Proceedings of the World Congress on Engineering 2011 Vol I
WCE 2011, July 6 - 8, 2011, London, U.K.

Further Results on the Game of Cutblock

Alessandro Cincotti

***Abstract*—The game of Cutblock is the three-player variant of Cutcake, a classical combinatorial game. Even though to determine the solution of Cutcake is trivial, solving Cutblock is challenging because of the identification of queer games, i.e., games where no player has a winning strategy. New results about the classification of the instances of Cutblock are presented.**

***Index Terms*—combinatorial game, Cutblock, queer game.**

I. Introduction

THE game of Cutcake [1] is a classical two-player combinatorial game. Every instance of this game is defined as a set of rectangles of integer side-lengths with edges parallel to the x- and y- axes. The two players are often called Left and Right. A legal move for Left is to divide one of the rectangles into two rectangles of integer side-length by means of a single cut parallel to the x-axis and a legal move for Right is to divide one of the rectangles into two rectangles of integer side-length by means of a single cut parallel to the y- axis. Players take turns making legal moves until one of them cannot move. In the normal play convention, the first player unable to move is the loser. We recall that in the game of Cutcake the outcome for an $l \times r$ rectangle depends on the dimensions of l and r as shown in Table I. For example, in the 8×7 rectangle Left has a winning strategy and in the 3×4 rectangle Right has a winning strategy but the 7×4 rectangle is a zero-game.

The game of Cutblock (a three-player version of Cutcake) was introduced by Propp in [2]. Every instance of this game is defined as a set of blocks of integer side-lengths, with edges parallel to the $x-$, $y-$, and $z-$axes. We use $[l, c, r]$ to indicate an l by c by r block. A legal move for Left is to divide one of the blocks into two blocks of integer side-length by means of a single cut perpendicular to the $x-$axis; analogously, we define the legal moves for Center and Right. Players take turns making legal moves in cyclic fashion (. . ., Left, Center, Right, Left, Center, Right, . . .). When one of the three players is unable to move, he/she leaves the game and the remaining players continue in alternation until one of them is unable to move. Then that player leaves the game and the remaining player is the winner.

The paper is organized as follows. In Section 2, we recall the main definitions of three-player partizan games. In Section 3, we report the previous results about the classification of Cutblock. In the fourth section, we show our new results and in the last section future work is indicated.

II. Three-player partizan games

For the sake of self-containment, we recall the basic definitions and main results of a theory for three-player partizan games. Such a theory is an extension of Conway's theory of partizan games [3] and, as a consequence, it is both a theory of games and a theory of numbers.

TABLE I
OUTCOME CLASSES IN CUTCAKE

	Left starts	Right starts
$\lfloor \log_2 l \rfloor > \lfloor \log_2 r \rfloor$	Left wins	Left wins
$\lfloor \log_2 l \rfloor < \lfloor \log_2 r \rfloor$	Right wins	Right wins
$\lfloor \log_2 l \rfloor = \lfloor \log_2 r \rfloor$	Right wins	Left wins

TABLE II
OUTCOME CLASSES FOR NUMBERS

Class	Left starts	Center starts	Right starts
$=$	Right wins	Left wins	Center wins
$>_L$	Left wins	Left wins	Left wins
$>_C$	Center wins	Center wins	Center wins
$>_R$	Right wins	Right wins	Right wins
$=_{LC}$	Center wins	Left wins	Center wins
$=_{LR}$	Right wins	Left wins	Left wins
$=_{CR}$	Right wins	Right wins	Center wins
$<_{CR}$	?	Left wins	Left wins
$<_{LR}$	Center wins	?	Center wins
$<_{LC}$	Right wins	Right wins	?
$<$	?	?	?

Definition 1: If L, C, R are any three sets of games previously defined then $\{L|C|R\}$ is a game. All games are constructed in this way.

Definition 2: Let $x = \{L|C|R\}$ and $y = \{L'|C'|R'\}$ be two games.

- $x \geq_L y$ iff ($y \geq_L$ no x^C, $y \geq_L$ no x^R and no $y^L \geq_L x$),
- $x \geq_C y$ iff ($y \geq_C$ no x^L, $y \geq_C$ no x^R and no $y^C \geq_C x$),
- $x \geq_R y$ iff ($y \geq_R$ no x^L, $y \geq_R$ no x^C and no $y^R \geq_R x$),

where x^L, x^C, x^R are respectively the typical elements of L, C, R and y^L, y^C, y^R are respectively the typical elements of L', C', R'.

