A Generalized Stochastic Model for Number of Events in Equilibrium Renewal Process (ERP)

V.K.Singh and B.P.Singh

Abstract - **If a system, producing point events has been running for a long time before it is first observed, one obtains an equilibrium renewal process (ERP). Such renewal processes are common in real life experiments where the experimenter enters the system abruptly at a fixed point t_0, a long time after the start of the process and counts the number of events of his interest upto the time point $t_0 + t$ .The present paper is aimed at to develop a generalized model for number of point events in ERP with k –point state space of the process. Some special cases of the model have been dealt with. Expressions for mean and variance of the model have been obtained for estimating parameters of the model.**

Key words - **Equilibrium renewal process, moments, state space, stochastic model**

I. INTRODUCTION

If in a process interest lies on the individual occurrences of the events themselves; events being distinguished only by their position in time, then the process may be looked upon as a point process. Consider such a point process consisting of a series of point events occurring in time, such as, stops of a machine taking running time of the machine as the time scale, fibre entanglements occurring along the length of a textile yarn, emissions from a radioactive source, mutations, conceptions to a married women, etc. In such a process, the starting point of the process may be characterized by $t = 0$ and one may be interested into the number of events occurring in the interval $(0, t]$. However, in most of the practical situations, experimenter starts taking observations for a fixed duration of time, starting from an abrupt point of time t_0, which generally becomes a quite distant point from $t = 0$. It is well- known in renewal theory that if a point process has been in effect for a long time and the count of observations along the time axis started at a later date then the distribution of events is quite different from ordinary or modified renewal process and such a process is termed as 'Equilibrium Renewal Process' (ERP) ([1] - [2]). Thus, if $X_i, (i = 1,2, \dots)$ is the random variable characterizing the waiting time between (i - 1)thand ith events and $X_2, X_3, \dots$ are independently and identically distributed (i.i.d.) random variables with probability density function (p.d.f.) $f(x)$ but X_1 has different distribution $f_1(x)$ given by $f_1(x) = [1 - F(x)]/\mu$; where $\mu = E(x_i)$ $(i = 2,3, \dots)$and $F(x)$ is the distribution function (d.f.) corresponding to $f(x)$, then the process is an ERP.

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If in any point process, the prime objective is to observe the inherent characteristics of the process by estimating the parameters then it would be advisable to prefer the data on inter-arrival times (I.A.T.) due to the reasons that such data is readily available and generally involves no memory bias. The present work is devoted to the development of a stochastic model for number of events in ERP under a set of realistic assumptions which generally holds true in a number of situations as mentioned above. Some particular cases of practical importance have been presented. Following the results of renewal theory, expressions for mean and variance of the model have been obtained.

II. BASIC ASSUMPTIONS

Let an experiment started at time t = 0 and the experimenter started to count the number of events of his interest in a continuous one-dimensional time point process from the point $t = t_0$ where t_0 is a distant point from $t = 0$, the starting point of the process. Let he carried out his experiment up to the point $t_0 + t$, measured from $t = 0$. If $t_r, (r = 2,3, \dots)$ represents the I.A.T. between (r-1)th and rthevents, measured from t_0; then according to ERP, $t_2, t_3, \dots$ are i.i.d. random variables and t_1 has a different distribution than that of $t_r, (r = 2,3, \dots)$. In order to develop stochastic model for $X(t_0, t)$, the number of events recorded during the period $(t_0, t_0 + t)$, we make the following assumptions :

A1: The state space of the process is discrete consisting of k states and $\theta_p, (p = 1,2, \dots, k)$ is the probability that process is in the state S_p, we observe that

$$0 < \theta_p < 1 \ \forall p \text{ and } \sum_{p=1}^{k} \theta_p = 1.$$

Let $\theta = col. (\theta_1, \theta_2, \dots, \theta_k)$ be a $k \times 1$ vector.

