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On the Electrical Current Trajectories for the Two-Dimensional Electrical Impedance Equation, When the Conductivity is a Cubic Polynomial Function

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Abstract—**Using the elements of Pseudoanalytic Function Theory, we analyze a set of electrical current trajectories for the two-dimensional Electrical Impedance Equation, obtained by means of formal powers, when the conductivity is a cubic polynomial separable-variables function. We show that the trajectories keep a pattern when changes in the conductivity take place.**

Keywords: Electrical Impedance Equation, pseudoanalytic functions

I. Introduction

The study of the Electrical Impedance Equation

$$\nabla \cdot (\sigma \nabla u) = 0, \tag{1}$$

where σ is the conductivity and u denotes the electric potential, is crucial for well understanding the Electrical Impedance Tomography problem. Since the relation between the two-dimensional case of (1) and a Vekua equation [11] was first noticed in [1], a complete new theory for the Electrical Impedance Equation is under construction.

One of the most important achievements was the possibility of expressing the general solution of (1) in analytic form, by means of Taylor series in formal powers, which is a mathematical tool reached from the Classical Pseudoanalytic Function Theory [3].

Since more powerful computational resources are now available, the numerical solutions of (1) can be virtually approached with arbitrary accuracy, if we are able to express σ as a separable-variables function [4], [5].

Following the idea of a numerical method posed for obtaining piecewise separable-variables functions, when a finite set of conductivity values is given [9], this work employs the electrical current traces provoked by a cubic polynomial conductivity σ, within the unitary circle. We intend to show that, at least from the point of view of the first formal powers, a pattern is preserved when changes in the conductivity appear.

We introduce some elements of the Pseudoanalytic Function Theory [3], [6], in order to construct in exact form the first formal powers with real coefficients equal to 1. Then, we trace a small set of electrical current paths, corresponding to a constant conductivity, in order to establish a reference to explore the traces provoked by two cases of inhomogeneous media.

We close discussing the existence of a pattern, remarking the possible relevance of this fact, when applying elements of Pseudoanalytic Function Theory to cases located, somehow, nearer to real medical imaging.

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II. Preliminaries

Let (F, G) be a pair of complex-valued functions such that

$$\textbf{Im}\left(\overline{F}G\right) > 0. \tag{2}$$

Thus, any complex-valued function W can be expressed by means of the linear combination of (F, G):

$$W = \phi F + \psi G,$$

where ϕ and ψ are real. A pair of functions satisfying (2) is called a *Bers generating pair* [3]. Indeed, it is possible to introduce the *derivative in the sense of Bers* of W, or the (F, G)-derivative, according to the following expression:

$$\partial_{(F,G)} W = (\partial_z \phi) F + (\partial_z \psi) G, \tag{3}$$

where $\partial_z = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$, and i denotes the standard imaginary unit $i^2 = -1$. But (3) will exist if and only if

$$(\partial_{\overline{z}} \phi) F + (\partial_{\overline{z}} \psi) G = 0, \tag{4}$$

where $\partial_{\overline{z}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$ (notice the operators ∂_z and $\partial_{\overline{z}}$ are traditionally introduced with the factor $\frac{1}{2}$, nevertheless it will result somehow more comfortable not to work with it in this paper).

By introducing the notations

$$A_{(F,G)} = \frac{\overline{F}\partial_z G - \overline{G}\partial_z F}{F\overline{G} - \overline{F}G}, \tag{5}$$

$$a_{(F,G)} = \frac{\overline{G}\partial_{\overline{z}} F - \overline{F}\partial_{\overline{z}} G}{F\overline{G} - \overline{F}G},$$

$$B_{(F,G)} = \frac{F\partial_z G - G\partial_z F}{F\overline{G} - \overline{F}G},$$

$$b_{(F,G)} = \frac{F\partial_{\overline{z}} G - G\partial_{\overline{z}} F}{F\overline{G} - \overline{F}G};$$

the equation (3) will turn into

$$\partial_{(F,G)} W = \partial_z W - A_{(F,G)} W - B_{(F,G)} \overline{W},$$

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and (4) will become

$$\partial_{\overline{z}}W - a_{(F,G)}W - b_{(F,G)}\overline{W} = 0, \tag{6}$$

where $\overline{W}$ denotes the complex conjugation of W.

The last expression is known as the *Vekua equation* [11], and it will play a central role in our further analysis. The functions defined in (5) are known as *characteristic coefficients* of the generating pair (F, G), and the complex-valued functions W fulfilling (6) are called (F, G)-pseudoanalytic.

