On Solvability of the Nonlinear Optimal Control Problem for Processes Described by the Semi-linear Parabolic Equations

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In the paper we investigate solvability of nonlinear optimal Thermal and Diffusion processes control problems defined by semi-linear parabolic equations when the source function nonlinearly depends on the controlling parameters, and nonlinear integral criterion of quality is minimized. We obtained the sufficient conditions of uniqueness of the solution of the boundary value problem and the adjoint boundary value problem of control process. It is established that optimal control is described as a solution of the complex structure nonlinear integral equation with additional condition in the form of the differential inequality, and the algorithm to determination of the optimal control was constructed.

semi-linear parabolic equations, weak generalized solution, nonlinear optimization problem, optimality conditions, differential inequality, nonlinear integral equation of optimal control, factorization method.

I. INTRODUCTION

IN this paper we investigate the solvability of the nonlinear optimal control problem for the process described by semi-linear parabolic equations. We consider the case when the source function depends on controlling parameters, and a nonlinear integral criterion of quality is minimized. We obtain sufficient conditions of uniqueness of the solution of the boundary value problem and its adjoint problem. It is established that the optimal control is described as the solution of the complex structure nonlinear integral equation with additional condition in the form of inequality, and the construction algorithm for the determination of the optimal control is given.

II. BOUNDARY PROBLEM OF CONTROLLED PROCESS AND ITS SOLUTION

Let us a consider controlled process describable by scalar function $V(t,x)$, which in the region $Q_T = Q\times(0,T)$ satisfies a semi-linear parabolic equation in $[7,16]$

$$V_t - AV = \varphi\left[t,x,V(t,x)\right] + g(x)f\left[t,u(t)\right] \tag{1.1}$$

and the initial conditions

$$V(0,x)=\psi(x),\ x\in Q \tag{1.2}$$

and the boundary conditions

$$\Gamma V(t,x) \equiv \sum_{i,j=1}^{n} a_{ij}(x)V_{x_j}(t,x)\cos(\nu,x_i) + a(x)V(t,x)=0, x\in\gamma, 0<t<T \tag{1.3}$$

Here $\varphi\left[t,x,V(t,x)\right]$ is a given function nonlinearly depending on controlled process state $V(t,x)\in H(Q_T)$; $f\left[t,u(t)\right]\in H(0,T)$ is a given function of external source nonlinearly depending on the control function $u(t)\in H(0,T)$; Q is bounded region of space R^n with piecewise smooth boundary γ; ν is outer exterior normal at the point $x\in\gamma$; $g(x)\in H(Q)$, $\psi(x)\in H(Q)$ are given functions; operator A is defined as

$$AV(t,x) \equiv \sum_{i,j=1}^{n}\left(a_{ij}(x)V_{x_j}(t,x)\right)_{x_j} - c(x)V(t,x) \tag{1.4}$$

is elliptic in the closed domain $\bar{Q}=Q\cup\gamma$; $a(x)\geq 0$; $c(x)\geq 0$ are bounded measurable functions; H is Hilbert space; T is a fixed constant.

Let ψ_n and g_n - Fourier coefficients of functions $\psi(x)$ and $g(x)$ respectively; $\{z_n(x)\}$- be a complete orthonormal system of eigen functions of the following problem in $H(Q)$ [14]

$$Az(x) = -\lambda z(x),\ x\in Q,$$
$$\Gamma z(x)=0,\ x\in\gamma, \tag{1.5}$$

$\{\lambda_n\}$- is a corresponding sequence of eigen values, where $\lambda_n\leq\lambda_{n+1}$ and $\lim\limits_{n\to\infty}\lambda_n=\infty$.

Definition 1.1 *A function* $V(t,x)\in H(Q_T)$, *at every fixed control* $u(t)\in H(Q_T)$ *which satisfy nonlinear integral equation*

$$V(t,x)=\sum_{n=1}^{\infty}\left[\psi_n e^{-\lambda_n t} + \int_0^t g_n e^{-\lambda_n(t-\tau)} f\left[\tau,u(\tau)\right]d\tau + \int_0^t e^{-\lambda_n(t-\tau)}\int_Q \varphi\left[\tau,\xi,V(\tau,\xi)\right]z_n(\xi)d\xi d\tau\right]z_n(x) \tag{1.6}$$

Manuscript received Dec 28, 2010; revised Jan 22, 2011.