A2: As soon as an event occurs, the system, producing events, is locked for a constant period of time depending upon the type of the event, during which no further event may occur. Let us term this period as 'dead time' or 'inoperative period'. After the inoperative period is over, the system becomes free to record another event after a random gap of time. Let $h_p, (p = 1,2, \dots, k)$ be the inoperative period associated with the state S_p. For avoiding arbitrariness in the choice of h_p values, we assume that $\{h_p : p = 1,2, \dots, k\}$ is an increasing sequence of time

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intervals with $h_0 = 0$. Let $h = col.(h_1, h_2, \ldots, h_k)$ be a $k \times 1$ vector.

A3: The probability of occurrence of an event in the time interval $(t, t+\Delta t)$ posterior to t_0 is $\lambda\Delta t + O(\Delta t), \lambda > 0$, if the event occurs prior to $t - h_p$, assuming that the system was in the state S_p

A4: The process is capable of producing point events during the entire observational period.

Remark 1

The blockage of the system for some period after the occurrences of each event, as mentioned in A2, is a common phenomenon in many of the processes. For example, an electronic counter measuring the emissions from a radioactive source, bank ATMs, married women having conceptions are subject to such blockages.

III. THE MODEL

Let the events be counted from t_0 and $Z_r = \sum_{i=1}^{r} t_i$ be the time of i[th] event. Then under the assumptions A1-A4, we have the following theorem:

THEOREM 1. The p.d.f. of $X(t_0, t)$ is given by

$$P_0 = 1 - Q_0(t) + Q_1(t) \qquad (3.1)$$

$$P_r = Q_{r-1}(t) - 2Q_r(t) + Q_{r+1}(t);\ r = 1, 2, \ldots, n \qquad (3.2)$$

where $n = t/h_1$ or $[(t + h_1)/h_1]$ according as t is a multiple of h_1 or not; $[q]$ stands for the greatest integer not exceeding q and

$$Q_r(t) = \frac{\lambda}{1+\lambda\bar{h}}\left[\sum_{a_1=0}^{b_r}\sum_{a_2=0}^{b_r-a_1}\sum_{a_3=0}^{b_r-a_1-a_2}\cdots\sum_{a_{k-1}=0}^{b_r-a_1-a_2-\cdots-a_{k-2}}\right.$$

$$\left\{\prod_{c=0}^{k-2}\binom{r-\sum_{j=0}^{c}a_j}{a_{c+1}}\theta_{k-c}^{a_{c+1}}\theta_1^{r-\sum_{j=0}^{c}a_j}\right\}$$

$$\left.\int_{\Phi_{r,k}}^{t}\left\{1 - e^{-\lambda(x-\Phi_{r,k})}\sum_{s=0}^{r-1}\frac{\lambda^s(x-\Phi_{r,k})^s}{s!}\right\}d_x\right],$$

$$x > \Phi_{r,k} \qquad (3.3)$$

with

$$b_r = min.\left[r, \frac{t - r h_1}{h_k - h_1}\right],$$

$$\Phi_{r,k} = \left\{r - \sum_{j=0}^{k-1}a_j\right\}h_1 + \sum_{d=1}^{k-1}a_d h_{k-d+1},$$

$a_0 = 0$, $\bar{h} = \theta' h$.

The term $\binom{r-\sum_{j=0}^{c}a_j}{a_{c+1}}$ in (3.3) stands for the combinations $(r - \sum_{j=0}^{c}a_j)c_{a_{c+1}}$. The proof of the theorem is presented in the Appendix.

IV. SOME PERTICULAR CASES

It is apparent that the model (3.1), (3.2) may be applied to a variety of situations due to flexibility of its probability expressions. However, in the absence of suitable estimation procedures for estimating all the parameters of the model, it might not be applicable to practical situations in its present form. We have, therefore, presented some of its particular cases for application purposes:

Case I: Model with two - point state space (i.e. k = 2)

In many point processes, the state space generally consists of only two states, viz., 'on' and 'off' positions in an electric circuit; 'defective' and 'non-defective' items in a manufacturing process, 'foetal wastages' and 'live births' in human reproduction process, etc. For such systems the expression (3.3) reduces to.