The following concepts can be found in [3].

Definition 1: Let (F_0, G_0) and (F_1, G_1) be two generating pairs, and let their characteristic coefficients fulfil

$$\begin{aligned} a_{(F_0,G_0)} &= a_{(F_1,G_1)}, \\ B_{(F_0,G_0)} &= -b_{(F_1,G_1)}. \end{aligned} \tag{7}$$

Then (F_1, G_1) will be called a successor pair of (F_0, G_0), as well (F_0, G_0) will be the predecessor pair of (F_1, G_1).

Definition 2: Let the set of generating pairs $\{(F_n, G_n)\}$, $n = 0, \pm1, \pm2, ...$ be such that every pair (F_{n+1}, G_{n+1}) is a successor of (F_n, G_n). Then the set $\{(F_n, G_n)\}$ will be called a Bers generating sequence. If $(F, G) = (F_0, G_0)$, we say that (F, G) is embedded within $\{(F_n, G_n)\}$.

Definition 3: Let $\{(F_n, G_n)\}$, $n = 0, \pm1, \pm2, ...$ be a Bers generating sequence. If there exist a number k such that for every n the equality $(F_n, G_n) = (F_{n+k}, G_{n+k})$ holds, we say that $\{(F_n, G_n)\}$ is a periodic sequence with period k.

Remark 1: It is very important to notice that if W is an (F_n, G_n)-pseudoanalytic function, its (F_n, G_n)-derivative will be (F_{n+1}, G_{n+1})-pseudoanalytic. This implies that if we wish to obtain the m-derivative in the sense of Bers of a (F_0, G_0)-pseudoanalytic function, we need to posses in explicit form all pairs belonging to the generating sequence $\{(F_n, G_n)\}$, going from $n = 0$ until $n = m$.

L. Bers also defined the (F, G)-integral of a function W. We refer the reader to [3] or [6] for examining the requirements for its existence, since in this work we will use exclusively (F, G)-integrable functions.

Definition 4: Let (F_0, G_0) be a generating pair. Its adjoint pair (F_0^*, G_0^*) will be defined according to the formulas

$$F_0^* = -\frac{2\overline{F}_0}{F_0\overline{G}_0 - \overline{F}_0 G_0}, \quad G_0^* = \frac{2\overline{G}_0}{F_0\overline{G}_0 - \overline{F}_0 G_0}.$$

Definition 5: The (F, G)-*integral* of W is defined as

$$\int_{z_0}^{z_1} W \ d_{(F,G)}z =$$

$$= F(z_1)\,\mathbf{Re}\int_{z_0}^{z_1} G^* W \ dz + G(z_1)\,\mathbf{Re}\int_{z_0}^{z_1} F^* W \ dz.$$

Particularly, if $W = \phi F + \psi G$ is (F, G)-pseudoanalytic, then

$$\int_{z_0}^{z} \partial_{(F,G)} W \ d_{(F,G)}z =$$

$$= W(z) - \phi(z_0) F(z) - \psi(z_0) G(z),$$

and due to $\partial_{(F,G)} F = \partial_{(F,G)} G = 0$, this expression represents the antiderivative in the sense of Bers of $\partial_{(F,G)} W$.

A. Taylor series in formal powers

It is well known that any analytic function φ with respect to the complex variable $\overline{z} = x - iy$, can be expressed by means of Taylor series $\varphi = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, where $a_n = \frac{1}{n!}\partial_{\overline{z}}^n \varphi(z_0)$, $\partial_{\overline{z}}^n = \frac{\partial^n}{\partial \overline{z}^n}$, $z = x + iy$, and z_0 is a fixed point in the complex plane.

L. Bers posed an analog representation for an (F_0, G_0)-pseudoanalytic function W employing what he called *formal powers*.

Definition 6: The formal power $Z_m^{(0)}(a_0, z_0; z)$, with coefficient a_0, center at z_0, and depending upon z is defined as

$$Z_m^{(0)}(a_0, z_0; z) = \lambda_m F_m + \mu_m F_m,$$

where the complex constants λ_m and μ_m fulfil

$$\lambda_m F(z_0) + \mu_m G(z_0) = a_0.$$

The formal powers with higher exponents are obtained according to the recursive formulas

$$Z_m^{(n)}(a_n, z_0; z) = n\int_{z_0}^{z} Z_{m-1}^{(n-1)}(a_n, z_0; z)\, d_{(F_m,G_m)}z.$$

Notice these integral operators are integrals in the sense of Bers.