A special case of games can be considered to define what we call *numbers*.

Definition 3: If L, C, R are any three sets of numbers previously defined and

1) no element of L is $\geq_L$ any element of $C \cup R$, and
2) no element of C is $\geq_C$ any element of $L \cup R$, and
3) no element of R is $\geq_R$ any element of $L \cup C$,

then $\{L|C|R\}$ is a number. All numbers are constructed in this way.

Definition 4: We define the sum of two numbers as follows:

$$x + y = \{x^L + y, x + y^L | x^C + y, x + y^C | x^R + y, x + y^R\}.$$

All the given definitions are inductive, so that to decide whether $x \geq_L y$ we check the pairs (x^C, y), (x^R, y), and (x, y^L).

All numbers can be classified into 11 outcome classes as shown in Table II.

For further details, please refer to [4].

A. Cincotti is with the School of Information Science, Japan Advanced Institute of Science and Technology, 1-1 Asahidai, Nomi, Ishikawa 923-1292 Japan (phone: +81-761-51-1293; e-mail: cincotti@jaist.ac.jp).

ISBN: 978-988-18210-6-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

TABLE III
CLASSIFICATION OF CUTBLOCK

Class	$[l, c, r]$
$=$	$[1, 1, 1]$
$>_L$	$\lceil\log_2 l\rceil > \lceil\log_2 c\rceil + \lceil\log_2 r\rceil$
$>_C$	$\lceil\log_2 c\rceil > \lceil\log_2 l\rceil + \lceil\log_2 r\rceil$
$>_R$	$\lceil\log_2 r\rceil > \lceil\log_2 l\rceil + \lceil\log_2 c\rceil$
$=_{LC}$	$\lceil\log_2 l\rceil = \lceil\log_2 c\rceil$, $r = 1$
$=_{LR}$	$\lceil\log_2 l\rceil = \lceil\log_2 r\rceil$, $c = 1$
$=_{CR}$	$\lceil\log_2 c\rceil = \lceil\log_2 r\rceil$, $l = 1$
$<_{CR}$	$\lceil\log_2 l\rceil = \lceil\log_2 c\rceil + \lceil\log_2 r\rceil$, $l, c, r > 1$
$<_{LR}$	$\lceil\log_2 c\rceil = \lceil\log_2 l\rceil + \lceil\log_2 r\rceil$, $l, c, r > 1$
$<_{LC}$	$\lceil\log_2 r\rceil = \lceil\log_2 l\rceil + \lceil\log_2 c\rceil$, $l, c, r > 1$
$<$	otherwise

TABLE IV
OUTCOME CLASSES FOR A_1 AND A_2

Player	A	A_1	A_2
Center/Right	$<$	$<_{CR}$	$<_{CR}$
Center/Right	$<$	$<_{CR}$	$>_L$
Center	$<$	$=_{LR}$	$=_{LR}$
Right	$<$	$=_{LC}$	$=_{LC}$
Center/Right	$<_{CR}$	$>_L$	$>_L$
Right	$=_{LR}$	$>_L$	$>_L$
Right	$>_L$	$>_L$	$>_L$

III. PREVIOUS RESULTS

Cincotti [5] presents a classification of the instances of Cutblock using a three-player extension of partizan games as shown in Table III.

The case $G <_{CR} 0$ is particular interesting. By previous results, we know that when Left makes the first move, either Left has a winning strategy or the game is queer, i.e., no player has a winning strategy.

In the next section, we present some sufficient conditions to guarantee a win for Left.

IV. NEW RESULTS

Theorem 1: Let $G = [l, c, r] <_{CR} 0$ be an instance of Cutblock. If $c = 2^x$, and $r = 2^y$ with $x, y \geq 1$, then Left has a winning strategy when he/she makes the first move.

