Unsteady MHD Laminar Boundary Layer Flow due to an Impulsively Stretching Surface

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***Abstract* - The unsteady, incompressible boundary layer flow caused by an impulsively stretching surface under the influence of a transverse magnetic field is investigated. The partial differential equations governing the laminar flow, under boundary-layer approximations, are non-dimensionalised using similarity transformations and then solved numerically using an efficient, implicit finite-difference scheme known as Keller-box method. The numerical solutions are obtained for all dimensionless time from initial unsteady state flow to final steady-state, uniformly valid in the whole spatial region. The numerical results for the surface shear stress are compared with those of the analytical approach results, and they are found to be in good agreement. It is observed that there is a smooth transition from the small time solution to the large time solution. The magnetic field significantly affects the flow field and skin friction coefficient. Indeed, skin friction coefficient found to decrease rapidly, initially, in small time interval before attaining a steady-state for large time.**

***Keywords*: Unsteady; Laminar boundary layer; impulsive – motion; Magnetic field.**

I. INTRODUCTION

Magneto-hydrodynamics (MHD) is the branch of continuum mechanics which deals with the study of electrically conducting fluids and electromagnetic forces. The field of MHD was initiated by Swedish physicist, Hannes Alfven for which he received in 1970 the Noble prize. The idea of MHD is that magnetic fields can induce currents in a moving conductive fluid, which create forces on the fluids, and also change the magnetic field itself. MHD problems arise in a wide variety of situations ranging from the explanation of the origin of Earth's magnetic field and the prediction of space weather to the damping of turbulent fluctuations in semiconductor melts during crystal growth and, even in the measurement of the flow rates of beverages in food industry. An interesting application of MHD to metallurgy lies in the purification of molten metals from non-metallic inclusions by the application of a transverse magnetic field.

In recent years, MHD flow problems have become more important industrially. Indeed, MHD laminar boundary layer behavior over a stretching surface is a significant type of flow having considerable practical applications in chemical engineering, electrochemistry and polymer processing.

In his pioneering work, Sakiadis [1] developed the flow field due to a flat surface, which is moving with a constant velocity in a quiescent fluid. Crane [2] extended the work of [1] for the two-dimensional problem where the surface velocity is proportional to the distance from the flat surface. As many natural phenomena and engineering problems are worth being subjected to MHD analysis, the effect of transverse magnetic field on the laminar flow over a stretching surface was studied by a number of researchers [3-5]

The studies reported above deal with steady flows. However, the flow problem will become unsteady due to impulsive change in the surface velocity of a moving stretching surface. The unsteady flow on a stretching surface is an important problem, since it is not always possible to maintain steady-state conditions. Pop and Na [6] and Nazar et.al [7] have considered the time-dependent boundary layer flow due to an impulsively stretching surface. Awang Kechil and Hashim [8] presented series solutions for unsteady boundary-layer flows due to impulsively stretched plate, in the absence of magnetic field.

The aim of the present paper is to investigate the unsteady MHD boundary layer development caused by an impulsively stretching surface in a constant pressure viscous flow, using numerical approach.

II. MATHEMATICAL FORMULATION

Let us consider the unsteady, laminar incompressible flow of a viscous electrically conducting fluid over a linearly stretched surface. The stretching surface is assumed to be electrically non-conducting. Prior to the time t=0, the surface is at rest in an unbounded quiescent fluid. At time t ≥ 0, the surface is suddenly stretched with velocity $U = ax$ along the distance (x) of the surface. The impulsive change in the surface velocity gives rise to unsteadiness in the flow field.

A transverse magnetic field of uniform strength $B(x)$ is applied in y-direction normal to the stretching surface throughout the fluid flow. It is assumed that the magnetic Reynolds number is small so that induced magnetic field is neglected, in comparison to the applied magnetic field. The boundary-layer equations based on conservation of mass and momentum, governing the unsteady two-dimensional flow on the impulsively stretching surface is:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (1)$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B^2(x)}{\rho}u \qquad (2)$$

subject to the boundary conditions

$$t \ge 0: u(x,y,t)=U=ax\ (a>0),\ v(x,y,t)=0\ at\ y=0$$
$$u(x,y,t)=0\ as\ y\to\infty \qquad (3)$$

and initial conditions

$$t<0:\ u(x,y,t)=v(x,y,t)=0$$

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Here u and v are velocity components along x and y-directions, respectively; σ, ρ and ν denote, respectively, electrical conductivity, density and kinematic viscosity; U is surface velocity and a is a positive constant.