$$Q_r(t) = \frac{\lambda}{1+\lambda\bar{h}} - r$$

$$+\frac{1}{1+\lambda\bar{h}}\sum_{a_1=0}^{r}\binom{r}{a_1}\theta_2^{a_1}\theta_1^{r-a_1}e^{-\lambda(t-a_1h_2-\overline{r-a_1}h_1)}$$

$$\sum_{s=0}^{r-1}\sum_{g=0}^{s}\frac{\lambda^g(t-a_1h_2-\overline{r-a_1}h_1)^g}{g!},$$

$$\text{if} \quad b_r = r \qquad (4.1)$$

$$= \sum_{a_1=0}^{r}\binom{r}{a_1}\theta_2^{a_1}\theta_1^{r-a_1}\left[\frac{\lambda(t-a_1h_2-\overline{r-a_1}h_1)}{1+\lambda\bar{h}} - \frac{1}{1+\lambda\bar{h}}\right.$$

$$+\frac{1}{1+\lambda\bar{h}}e^{-\lambda(t-a_1h_2-r-a_1h_1)}$$

$$\left.\sum_{s=0}^{r-1}\sum_{g=0}^{s}\frac{\lambda^g(t-a_1h_2-\overline{r-a_1}h_1)^g}{g!}\right],$$

$$\text{if} \quad b_r \neq r \qquad (4.2)$$

with $\bar{h} = \sum_{i=1}^{k}\theta_i h_i$.

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Case II: Model with three - point state space (i.e. k = 3)

There are certain stochastic processes which comprises of three - point state space. For instance, a valve which is subject to failure, may be inspected each day and be classified as being in one of the three states : satisfactory, unsatisfactory, failed; a conception to a married women may result in a foetal wastage, still birth or live birth, etc. For such systems, the expression (3.3) may be obtained as

$$Q_r(t) = \frac{\lambda t}{1+\lambda\bar{h}} - r$$

$$+\frac{1}{1+\lambda\bar{h}}\sum_{a_1=0}^{r}\sum_{a_2=0}^{r-a_1}\binom{r}{a_1}\binom{r-a_1}{a_2}\theta_3^{a_1}\theta_2^{a_2}\theta_1^{r-a_1-a_2}$$

$$.\,e^{-\lambda\left(t-a_1h_3-a_2h_2-\overline{r-a_1-a_2}h_1\right)}$$

$$\sum_{s=0}^{r-1}\sum_{g=0}^{s}\frac{\lambda^g\left(t-a_1h_3-a_2h_2-\overline{r-a_1-a_2}\,h_1\right)^g}{g!}$$

if $b_r = r$ (4.3)

$$=\sum_{a_1=0}^{b_r}\sum_{a_2=0}^{b_{r-a_1}}\binom{r}{a_1}\binom{r-a_1}{a_2}\theta_3^{a_1}\theta_2^{a_2}\theta_1^{r-a_1-a_2}$$

$$\Bigg[\frac{\lambda\left(t-a_1h_3-a_2h_2-\overline{r-a_1-a_2}\,h_1\right)}{1+\lambda\bar{h}}$$

$$-\frac{1}{1+\lambda\bar{h}}+\frac{1}{1+\lambda\bar{h}}e^{-\lambda\left(t-a_1h_3-a_2h_2-\overline{r-a_1-a_2}h_1\right)}$$

$$\sum_{s=0}^{r-1}\sum_{g=0}^{s}\frac{\lambda^g\left(t-a_1h_3-a_2h_2-\overline{r-a_1-a_2}\,h_1\right)^g}{g!}$$

if $b_r \neq r$,

with $\bar{h} = \sum_{i=1}^{3}\theta_i h_i$. (4.4)

Remark 2

It is clear from above that with increasing value of k, the number of unknown parameters of the model also increases. Thus, for k=2, we have four parameters, λ, h_1, h_2 and θ_1 to be estimated while with k=3, there are six parameters, $\lambda, h_1, h_2, h_3, \theta_1$ and θ_2. The number of parameters with k - point state space would be, therefore, 2k. In the absence of suitable estimation procedures for estimating all the parameters of the model, it would be, therefore, advisable to assume known values of some of the parameters and to estimate rest of the parameters so as to apply the model to some practical situations. Assuming some of the parameters to be known from past experience or otherwise, the application of the model with k = 2 and 3 have been demonstrated in [3] and [4] respectively.