Remark 2: The formal powers $Z_m^{(n)}(a_n, z_0; z)$ posses the following properties:

1) $Z_m^{(n)}(a_n, z_0; z)$ are (F_m, G_m)-pseudoanalytic.
2) If a_1 and a_2 are real constants, we have

$$Z_m^{(n)}(a_1 + ia_2, z_0; z) =$$

$$= a_1 Z_m^{(n)}(1, z_0; z) + a_2 Z_m^{(n)}(i, z_0; z).$$

3) $Z_m^{(n)}(a_n, z_0; z) \to a_n (z - z_0)^n$ when $z \to z_0$.

Then, any (F_0, G_0)-pseudoanalytic function W can be represented through these formal powers

$$W = \sum_{n=0}^{\infty} Z_m^{(n)}(a_n, z_0; z), \tag{8}$$

where

$$a_n = \frac{1}{n!}\partial_{(F_m,G_m)}^n W(z_0).$$

From this point of view, it is possible to assert that the expansion (8) is an analytical representation for the general solution of the Vekua equation (6).

III. Pseudoanalytic function theory and the two-dimensional Electrical Impedance Equation

K. Astala and L. Païvarïnta showed in [1] that the two-dimensional Electrical Impedance Equation (1) was fully equivalent to a special kind of Vekua equation, and this was the departure point for giving a positive answer to the Electrical Impedance Tomography problem in the plane.

Using this relation, Kravchenko et al. [7] published what could well be considered one of the first general solutions of (1) in exact form, for a certain class of conductivity functions σ.

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After that, the relation between (1) and a the Vekua equation was rediscovered in a variety of works (see e.g. [6] and [10]). In order to achieve our purposes, we will employ the techniques described in [9]. Basically, when σ is a separable-variables function

$$\sigma = \sigma_1(x_1)\,\sigma_2(x_2),$$

introducing the notations

$$W = -\sqrt{\sigma}\frac{\partial}{\partial x_2}u - i\sqrt{\sigma}\frac{\partial}{\partial x_1}u, \tag{9}$$

$$\partial_{\overline{z}} = \frac{\partial}{\partial x_2} + i\frac{\partial}{\partial x_1},$$

$$p = \frac{\sqrt{\sigma_1(x_1)}}{\sqrt{\sigma_2(x_2)}};$$

the two-dimensional Electrical Impedance Equation (1) will turn into the Vekua Equation

$$\partial_{\overline{z}}W - \frac{\partial_{\overline{z}}p}{p}\overline{W} = 0. \tag{10}$$

Very interesting works have been dedicated to the numerical construction of the formal powers for this equation [5], specifically for the case when $\sigma = e^{k_1x_1+k_2x_2}$, where k_1, k_2 are real constants. Moreover, based upon the analytic approach of its first formal powers, with coefficients 1 and i, [9] posed an analysis of current paths, intending to show that a pattern is kept when comparing qualitatively the electrical current trajectories of inhomogeneous media, with those traced in the homogeneous case.

Indeed, trying to make wider the bridge between this new branch of the mathematical theory for the Electrical Impedance Equation, and its applications for the Electrical Impedance Tomography, also in [9] was posed a technique for approaching separable-variables conductivity functions, given a matrix of conductivity values associated with a two-dimensional domain, a very natural case in medical imaging.

The proposal is based onto the construction of a piecewise function, obtained by applying standard cubic polynomial interpolation. The preliminary trials [8] indicate that it can be useful as a departure point for more specific experimental designs.

The goal is, of course, the numerical approach of the formal powers using piecewise functions, in order to approach the solutions of boundary value problems, a critical matter in the procedure for approaching the solution of a tomography problem.

Moreover, already in [4] was shown that by applying the well known Gram-Scmidt method to the formal powers, it is possible to obtain a complete orthonormal system for the electric potentials at the boundary.

Among all these interesting facts, the qualitative behavior of the electrical current paths, can be a useful tool to prevent unstable behaviors of the numerical calculations, a major challenge always present in Electrical Impedance Tomography.