The problem due to impulsive motion considered here should be formulated mathematically in such a way that for small time it should be governed by Rayleigh type of equation and for large time, by Crane type of equation. This implies that we have to select a scaling of y-coordinate which behaves like $y/(\nu t)^{1/2}$ for small time and as $(a/\nu)^{1/2} y$ for large time. Williams and Rhyne [9] have found such similarity transformations and they are expresses as

$$\eta = (a/\nu)^{1/2} \xi^{-1/2} y, \ \ \xi = 1 - \exp(-t^*), t^* = at$$

$$u = \frac{\partial \psi}{\partial y}, \ \ v = -\frac{\partial \psi}{\partial x}, \ \ \psi(x,y,t) = (a\nu)^{1/2} \xi^{1/2} x f(\eta,\xi)$$

$$u = axf'(\eta,\xi), v = -(a\nu)^{1/2} \xi^{1/2} f(\eta,\xi),$$

$$f' = \frac{\partial f}{\partial \eta}, M = \frac{\sigma B^2(x)}{a\rho} \qquad (4)$$

Using the above transformations in (1) and (2), we find that (1) is identically satisfied and (2) becomes

$$\frac{\partial^3 f}{\partial \eta^3} + \frac{1}{2}(1-\xi)\eta \frac{\partial^2 f}{\partial \eta^2} + \xi \left[f\left(\frac{\partial^2 f}{\partial \eta^2}\right) - \left(\frac{\partial f}{\partial \eta}\right)^2 - M\left(\frac{\partial f}{\partial \eta}\right)\right]$$

$$= \xi(1-\xi)\frac{\partial^2 f}{\partial \eta \partial \xi} \qquad (5)$$

The boundary conditions (3) becomes

$$f(0,\xi) = 0, \ \ \left.\frac{\partial f}{\partial \eta}\right|_{(0,\xi)} = 1; \ \ \left.\frac{\partial f}{\partial \eta}\right|_{(+\infty,\xi)} = 0 \qquad (6)$$

Here η and ξ are the transformed dimensionless independent variables; t* is the dimensionless time; ψ is the stream function; f is the dimensionless stream function; f' is the dimensionless velocity; M is the dimensionless magnetic field parameter.

The parameter of engineering interest is the skin friction coefficient (which indicates physically the surface shear stress) given by

$$C_f(x,\xi) = \frac{\tau_w}{\rho U^2/2} = (\xi \mathrm{Re}_x)^{-1/2} f''(0,\xi), 0 \le \xi \le 1 \qquad (7)$$

where the wall shear stress τ_w is given by

$$\tau_w = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0} \qquad (8)$$

with μ is the dynamic viscosity and $\mathrm{Re}_x = (ax^2/\nu)$ is the local Reynolds number.

The unsteady case can be divided into two cases:

1. Initial unsteady state flow (ξ = 0):-

When ξ = 0, corresponding to t* = 0, (5) becomes Rayleigh ordinary differential equation viz.,

$$\frac{\partial^3 f}{\partial \eta^3} + \left(\frac{\eta}{2}\right)\frac{\partial^2 f}{\partial \eta^2} = 0 \qquad (9)$$

subject to boundary conditions

$$f(0,0) = 0, \ \ f'(0,0) = 1, \ \ f'(+\infty,0) = 0 \qquad (10)$$

The above equation with boundary conditions (10) admits closed form solution and indeed, it has theexact solution

$$f(\eta,0) = \eta\, erfc(\eta/2) + \left(2/\sqrt{\pi}\right)\left[1 - \exp(-\eta^2/4)\right] \qquad (11)$$

where

$$erfc(\eta) = \left(2/\sqrt{\pi}\right)\int_{\eta}^{+\infty} \exp(-z^2)dz \qquad (12)$$

is the complementary error function.

2. Final steady state flow (ξ = 1):-

When ξ = 1, corresponding to t* →+∞, (5) becomes Crane type ordinary differential equation viz.,

$$\frac{\partial^3 f}{\partial \eta^3} + f\left(\frac{\partial^2 f}{\partial \eta^2}\right) - \left(\frac{\partial f}{\partial \eta}\right)^2 - M\left(\frac{\partial f}{\partial \eta}\right) = 0 \qquad (13)$$

with boundary conditions (10). The exact solution of (13) is given by

$$f(\eta,1) = \beta^{-1}\left[1 - \exp(-\beta\eta)\right] \qquad (14)$$

where $\beta = (1+M)^{1/2}$.