Proof: By hypothesis, we have

$$\lfloor\log_2 l\rfloor = \lfloor\log_2 c\rfloor + \lfloor\log_2 r\rfloor$$

Let us assume that Left moves

$$[l, c, r] \to [l_1, c, r] + [l_2, c, r]$$

where $l_1 = \lfloor l/2\rfloor$ and $l_2 = \lceil l/2\rceil$. We observe that $\lfloor\log_2 l_1\rfloor = \lfloor\log_2 l\rfloor - 1$ therefore, $[l_1, c, r] < 0$. Moreover, either $\lfloor\log_2 l_2\rfloor = \lfloor\log_2 l\rfloor - 1$ or $\lfloor\log_2 l_2\rfloor = \lfloor\log_2 l\rfloor$. Therefore, either $[l_2, c, r] < 0$ or $[l_2, c, r] <_{CR} 0$.

When Center or Right moves in a block A, then he/she will create two new blocks A_1 and A_2 as shown in Table IV. As a consequence, at the end of the first turn, i.e., after that Center and Right have made their first move, $G = B_1 + B_2 + B_3 + B_4$ as shown in Table V.

- In the first 8 cases, $G = B_1 + B_2 + B_3 + B_4 >_L 0$ therefore, Left has a winning strategy.
- In the cases 9, 10, and 11, $B_2 + B_3 + B_4 >_L 0$ therefore, Left can continue to play in this sub-game until Center or Right will be forced to play in B_1 and we will have $G >_L 0$.
- In the case 12, Left can play $B_1 \to B_{11} + B_{12} >_C 0$ therefore, Center has two moves of advantage but only one after that he/she plays in B_{11} or B_{12}. We note that to play $B_2 \to B_{21} + B_{22} >_L 0$ is not a good move for Center because it will give two moves of advantage to Left. Right has to play $B_3 \to B_{31} + B_{32} >_L 0$ or $B_4 \to B_{41} + B_{42} >_L 0$. In both cases, Left will have two moves of advantage and, at the end, he/she will be able to win the game.
- In the case 13, Left can play $B_3 \to B_{31} + B_{32} >_C 0$ therefore, Center has two moves of advantage but only one after that he/she plays in B_{31} or B_{32}. We note that to play in B_1, B_2, or B_4 is not a good move for Center because it will give two moves of advantage to Left. Right has to play $B_1 \to B_{11} + B_{12} >_L 0$ or $B_2 \to B_{21} + B_{22} >_L 0$. In both cases, Left will have two moves of advantage and, at the end, he/she will be able to win the game.
- The case 14 is similar to the case 13.
- In the case 15, Left can play $B_1 \to B_{11} + B_{12} < 0$. If Center and Right do not play in B_2, B_3, and B_4 to avoid to give Left two moves of advantage, then they must play in B_{11} and B_{12} therefore, at the end, the game G will become similar to the case 13 or 14.

■

TABLE V
OUTCOME CLASSES FOR B_1, B_2, B_3, AND B_4

Case	B_1	B_2	B_3	B_4
1	$<_{CR}$	$<_{CR}$	$<_{CR}$	$>_L$
2	$<_{CR}$	$<_{CR}$	$>_L$	$>_L$
3	$<_{CR}$	$=_{LC}$	$=_{LC}$	$>_L$
4	$<_{CR}$	$=_{LR}$	$=_{LR}$	$>_L$
5	$<_{CR}$	$=_{LR}$	$>_L$	$>_L$
6	$<_{CR}$	$>_L$	$>_L$	$>_L$
7	$=_{LC}$	$=_{LC}$	$>_L$	$>_L$
8	$=_{LR}$	$=_{LR}$	$>_L$	$>_L$
9	$<$	$<_{CR}$	$>_L$	$>_L$
10	$<$	$=_{LR}$	$>_L$	$>_L$
11	$<$	$>_L$	$>_L$	$>_L$
12	$=_{LC}$	$=_{LC}$	$=_{LR}$	$=_{LR}$
13	$<_{CR}$	$<_{CR}$	$=_{LC}$	$=_{LC}$
14	$<_{CR}$	$<_{CR}$	$=_{LR}$	$=_{LR}$
15	$<_{CR}$	$<_{CR}$	$<_{CR}$	$<_{CR}$

Theorem 2: Let $G = [l, c, r] <_{CR} 0$ be an instance of Cutblock. If $2^{x+1} + 2^x \leq l < 2^{x+2}$, and $c = 2^y$ with $x, y \geq 1$, then Left has a winning strategy when he/she makes the first move.

