bar{u}_2, \bar{u}_3)$ *is given by*

$$\bar{u}_1 = \frac{1}{c_1} P_1, \quad \bar{u}_2 = \frac{1}{c_2} P_2, \quad \bar{u}_3 = \frac{1}{c_3} P_3$$

where

$$\dot{P}_1 = \frac{1}{c_3} P_2 P_3 \tag{17}$$

$$\dot{P}_2 = -\frac{1}{c_3} P_1 P_3 \tag{18}$$

$$\dot{P}_3 = \left(\frac{1}{c_2} - \frac{1}{c_1}\right) P_1 P_2. \tag{19}$$

Proof. The reduced Hamiltonian (on $\mathfrak{se}(2)^* = \mathbb{R}^3$) is

$$H = \frac{1}{2}\left(\frac{1}{c_1} P_1^2 + \frac{1}{c_2} P_2^2 + \frac{1}{c_3} P_3^2\right). \tag{20}$$

The desired result now follows from **Proposition 1** and (11)-(12)-(13). □

It follows that the extremal trajectories (i.e., the solution curves of the reduced Hamilton equations) are the intersections of the ellipsoids $\frac{1}{c_1} P_1^2 + \frac{1}{c_2} P_2^2 + \frac{1}{c_3} P_3^2 = 2H$ and the cylinders $P_1^2 + P_2^2 = C$.

Remark. When $c = c_1 = c_2$, the reduced Hamilton equations (17)-(18)-(19) have the solutions

$$\begin{aligned} P_1(t) &= \sqrt{k_1} \sin\left(\frac{k_2}{c} t\right) \\ P_2(t) &= \sqrt{k_1} \cos\left(\frac{k_2}{c} t\right) \\ P_3(t) &= k_2, \end{aligned}$$

where $k_1 = P_1^2(0) + P_2^2(0)$ and $k_2 = P_3(0)$. In the general case, these equations can be explicitly integrated by Jacobi elliptic functions.

4 Stability

The equilibrium states are

$$P_{e1}^M = (M, 0, 0), \quad P_{e2}^M = (0, M, 0), \quad P_{e3}^M = (0, 0, M)$$

(here, $M \in \mathbb{R} \setminus \{0\}$) and the origin $(0, 0, 0)$.

Proposition 3. *The equilibrium state* $P_{e1}^M = (M, 0, 0)$ *has the following behaviour:*

(i) If $c_1 < c_2$, then it is unstable.

(ii) If $c_1 > c_2$, then it is nonlinearly stable.

Proof. (i) The matrix of the linearization of the system (at P_{e1}^M) is

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{c_3} M \\ 0 & \frac{c_2 - c_1}{c_1 c_2} M & 0 \end{bmatrix}$$

with eigenvalues

$$\lambda_1 = 0, \quad \lambda_{2,3} = \pm\sqrt{\frac{c_2 - c_1}{c_1 c_2 c_3}} M.$$

Since $c_1 < c_2$, it follows that either $\Re(\lambda_2)$ or $\Re(\lambda_3)$ is positive, hence (the equilibrium state) P_{e1}^M is unstable.

(ii) Let H_ψ be the (energy-Casimir) function given by

$$H_\psi(P_1, P_2, P_3) = \frac{1}{2c_1} P_1^2 + \frac{1}{2c_2} P_2^2 + \frac{1}{2c_3} P_3^2 + \psi\left(P_1^2 + P_2^2\right),$$

where $\psi \in C^\infty(\mathbb{R}, \mathbb{R})$. The first variation

$$\begin{aligned} \delta H_\psi &= \frac{1}{c_1} P_1 \delta_1 + \frac{1}{c_2} P_2 \delta_2 + \frac{1}{c_3} P_3 \delta_3 + \\ &\quad + (P_1 \delta_1 + P_2 \delta_2)\, \dot{\psi}\left(\frac{1}{2}(P_1^2 + P_2^2)\right) \end{aligned}$$

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equals zero at P_{e1}^M if and only if

$$\dot{\psi}\left(\frac{1}{2}M^2\right) = -\frac{1}{c_1}\cdot \tag{21}$$

The second variation (at P_{e1}^M)

$$\delta^2 H_\psi = M^2\,\ddot{\psi}\left(\frac{1}{2}M^2\right)\delta_1^2 + \left(\frac{1}{c_2} - \frac{1}{c_1}\right)\delta_2^2 + \frac{1}{c_3}\delta_3^2$$

is positive definite if and only if

$$\ddot{\psi}\left(\frac{1}{2}M^2\right) > 0. \tag{22}$$

The function

$$\psi(x) = x(x - c_1 - M^2)$$

satisfies the conditions (21) and (22). Hence, by the energy-Casimir method, the (equilibrium state) P_{e1}^M is nonlinearly stable. □

In a similar manner, we can prove the following result.

