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A Better Approximation to the Solution of Burger-Fisher Equation

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Abstract—**The Burger-Fisher equations occur in various areas of applied sciences and physical applications, such as modeling of gas dynamics, financial mathematics and fluid mechanics. In this paper, this equation has been solved by using a different numerical approach that shows rather rapid convergence than other methods. Illustrative examples suggest that it is a powerful series approach to find numerical solutions of Burger-Fisher equations.**

Index Terms—**Reduced differential transform method, Variational iteration method, Burger-Fisher Equation.**

I. INTRODUCTION

THE Burger-Fisher equation has important applications in various fields of financial mathematics, gas dynamic, traffic flow, applied mathematics and physics applications[8-16]. This equation shows a prototypical model for describing the interaction between the reaction mechanism, convection effect, and diffussion transport[7]. The Burger-Fisher equation uncovers Johannes Martinus Burgers (1895-1981) and Ronald Aylmer Fisher (1890-1962).

In this paper, our aim is to solve the Burger-Fisher equation using Reduced Differential Transformation Method (RDTM)[1]-[5] and to compare the results with those of the exact solution.

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The standart Burger-Fisher equation[6] can be written as

$$u_t - u_{xx} + \alpha u^{\gamma} u_x + \beta u(u^{\gamma} - 1) = 0, \qquad 0 \le x \le 1, t \ge 0, \qquad (1.1)$$

$$u(x,0) = \left(\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{\alpha\gamma}{2(1+\gamma)} x \right) \right)^{\frac{1}{\gamma}} \qquad (1.2)$$

$$u(x,t) = \left(\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{\alpha\gamma}{2(1+\gamma)} \left[x - \left(\frac{\alpha^2 + \beta(1+\gamma)^2}{\alpha(1+\gamma)} \right) t \right] \right) \right)^{\frac{1}{\gamma}} \qquad (1.3)$$

where, α, β, γ are non-zero parameters and $u_k(x) = \frac{\partial^k}{\partial x^k}$.

The proposed method in the solution process of this equation has been successfully applied to solve many types of linear and nonlinear equation as Kawahara, Gas dynamics, Nonlinear dispersive $K(m,n)$, Generalized Hirota-Satsuma coupled KdV and Coupled Modified KdV equations ([1]-[5]).

II. ANALYSIS OF THE METHOD

The basic definition of RDTM and that of its inverse can be given respectively as follow [3]:

Definition 2.1. If two dimensional function $u(x,t)$ is analytic over a specified interval of time t and spatial dimension x , then we define

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x,t) \right]_{t=0} \qquad (2.1)$$

where the t-dimensional spectrum function $U_k(x)$ is called the transformed function of u. Throughout this paper, the lowercase $u(x,t)$ represents the original function while the uppercase $U_k(x)$ stands for the transformed function with respect to time variable t.

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Definition 2.2. The differential inverse transform of $U_k(x)$ is defined as follows:

$$u(x,t)=\sum_{k=0}^{\infty} U_k(x)t^k \qquad (2.2)$$

Then combining equation (2.1) and (2.2) we write

$$u(x,t)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0} t^k \qquad (2.3)$$

Some basic operational rules of the RDTM that can be obtained from definitions (2.1) and (2.2) , are summarized in Table 1.