Proof: By hypothesis, we have

$$\lfloor\log_2 l\rfloor = \lfloor\log_2 c\rfloor + \lfloor\log_2 r\rfloor$$

Let us assume that Left moves

$$[l, c, r] \to [l_1, c, r] + [l_2, c, r]$$

where $l_1 = 2^{x+1}$ and $l_2 = l - l_1$. We observe that $\lfloor\log_2 l_1\rfloor = \lfloor\log_2 l\rfloor$ therefore, $[l_1, c, r] <_{CR}$. Moreover, $\lfloor\log_2 l_2\rfloor = \lfloor\log_2 l\rfloor - 1$ therefore, $[l_2, c, r] < 0$.

When Center moves in a block A, then he/she will create two new blocks A_1 and A_2 as shown in Table VI. As a consequence, after that Left and Center have made their first move, $G = B_1 + B_2 + B_3$ as shown in Table VII.

- In the first case, $G = B_1 + B_2 + B_3 >_L 0$ therefore, Left has a winning strategy.

ISBN: 978-988-18210-6-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

TABLE VI
OUTCOME CLASSES FOR A_1 AND A_2

A	A_1	A_2
$<$	$<_{CR}$	$<_{CR}$
$<$	$<_{CR}$	$>_L$
$<$	$=_{LR}$	$=_{LR}$
$<_{CR}$	$>_L$	$>_L$

TABLE VII
OUTCOME CLASSES FOR B_1, B_2, B_3

Case	B_1	B_2	B_3
1	$<_{CR}$	$<_{CR}$	$>_L$
2	$<_{CR}$	$<_{CR}$	$<_{CR}$
3	$<_{CR}$	$=_{LR}$	$=_{LR}$
4	$<$	$>_L$	$>_L$

- In the second and third case, $G = B_1 + B_2 + B_3 <_{CR} 0$ therefore, Left has a winning strategy because Right has to play.
- The last case is similar to the cases 9, 10, and 11 of the previous theorem.

■

By symmetry the following four theorems hold.

Theorem 3: Let $G = [l, c, r] <_{LR} 0$ be an instance of Cutblock. If $l = 2^x$, and $r = 2^y$ with $x, y \geq 1$, then Center has a winning strategy when he/she makes the first move.

Theorem 4: Let $G = [l, c, r] <_{LR} 0$ be an instance of Cutblock. If $2^{x+1} + 2^x \leq c < 2^{x+2}$, and $r = 2^y$ with $x, y \geq 1$, then Center has a winning strategy when he/she makes the first move.

Theorem 5: Let $G = [l, c, r] <_{LC} 0$ be an instance of Cutblock. If $l = 2^x$, and $c = 2^y$ with $x, y \geq 1$, then Right has a winning strategy when he/she makes the first move.

Theorem 6: Let $G = [l, c, r] <_{LC} 0$ be an instance of Cutblock. If $2^{x+1} + 2^x \leq r < 2^{x+2}$, and $l = 2^y$ with $x, y \geq 1$, then Right has a winning strategy when he/she makes the first move.

V. FUTURE WORK

In this paper, we have presented some sufficient conditions to guarantee a win for Left in the game $G <_{CR} 0$ when he/she makes the first move. Future work will concern the resolution of the following open problems:

- To find some sufficient conditions to identify queer games in the case $G <_{CR} 0$.
- To find some sufficient conditions to identify either a winner or a queer game in the case $G < 0$.

REFERENCES

[1] E. R. Berlekamp, J. H. Conway, and R. K. Guy, *Winning Ways For Your Mathematical Plays*. A K Peters, 2001.
[2] J. Propp, "The Game of Cutblock and Surreal Vectors," `http://jamespropp.org/surreal/text.ps.gz`.
[3] J. H. Conway, *On Numbers and Games*. A K Peters, 2001.
[4] A. Cincotti, "Three-player partizan games," *Theoretical Computer Science* vol. 332, no. 1-3, pp. 367–389, 2005.
[5] A. Cincotti, "The game of Cutblock," *INTEGERS: Electronic Journal of Combinatorial Number Theory* vol. 8, #G06, 2008.

ISBN: 978-988-18210-6-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)