Proposition 4. *The equilibrium state* $P_{e2}^M = (0, M, 0)$ *has the following behaviour:*

(i) If $c_1 > c_2$, *then it is unstable.*

(ii) If $c_1 < c_2$, *then it is nonlinearly stable.*

The following result holds, but the proof will be omitted. (In this case, stronger methods for studying nonlinear stability are required as the energy-Casimir method does not work.)

Proposition 5. *The equilibrium state* $P_{e3}^M = (0, 0, M)$ *is nonlinearly stable.*

Remark. It turns out that the origin $(0, 0, 0)$ is also nonlinearly stable.

5 Final Remark

Invariant optimal control problems on other interesting matrix Lie groups (of low dimension) can also be considered. Further work (particularly, on control and stability) is in progress.

References

[1] Agrachev, A.A., Sachkov, Y.L., *Control Theory from the Geometric Viewpoint*, Springer-Verlag, 2004.

[2] Bloch, A.M., *Nonholonomic Mechanics and Control*, Springer-Verlag, 2003.

[3] Brockett, R.W., "System Theory on Group Manifolds and Coset Spaces," *SIAM J. Control*, V10, N2, pp. 265-284, 5/72

[4] Holm D.D., Marsden J.E., Ratiu T., Weinstein A., "Nonlinear Stability of Fluid and Plasma Equilibria," *Phys. Rep.*, V123, N1&2, pp.1-116, 7/85.

[5] Jurdjevic, V., "Non-Euclidean Elastica," *Amer J. Math.*, V117, N1, pp. 93-124, 2/95

[6] Jurdjevic, V. *Geometric Control Theory*, Cambridge University Press, 1997.

[7] Jurdjevic, V., Sussmann, H.J., "Control Systems on Lie Groups," *J. Diff. Equations*, V12, N2, pp. 313-329, 9/72

[8] Krishnaprasad, P.S., "Optimal Control and Poisson Reduction," *IEEE Conf on Decision & Control*, San Antonio, USA, 12/93 (Workshop on Mechanics, Holonomy and Control)

[9] Leonard, N.E., Krishnaprasad, P.S., "Motion Control of Drift-Free, Left-Invariant Systems on Lie Groups," *IEEE Trans on Automatic Control*, V40, N9, pp. 1539-1554, 9/95

[10] Marsden, J.E., Ratiu, T.S., *Introduction to Mechanics and Symmetry*, Second Edition, Springer-Verlag, 1999.

[11] Puta, M., *Hamiltonian Mechanical Systems and Geometric Quantization*, Kluwer, 1993.

[12] Puta, M., "Optimal Control Problems on Matrix Lie Groups," *Quad. Sem. Top. Alg. e Diff.*, Univ. Roma "La Sapienza", 1996.

[13] Puta, M., "Stability and Control in Spacecraft Dynamics,"' *J. Lie Theory*, V7, N2, pp. 269-278, e/97

[14] Puta M., Birtea P., Lăzureanu C., Pop C., Tudoran R., "Control, Integrability and Stability in some Concrete Mechanical Problems on Matrix Lie Groups," *Quad. Sem. Top. Alg. e Diff.*, Univ. di Roma "La Sapienza", 1998.

[15] Remsing, C.C., "Optimal Control and Hamilton-Poisson Formalism," *Int. J. Pure Appl. Math.*, V59, N1, pp. 11-17, 1/10

[16] Remsing, C.C., "Control and Integrability on $\mathsf{SO}(3)$," *Conf on Applied & Engineering Mathematics*, London, UK, pp. 1705-1710, 6/10

[17] Remsing, C.C., "Integrability and Optimal Control," *Symp on Mathematical Theory of Networks & Systems*, Budapest, Hungary, pp. 1749-1754, 7/10

[18] Sachkov, Y.L., "Maxwell Strata in the Euler Elastic Problem," *J. Dynam. Control Syst.*, V14, N2, pp. 169-234, 4/08

[19] Walsh, G., Montgomery, R., Sastry, S., "Optimal Path Planning on Matrix Lie Groups," *IEEE Conf on Decision & Control*, Lake Buena Vista, USA, pp. 1258-1263, 12/94

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