TABLE I
BASIC OPERATIONS IN RDTM

Function	Transformed Form
$u(x,t)$	$U_k(x)=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}$
$w(x,t)=u(x,t)\pm v$	$W_k(x)=U_k(x)\pm V_k(x)$
$w(x,t)=\alpha u(x,t)$	$W_k(x)=\alpha U_k(x)$ (α is a constant)
$w(x,y)=x^m t^n$	$W_k(x)=x^m\delta(k-n)$
$w(x,y)=x^m t^n u(x,t$	$W_k(x)=x^m U(k-n)$
$w(x,t)=u(x,t)v(x$	$W_k(x)=\sum_{r=0}^{k}V_r(x)U_{k-r}(x)=\sum_{r=0}^{k}U_r(x)V_{k-r}(x)$
$w(x,t)=\frac{\partial^r}{\partial t^r}u(x,t)$	$W_k(x)=\frac{\Gamma(k+r+1)}{\Gamma(k+1)}U_{k+r}(x),\ where$ $\Gamma(x+1)=\int_0^\infty e^t t^x dt\ \ is\ the\ gamma\ function$
$w(x,t)=\frac{\partial}{\partial x}u(x,t)$	$W_k(x)=\frac{\partial}{\partial x}U_k(x)$
$Nu(x,t)$	#Nonlinear function NF:=Nu(x,t): odr:=3: # Order u[t]:=sum(u[b]*t^b,b=0..odr): NF:=subs({Nu(x,t)=u[t]},NF): s:=expand(NF,t): dt:=unapply(s,t): for i from 0 to odr do n[i]:=((D@@i)(dt)(0)/i!): print(N[i],n[i]); # Transform Function

A detailed analysis of these operations can be seen in [17].

From the above definitions, it is clear that the idea behind the method stems from the concept of Taylor series expansion.

For the purpose of illustration of the proposed method, we write the gas dynamics equation in the standard operator form

$$L(u(x,t))+R(u(x,t))+N(u(x,t))=g(x,t) \qquad (2.4)$$

with initial condition

$$u(x,0)=f(x) \qquad (2.5)$$

where $L(u(x,t))=u_t(x,t)$ is a linear operator which has partial derivatives, $R(u(x,t))=u(x,t)$, $N(u(x,t))=\frac{1}{2}u_x^2(x,t)+u^2(x,t)$ is a nonlinear term and $g(x,t)$ is an inhomogeneous term.

According to the RDTM, we can construct the following recursive formula:

$$(k+1)U_{k+1}(x)=G_k(x)-N(U_k(x))-R(U_k(x)) \qquad (2.6)$$

where $R(U_k(x)), N(U_k(x))$ and $G_k(x)$ are the transformations of the functions $R(u(x,t)),\ N(u(x,t))$ and $g(x,t)$ respectively.

For the easy to follow of the reader, we can give the first few nonlinear term are

$$N_0=\frac{\partial}{\partial x}\left(\frac{U_0^2(x)}{2}\right)+U_0^2(x)$$

$$N_1=\frac{\partial}{\partial x}\left(\frac{2U_0(x)U_1(x)}{2}\right)+2U_0(x)U_1(x)$$

$$N_2=\frac{\partial}{\partial x}\left(\frac{2U_0(x)U_2(x)+U_1^2(x)}{2}\right)+2U_0(x)U_2(x)+U_1^2(x)$$

From initial condition (1.2), we write

$$U_0(x)=f(x) \qquad (2.7)$$

Substituting (2.7) into (2.6) and by a straight forward iterative calculations, we get the following $U_k(x)$ values.

Then the inverse transformation of the set of values $\{U_k(x)\}_{k=0}^{n}$ gives the approximation solution as,

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$$\tilde{u}_n(x,t) = \sum_{k=0}^{n} U_k(x) t^k \tag{2.8}$$

where n is order of approximation solution. Therefore, the exact solution of problem is given by

$$u(x,t) = \lim_{n\to\infty} \tilde{u}_n(x,t) \tag{2.9}$$

III. APPLICATIONS

In order to illustrate the efficiency and accuracy of RDTM for Burger-Fisher equations, we work on the following two examples.