III. RESULTS AND DISCUSSION

The nonlinear partial differential equation (5) subject to boundary conditions (6) is solved numerically using an implicit finite-difference scheme known as Keller-box method, as described in [10]. This method is unconditionally stable and has a second order convergence.

The method has the following four main steps:

- Reduce (5) to a first order equation;
- Write the difference equations using central differences;
- Linearize the resulting algebraic equations by Newton's method and write them in matrix-vector form;
- Solve the linear system by block-tridiagonal-elimination technique.

To conserve the space, the details of the entire solution procedure of Keller-box method are not presented here.

Numerical computations were carried out for different values of the magnetic parameter M. The step size Δη in η-direction and the position of the edge of the boundary layer η_∞ have been adjusted to maintain the necessary accuracy. The values of Δη between 0.001 to 0.1 were used so that numerical solutions obtained are independent of Δη chosen, at least to four decimal places. However, a uniform grid Δη=0.01 was found to be satisfactory for a convergence criterion of 10^{-5} which gives accuracy to four decimal places. On other hand, the boundary layer thickness η_∞ between 6 to 10 was chosen where the infinity boundary-conditions are achieved.

In order to assess the accuracy of our method, the numerical values of $f''(0,\xi)$ for the range 0 ≤ ξ ≤ 1 obtained in this study, in the absence of magnetic field (M = 0), have been

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compared in Table 1, with those of Awang Kechil and Hashim[8]. Our numerical results are in good agreement with those of [8].

Table 1. Comparison of $f''(0,\xi)$ with those of Awang Kechil and Hashim [8] when M=0.0

ξ	Present Results	Awang Kechil and Hashim [8]
0	-0.5643740	-0.5643740
0.1	-0.6106120	-0.6150550
0.3	-0.7115610	-0.7115696
0.5	-0.8004117	-0.8018198
0.7	-0.8873160	-0.8856581
0.8	-0.9200550	-0.9252701
0.9	-0.9623398	-0.9633761
1.0	-1.0000000	-1.0000000

Further, we have compared our analytical solutions (exact solutions) of surface shear stress [$f''(0,\xi)$] for the steady state flow (ξ=1.0), in Table 2, in the presence of magnetic field M ($0\leq M \leq 1$). It is clear from the Table 2 that numerical results obtained by the Keller-box method are almost identical with those of exact solutions. Further, it is found that $f''(0,\xi)$ increases with the increase of magnetic field M. This is because when M increases, the Lorentz force produces more resistance to the transport phenomena which leads to the deceleration of the flow, enhancing the surface shear stress.

Table 2. Comparison of surface shear stress $f''(0,\xi)$ with exact solution (Analytical solution)

	ξ = 1.0	
M	Present Results	Exact solution
0.0	-1.000000	-1.000000
0.2	-1.095881	-1.095440
0.4	-1.183360	-1.183215
0.5	-1.224865	-1.224740
0.6	-1.265045	-1.264911
0.8	-1.341854	-1.341640
1.0	-1.414519	-1.414213

The variation of dimensionless velocity profile $f'(\xi,\eta)$ is illustrated in Fig.1, for different times, under the influence of uniform magnetic field M (M = 0.5).

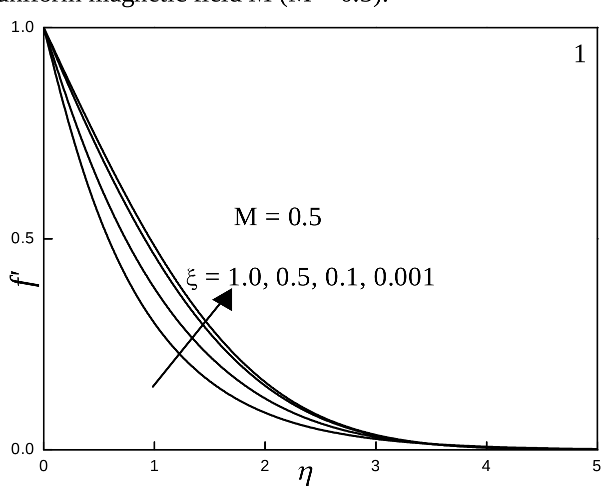

Fig.1 Velocity profiles for various values of ξ (M = 0.5)

It is evident from this figure that increase in ξ results in the reduction of momentum boundary layer thickness and thereby enhancing the velocity gradient at the surface. Further, the velocity profiles decrease monotonically with the distance from the surface and finally become zero for away from it, satisfying the boundary conditions asymptotically, and thus supporting the numerical results obtained.