A. Formal powers in analytic form and electrical current trajectories

The interpolation method posed in [9] considers, for every subsection of the domain of interest, a conductivity function of the form

$$\sigma(x_1,x_2) = (\alpha x_1+\beta)\left(ax_2^3+bx_2^2+cx_2+d\right). \tag{11}$$

According to the notations introduced in the section of Preliminaries, we will have that

$$\begin{aligned} p &= \frac{\sqrt{\alpha x_1+\beta}}{\sqrt{ax_2^3+bx_2^2+cx_2+d}}, \\ F_0 &= \frac{\sqrt{\alpha x_1+\beta}}{\sqrt{ax_2^3+bx_2^2+cx_2+d}}, \\ G_0 &= i\frac{\sqrt{ax_2^3+bx_2^2+cx_2+d}}{\sqrt{\alpha x_1+\beta}}; \end{aligned}$$

and that

$$F_0^* = -i\frac{\sqrt{\alpha x_1+\beta}}{\sqrt{ax_2^3+bx_2^2+cx_2+d}},$$

$$G_0^* = \frac{\sqrt{ax_2^3+bx_2^2+cx_2+d}}{\sqrt{\alpha x_1+\beta}}.$$

Notice the generating pair (F_0, G_0) is embedded into a periodic generating sequence with period 2 [6]. Thus, we can approach in exact form the first and the second formal powers with coefficients 1, centers at $z_0 = 0$, depending upon $z = x_2 + ix_1$.

We will have

$$Z_0^{(0)}(1,0;z) = \frac{\sqrt{\alpha x_1+\beta}}{\sqrt{ax_2^3+bx_2^2+cx_2+d}},$$

and

$$\begin{aligned} Z_0^{(0)}(1,0;z) = {} & \frac{\sqrt{\alpha x_1+\beta}}{\sqrt{ax_2^3+bx_2^2+cx_2+d}}\cdot \\ & \cdot\left(\frac{a}{4}x_2^4+\frac{b}{3}x_2^3+\frac{c}{2}x_2^2+dx_2\right)+ \\ & +i\frac{\sqrt{ax_2^3+bx_2^2+cx_2+d}}{\sqrt{\alpha x_1+\beta}}\cdot \\ & \cdot\left(\frac{\alpha}{2}x_1^2+\beta x_1\right). \end{aligned} \tag{12}$$

B. Electrical current trajectories

In order to establish a comparison point, let us suppose the conductivity σ is constant. The Vekua equation (10) will reduce to the Cauchy-Riemann equation $\partial_{\overline{z}}W = 0$.

Let us consider the second power of the Taylor series $z = x_2+ix_1$ (the first term does not reach any interesting trajectory, since only possesses one spatial component).

According to the differential Ohm's Law $\overrightarrow{j} = -\sigma\nabla u$ and to the notations (9), we have that

$$\overrightarrow{j} = \left(\sqrt{\sigma}\mathbf{Re}z, \sqrt{\sigma}\mathbf{Im}z\right).$$

In Figure 1, we pose the example of eight current traces that intersect the radius of the unitary circle at the angles $0, \frac{\pi}{2}, \pi$ and $\frac{3}{2}\pi$. The reader should keep in mind that this illustration,

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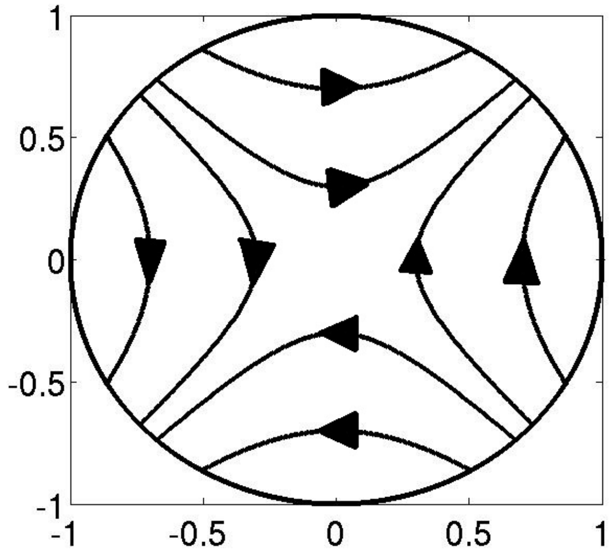

Fig. 1. Electrical current traces for the homogeneous case.