V. MOMENTS OF THE MODEL

In order to apply the model to some empirical data for testing its suitability, it is necessary to know the values of the parameters involved or to estimate them by some suitable mathematical or statistical methods. One such method is to derive the expressions of moments of the model and then to apply the method of moments for estimating the underlying parameters.

It is obvious that due to complex nature of the model, the higher order moments of the model are difficult to obtain. However, the mean and variance of the model may respectively be obtained as

$$E[X(t_0,t)] = \sum_{r=0}^{n} r P_r = Q_0(t) = \frac{\lambda t}{1+\lambda\bar{h}} \qquad (5.1)$$

and

$$V[X(t_0,t)] = \sum_{r=0}^{n} r^2 P_r - E[X(t_0,t)]^2 \qquad (5.2)$$

respectively. For large t, an approximate expression for $V[X(t_0,t)]$ in ERP is given in [1] as

$$V[X(t_0,t)] = \frac{\sigma^2 t}{\mu^3} + \left[\frac{1}{6} + \frac{\sigma^4}{2\mu^4} - \frac{\mu_3}{3\mu^3}\right] + 0(1) \qquad (5.3)$$

where

$\sigma^2 = E[X_i - \mu]^2$; $\mu_3 = E[X_i - \mu]^3$; $(i = 2, 3, \ldots)$ and 0(1) is a term of order zero.

Using the p.d.f. of t_r, (r =2, 3, …), we get

$$\mu = E(t_r) = \sum_{p=1}^{k}\theta_p\left(\frac{1}{\lambda}+h_p\right) = \frac{1}{\lambda}+\bar{h} \qquad (5.4)$$

$$\mu_2' = \sum_{p=1}^{k}\left\{\int_{h_p}^{\infty} t^2\theta_p\,\lambda e^{-\lambda(t-h_p)}d_t\right\}$$

$$= \frac{2}{\lambda^2}+\frac{2}{\lambda}\bar{h}+\sum_{p=1}^{k}\theta_p h_p^2 \qquad (3.5)$$

so that

$$\sigma^2 = \frac{1}{\lambda^2}+\sum_{p=1}^{k}\theta_p h_p^2 - \bar{h}^2 \qquad (5.6)$$

and

$$\mu_3' = \frac{6}{\lambda^3}+\frac{6}{\lambda^2}\bar{h}+\frac{3}{\lambda}\sum_{p=1}^{k}\theta_p h_p^2+\sum_{p=1}^{k}\theta_p h_p^3 \qquad (5.7)$$

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μ_3 can be evaluated with (5.4), (5.5) and (5.7).

VI. CONCLUDING REMARKS

The suggested stochastic model for the number of renewals in a continuous time point process has a wider application in many of the random experiments where the experimenter suddenly enters into the running system with a view to take observation for a fixed time interval and thereby estimating the characteristics of the system. Examples of some of the stochastic processes where this model has applications are already mentioned in the above paragraphs. The applicability of some particular cases of the model has already been tested in human reproduction processes where conceptions occurred to a married female in certain time interval were noted down. Since the probability expressions are in closed form, standard methods of estimation could be developed for estimating some of the underlying parameters. In other similar stochastic situations also the model may be applied if suitable empirical data are available.

REFERENCES

[1] D. R. Cox. and H. D. Miller, H.D. : "The Theory of Stochastic Processes", 2nd Edition, *Methuen and Company Limited*, London, 1970, pp. 340 – 345.

[2] S. Biswas : "Stochastic Processes in Demography and Applications", *Wiley Eastern Limited*, New Delhi, India, 1988, pp. 75 – 79.

[3] S. N. Singh and R. C. Yadava, R.C. : "A generalized probability model for an equilibrium birth process", *Demography India*, Vol. 6, pp. 163-173, 1970.

[4] S. N. Singh, V. K. Singh and R. N. Singh, R.N. : "A probability model for number of conceptions in equilibrium birth process", *Journal of the Indian Society of Agricultural Statistics,* Vol. 37, No. 2, pp. 145-153, 1985.