Example 3.1. Let us consider the following Burger-Fisher[7] equation, for $\alpha = -1, \beta = \gamma = 1$

$$u_t - u_{xx} - uu_x + u(u-1) = 0 \tag{3.1}$$

with initial condition

$$u(x,0) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-x}{4}\right) \tag{3.2}$$

Then, by using the basic properties of the RDTM, we can find the transformed form of equation as,

$$(k+1)U_{k+1}(x) - \frac{\partial^2}{\partial x^2}U_k(x) - U_k(x)\frac{\partial}{\partial x}U_k(x) - U_k(x) + N_k(x) = 0 \tag{3.3}$$

where $N_k(x)$ is transformed form of $u^2(x,t)$.Using the initial condition (3.2), we have

$$U_0(x) = \frac{1}{2} + \frac{1}{2}\tanh\left(-\frac{x}{4}\right) \tag{3.4}$$

Now, substituting (3.1) into (3.2), we obtain the following values $U_k(x)$ for $k = 0..4$, successively

$$U_1(x) = \frac{1}{4}\frac{1}{\cosh\left(\frac{x}{4}\right)^2}, U_2(x) = \frac{1}{8}\frac{\sinh\left(\frac{x}{4}\right)}{\cosh\left(\frac{x}{4}\right)^3}, U_3(x) = \frac{1}{48}\frac{-3+2\cosh\left(\frac{x}{4}\right)^2}{\cosh\left(\frac{x}{4}\right)^4}$$

$$U_4(x) = \frac{1}{96}\frac{\sinh\left(\frac{x}{4}\right)\left(-3+\cosh\left(\frac{x}{4}\right)^2\right)}{\cosh\left(\frac{x}{4}\right)^5},$$

Then, the inverse transformation gives 4-terms approximation as,

$$\tilde{u}_4(x,t) = \sum_{k=0}^{4} U_k(x) t^k$$

$$= \frac{1}{48}\frac{-24\cosh\left(\frac{x}{4}\right)^4 + 24\sinh\left(\frac{x}{4}\right)\cosh\left(\frac{x}{4}\right)^3 - 12t\cosh\left(\frac{x}{4}\right)^2 - 6\sinh\left(\frac{x}{4}\right)t^2\cosh\left(\frac{x}{4}\right) + 3t^3 - 2t^3\cosh\left(\frac{x}{4}\right)^2}{\cosh\left(\frac{x}{4}\right)^5} \tag{3.5}$$

It is also noted here that the convergence of the approach can be increased by considering further terms in the the series solution Therefore, the exact solution of problem can be given by

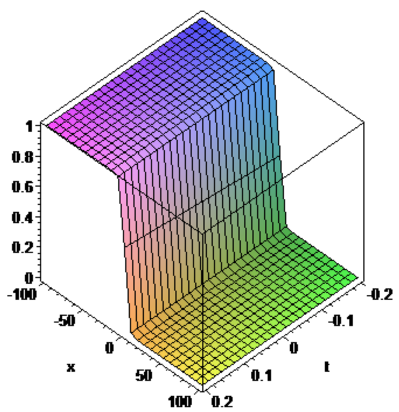
$$u(x,y) = \lim_{n\to\infty} \tilde{u}_n(x,y).$$

The graphical comparison of the above solution with variational iteration method (VIM)[11] has been given in Figure 1. It should be indicated here that all computations throughout this paper are performed in Maple 13 environment. The exact solution of the problem (3.1) turns to be

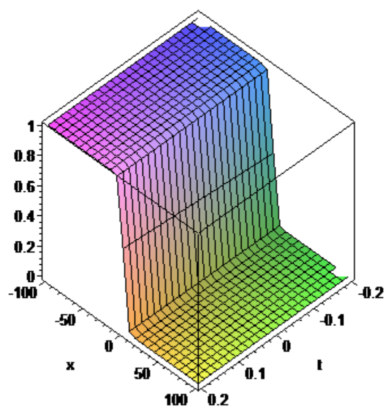
$$u(x,t) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{4} + \frac{5t}{8}\right)$$

Absolute errors of this approximation and VIM solution is also compared in Table 2.