The effect of magnetic field parameter (M) on the skin friction coefficient [$C_f(Re_x)^{1/2}$] is shown in Fig 2. At ξ = 0 (i.e., at t*= 0, at the start of impulsive motion), the velocity is independent of M, while the effect of M becomes more important as ξ increases. For a fixed ξ (ξ > 0), $C_f(Re_x)^{1/2}$ decrease with the magnetic parameter M and the effect of M becomes most significant at ξ = 1.0(i.e., as $t^* \to\infty$) when the steady state is reached. Further, the skin friction coefficient strongly depending on ξ, (See (7)) is found to decrease rapidly in a small time interval (0<ξ<0.4) after the start of the impulsive motion and reach the steady state near ξ = 0.7. There is a smooth transition from the small-time solution (unsteady state) to the large-time solution (steady state).

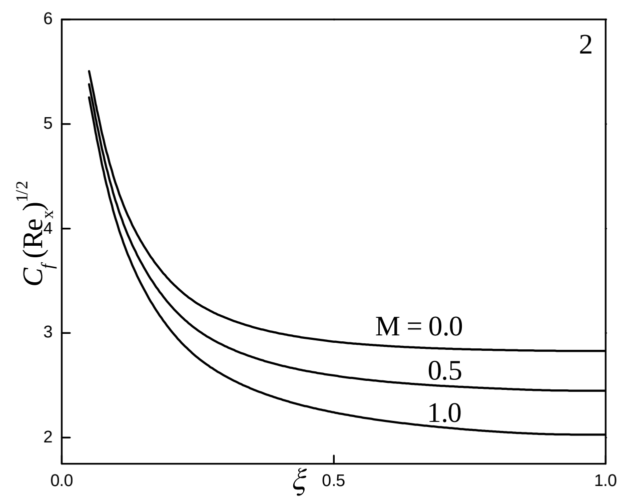

Fig. 2. Effect of magnetic field (M) on skin friction coefficient

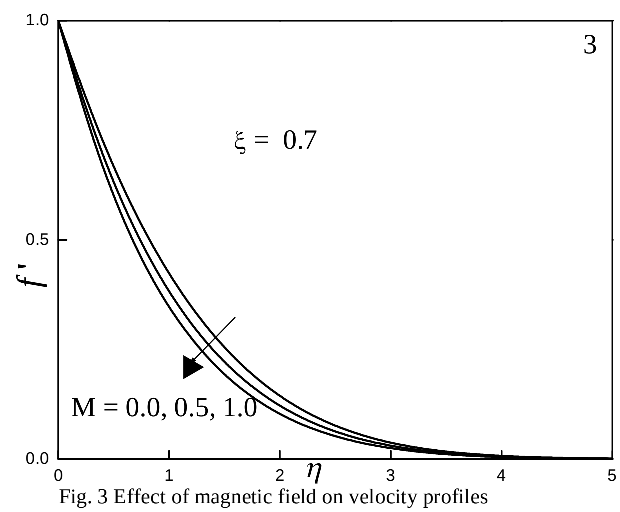

Fig. 3 Effect of magnetic field on velocity profiles

The effect of magnetic field on the corresponding velocity profile $f'(\xi,\eta)$ is shown in Fig 3. When ξ=0.7 (i.e., when steady state is reached), the magnetic field increases the velocity gradient at the surface resulting in the reduction of momentum boundary layer thickness.

IV. CONCLUSIONS

Numerical solutions of unsteady laminar boundary-layer flow caused by an impulsively stretching surface in the presence of a transverse magnetic field have been obtained. The present numerical results are compared with those of the analytical approach results and found them in good agreement. The magnetic field exerts significant influence on skin friction coefficient and reduces the momentum boundary layer thickness. There is a smooth transition from the small-time solution to the large-time solution.

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