[5] V. K. Singh : "A generalized stochastic model for the most recent inter- arrival time", *Communications in Statistics – Theory and Methods*, Vol. 26, No. 6, pp. 1411-1424, 1997.

APPENDIX

If t_0 is at a considerable distance from the origin, the process is in ERP, hence the p.d.f. of t_1 and $t_r\,(r>1)$ under the assumptions are respectively.

$$f_1(x) = \frac{1-F(x)}{\mu}$$

and

$$f(x) = \begin{cases} 0 & ; \quad 0 < x < h_1 \\ \sum_{r=1}^{s-1} \theta_r \lambda e^{-\lambda(x-h_r)} & ; \quad h_{s-1} < x < h_s \quad (s = 2,3,\ldots,k) \\ \sum_{r=1}^{k} \theta_r \lambda e^{-\lambda(x-h_r)} & ; \quad x > h_k \end{cases} \tag{A.1}$$

where $\mu = E(t_r), (r = 2,3,\ldots)$ and $F(x) = \int_0^x f(z)\,dz$.
We observe that

$$\mu = \frac{1}{\lambda} + \bar{h} \tag{A.2}$$

and

$$F(x) = \begin{cases} 0 & ; \quad 0 < x < h_1 \\ \sum_{r=1}^{s-1} \theta_r \{1 - e^{-\lambda(x-h_r)}\} & ; \quad h_{s-1} < x < h_s \quad (s = 2,3,\ldots,k) \\ 1 - \sum_{r=1}^{k} \theta_r \lambda e^{-\lambda(x-h_r)} & ; \quad x > h_k \end{cases} \tag{A.3}$$

Since $t_1, t_2, t_3, \ldots$ are i.i.d. random variables, the d.f. of the waiting times of r^{th} event since the point t_0, $F_r^*(t)$, will be

$$F_r^*(t) = F_1(t) * F^{(r)}(t) \tag{A.4}$$

where $F_1(t)$ is the d.f. corresponding to $f_1(x)$ and $F^{(r)}(t)$ is the r-fold convolution of $F(t)$ with itself. * stands for the convolution of distribution functions. Let $L(s)$ be the Laplace transform of $f(x)$, then the same for Z_r will be

$$\frac{\{1-L(s)\}\{L(s)\}^{r-1}}{\mu s} \tag{A.5}$$

Inverting (A.5), the p.d.f. of Z_r, can be obtained as

$$F_r^*(t) = \frac{1}{\mu} F^{(r-1)}(t) - \frac{1}{\mu} F^{(r)}(t) \tag{A.6}$$

where $F^{(r)}(t)$ is

$$F^{(r)}(t) = \sum_{a_1=0}^{b_r} \sum_{a_2=0}^{b_{r-a_1}} \sum_{a_3=0}^{b_{r-a_1-a_2}} \cdots \sum_{a_{k-1}=0}^{b_{r-a_1-a_2-\cdots,a_{k-2}}}$$

$$\left\{ \prod_{s=0}^{k-2} \binom{r - \sum_{j=0}^{s} a_j}{a_{s+1}} \theta_{k-s}^{a_s+1} \theta_1^{r-\sum_{j=0}^{s} a_j} \right\}$$

$$\left\{ 1 - e^{-\lambda(x-\Phi_{r,k})} \sum_{s=0}^{r-1} \frac{\lambda^s (x - \Phi_{r,k})^s}{s!} \right\}$$

Similar expression for $F^{(r)}(t)$ has also been derived in [5].

Now, following [2], we have

$$Q_r(t) = \int_0^t \frac{F^{(r)}(t)}{\mu} dt$$

and hence (3.1) and (3.2) follow. It can be seen that the total number of events n, during the interval $t_0, t_0 + t$ of length t can be exceed t/h_1 or $\{(t+h_1)/h_1\}$ according as t is a multiple of h_1 or not. From (A6) it is easy to observe that

$$Q_0(t) = \frac{\lambda t}{(1+\lambda\bar{h})}$$

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