(a)



(b)



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(c)

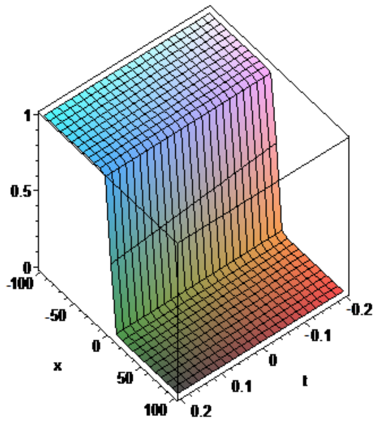

Fig. 1. (a) numerical results by 4-terms RDTM, (b) numerical results for u(x, t) by 3 terms-VIM, (c) graph of exact solution for example 3.1.

Example 3.2. Let us consider the following Burger-Fisher [7] equation, for $\alpha=-1,\beta=1,\gamma=2,$

$$u_t-u_{xx}-u^2u_x+u(u^2-1)=0 \tag{3.6}$$

with initial condition

$$u(x,0)=\sqrt{\frac{1}{2}+\frac{1}{2}\tanh\left(-\frac{x}{3}\right)} \tag{3.7}$$

Then, by using the basic properties of the RDTM, we can find the transformed form of equation as

$$(k+1)U_{k+1}(x)-\frac{\partial^2}{\partial x^2}U_k(x)-N_k(x)\frac{\partial}{\partial x}U_k(x)-U_k(x)+\tilde{N}_k(x)=0 \tag{3.8}$$

where $N_k(x)$, $\tilde{N}_k(x)$ is transformed form of $u^2(x,t)$, $u^3(x,t)$. Using the initial condition (3.7), we have

$$U_0(x)=\sqrt{\frac{1}{2}+\frac{1}{2}\tanh\left(-\frac{x}{3}\right)} \tag{3.9}$$

Now, substituting (3.1) into (3.2), we obtain the following $U_k(x)$ values successively

$$U_1(x)=\frac{1}{4}\sqrt{2-2\tanh\left(\frac{x}{3}\right)}\left(1+\tanh\left(\frac{x}{3}\right)\right)$$

$$U_2(x)=\frac{1}{16}\sqrt{2-2\tanh\left(\frac{x}{3}\right)}\left(-1+2\tanh\left(\frac{x}{3}\right)+3\tanh\left(\frac{x}{3}\right)^2\right)$$

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Finally, n -term approximate solution of problem (3.6) with (3.7) by the inverse transform of $U_k(x)$ gives

$$\tilde{u}(x,t)=\sum_{k=0}^{n}U_k(x)t^k \tag{3.10}$$

Then, the exact solution can be written as

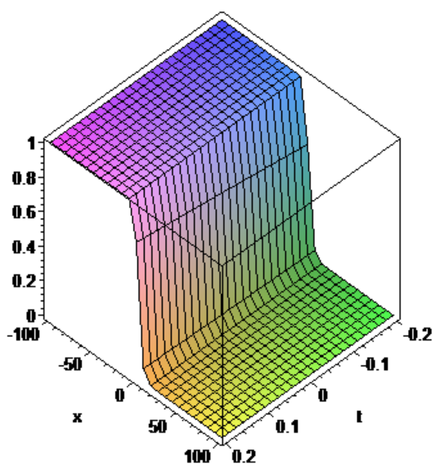
$$u(x,y)=\lim_{n\to\infty}\tilde{u}_n(x,y)$$

which is known to be

$$u(x,t)=\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{x}{4}+\frac{5t}{8}\right)$$

The graphical comparison of the 6-term RDTM and exact solution is given in Figure 2 and the absolute errors of RDTM for different x and t values are presented in Table 2.

(a)



(b)

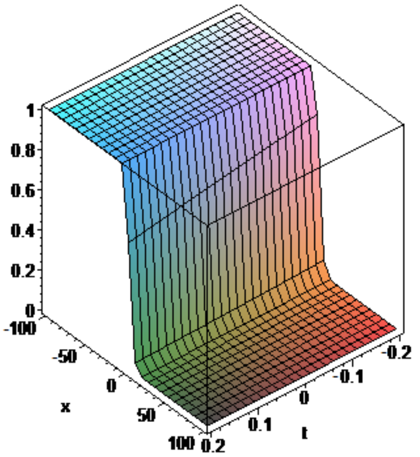


Fig. 2. Graphical representation of u(x, t), (a) by 6-term RDTM, and (b) by exact solution.