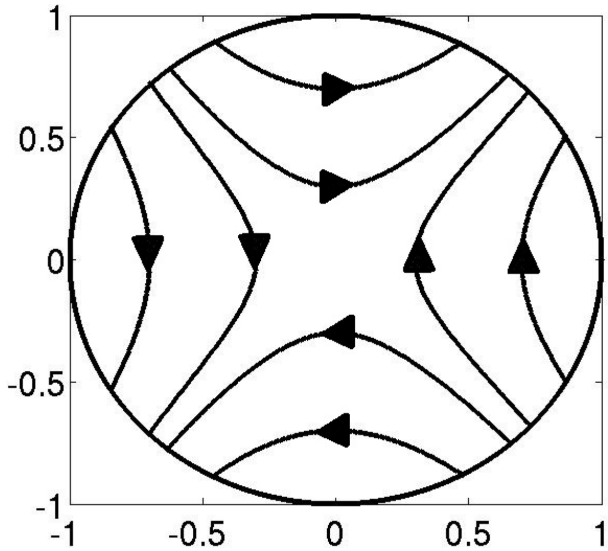

Fig. 2. Electrical current traces for the first inhomogeneous case.

as the rest will, only remarks the electrical current trajectories, without taking into account their magnitudes.

We can identify four *groups* of current paths, and this pattern is the one we might take as our main reference.

Let us consider now the formal power (12). The electrical current vector will be

$$\overrightarrow{j}=\left(\begin{array}{c}\sqrt{\sigma}\mathbf{Re}Z_0^{(0)}(1,0;z)\\ \sqrt{\sigma}\mathbf{Im}Z_0^{(0)}(1,0;z)\end{array}\right)=$$
$$=\left(\begin{array}{c}(\alpha x_1+\beta)\left(\frac{a}{4}x_2^4+\frac{b}{3}x_2^3+\frac{c}{2}x_2^2+dx_2\right)\\ \left(\frac{\alpha}{2}x_1^2+\beta x_1\right)\left(ax_2^3+bx_2^2+cx_2+d\right)\end{array}\right).$$

Figure (2) contains the traces corresponding to the coefficients $\alpha=1,\beta=3,a=1,b=11,c=1,d=25$. The set of values were selected only to ensure the absence of zero-valued points within the unitary circle. As the reader can appreciate, remarking again the illustration does not contain information concerning to the electrical current magnitudes, the pattern of the traces is very similar to the homogeneous case.

Of course, when medical imaging is considered, the difference between the conductivities of the human tissues, will be very significant (see e.g. [2]). Thus, at least from the numerical point of view, we may have the possibility of finding values very close to zero.

In order to test the proposal, let us select the set of values $\alpha=1,\beta=3,a=1,b=11,c=1,d=10^{-6}$. This will force the appearance of an axis, parallel and very close to x_1, on which the conductivity will almost vanish numerically. The reader will appreciate in Figure (3) that the current trajectories are indeed altered when crossing upon the mentioned axis, but we might say that the pattern does not experiment considerable changes.

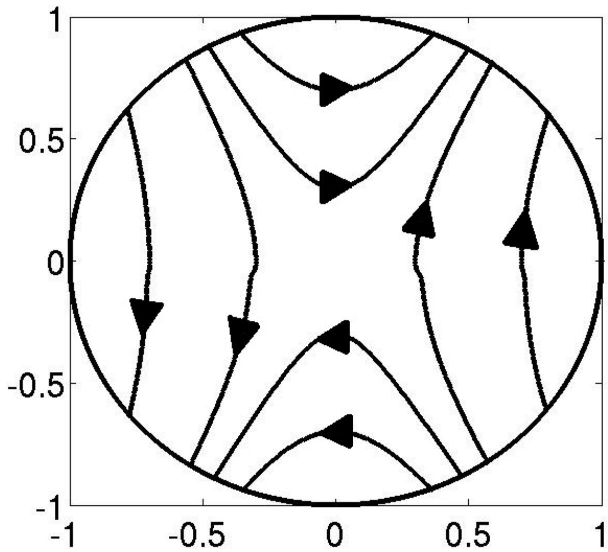

Fig. 3. Electrical current traces for the second inhomogeneous case.

IV. Conclusions

The study of patterns found when observing the electrical current trajectories, traced within an inhomogeneous media, could well reach useful information for preventing undesirable numerical behavior, when applying Pseudoanalytic Function Theory to the study of the Electrical Impedance Equation.

Specifically, when considering cubic polynomial separable-variables conductivity functions, the analysis shows the current patterns keep a *stable* behavior when changes occur to the conductivity σ, at least from the point of view of the electrical current paths.

Since this case is precisely the one corresponding to the numerical method that approaches piecewise separable-variables conductivity functions, the results presented in this work indicate that this approach can work as a good departure point for the construction of more detailed experimental methods.

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On the light of this information, we may remark the Pseudoanalytic Function Theory is becoming a useful tool for engineering cases.

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