TABLE II

COMPARISON OF ABSOLUTE ERRORS OF THE SOLUTIONS OF BURGER-FISHER (B-F) EQUATION, BY RDTM AND BY HE'S VIM FOR DİFFERENT VALUES OF X AND T , ASSUMİNG $\alpha = \beta = 0.001$ AND $\gamma = 1$

x	t	$Exact$	$RDTM$	VIM	$RDTM\,(AbsoluteError)$	$VIM\,(AbsoluteError)$
0.01	0.02	0.5000000006	0.5000050000	0.5025031108	0.49994×10^{-5}	0.0025031102
	0.04	0.4999975006	0.5000075000	0.5025056144	0.99994×10^{-5}	0.0025081138
	0.06	0.4999950006	0.5000100000	0.5025081176	1.49994×10^{-5}	0.0025131170
	0.08	0.4999925006	0.5000125000	0.5025106212	1.99994×10^{-5}	0.0025181206
0.04	0.02	0.5000075025	0.5000125000	0.5100036984	0.49975×10^{-5}	0.0099961959
	0.04	0.5000050025	0.5000150000	0.5100061924	0.99975×10^{-5}	0.0100011899
	0.06	0.5000025025	0.5000175000	0.5100086932	1.49975×10^{-5}	0.0100061907
	0.08	0.5000000025	0.5000200000	0.5100111940	1.99975×10^{-5}	0.0100111915
0.08	0.02	0.5000175050	0.5000225000	0.5199969384	0.49950×10^{-5}	0.0199794334
	0.04	0.5000150050	0.5000250000	0.5199994288	0.99950×10^{-5}	0.0199844238
	0.06	0.5000125050	0.5000275000	0.5200019304	1.49950×10^{-5}	0.0199894254
	0.08	0.5000100050	0.5000300000	0.5200044208	1.99950×10^{-5}	0.0199944158

TABLE III

RESULTS OF RDTM AND THEİR ERRORS FOR $\alpha = \beta = 0.001$ AND $\gamma = 2$

t	x	Exact	RDTM	Absolute Error
0.01	0.02	0.7071079600	0.7071126733	$0.47133\text{x}10^{-5}$
	0.04	0.7071056030	0.7071150301	$0.94271\text{x}10^{-5}$
	0.06	0.7071032460	0.7071173875	$1.41415\text{x}10^{-5}$
	0.08	0.7071008890	0.7071197443	$1.88553\text{x}10^{-5}$
0.04	0.02	0.7071185680	0.7071232797	$0.47117\text{x}10^{-5}$
	0.04	0.7071162105	0.7071256365	$0.94260\text{x}10^{-5}$
	0.06	0.7071138540	0.7071279939	$1.41399\text{x}10^{-5}$
	0.08	0.7071114970	0.7071303506	$1.88536\text{x}10^{-5}$
0.08	0.02	0.7071327110	0.7071374214	$0.47104\text{x}10^{-5}$
	0.04	0.7071303540	0.7071397781	$0.94241\text{x}10^{-5}$
	0.06	0.7071279975	0.7071421355	$1.41380\text{x}10^{-5}$
	0.08	0.7071256400	0.7071444921	$1.88521\text{x}10^{-5}$

IV. CONCLUSIONS

In this study, RDTM has been successfully applied to find the solution of Burger-Fisher equation. As known, RDTM can be successfully designed to obtain approximate series solution of the time-dependent equations. It is seen, from this study, that the solution of Burger-Fisher equation by the RDTM leads to better results than the existing approaches as VIM. Therefore, the proposed method is a powerful, effective and at the same time, efficient method regarding algorithmic simplicity in the computer environment. It is also noteworthy that the results of the RDTM are in rather good agreement with the exact solutions of illustrative examples which are deliberately chosen for comparison reasons.

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