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is said to be a weak generalized solution of the boundary value problem (1.1)-(1.4) .

Let us discuss the existence and the uniqueness of the solution of nonlinear integral equation (1.6).

Lemma 1.1. *A function*

$$h(t,x)=\sum_{n=1}^{\infty}\left[\psi_n e^{-\lambda_n t}\right]z_n(x) \tag{1.7}$$

is the element of the space $H(Q_T)$.

Lemma 1.2. F*or every* $V(t,x)\in H(Q_T)$ *let the function* $\varphi(\bullet,V)$ *is the element of the space* $H(Q_T)$. *Then the operator* $K_0[V]$, *defined by*

$$K_0\left[V(t,x)\right]= \sum_{n=1}^{\infty}\int_0^t e^{-\lambda_n(t-\tau)}\int_Q \varphi\left[\tau,\xi,V(\tau,\xi)\right]z_n(\xi)\,d\xi\, d\tau\, z_n(x) \tag{1.8}$$

is a mapping the space $H(Q_T)$ *into itself.*

Lemma 1.3. F*or every control* $u(t)\in H(0,T)$ *let the function* $f(t,u(t))$ *is an element of the space* $H(0,T)$. *Then the operator* $F\left[t,\,x,\,u(t)\right]$, *defined by*

$$F\left[t,\,x,\,u(t)\right]= =\sum_{n=1}^{\infty}\int_0^t e^{-\lambda_n(t-\tau)}g_n f\left[\tau,\,u(\tau)\right]d\tau\, z_n(x) \tag{1.9}$$

is a mapping from $H(0,T)$ *into* $H(Q_T)$.

Now we represent the integral equation (1.6) in the operator form using (1.7), (1.8) and (1.9)

$$V=K[V], \tag{1.10}$$

where operator

$$K[V]=h+K_0[V]+F$$

is a mapping the space $H(Q_T)$ into itself at any fixed $u(t)\in H(0,T)$, according to Lemmas 1.1 – 1.3.

Theorem 1.1. *Let the function* $\varphi\left[t,x,V(t,x)\right]$ *for a* $V(t,x)\in H(Q_T)$ *be the element of the space* $H(Q_T)$ *and satisfy the condition*

$$\varphi_0=\sup_{(t,x)\in Q_t}\left|\frac{\partial\varphi[t,x,V]}{\partial V}\right|>0\,. \tag{1.11}$$

Then the operator equation 1.10 *has the unique solution in the space* $H(Q_T)$ *if*

$$\lambda_0 T\varphi_0<1\,. \tag{1.12}$$

Proof. Under condition 1.11 the function $\varphi\left[t,x,V(t,x)\right]$ satisfies Lipschitz condition

$$\left|\varphi\left[t,x,V(t,x)\right]-\varphi\left[t,x,\bar V(t,x)\right]\right|\le$$
$$\le\varphi_0\left|V(t,x)-\bar V(t,x)\right|.$$

By the means of the norm of the space $H(Q_T)$

$$\left\|\varphi\left[t,x,V(t,x)\right]-\varphi\left[t,x,\bar V(t,x)\right]\right\|_{H(Q_T)}\le \le\varphi_0\left\|V(t,x)-\bar V(t,x)\right\|_{H(Q_T)}. \tag{1.13}$$

According to the inequality

$$\left\|K[V]-K\left[\bar V\right]\right\|_{H(Q_T)}=\left\|K_0[V]-K_0\left[\bar V\right]\right\|_{H(Q_T)}\le$$
$$\le T\lambda_0\left\|\varphi\left[t,x,V(t,x)\right]-\varphi\left[t,x,\bar V(t,x)\right]\right\|_{H(Q_T)}\le$$
$$\le\varphi_0 T\lambda_0\left\|V(t,x)-\bar V(t,x)\right\|_{H(Q_T)}$$

and by the fulfillment of the condition (1.12) the operator $K[V]$ is the contracting operator for a fixed control $u(t)\in H(0,T)$. Therefore the operator equation (1.10) in the space $H(Q_T)$ has the unique solution [12], which can be found by the fixed point iterations method

$$V_n(t,\,x)=K\left[V_{n-1}(t,\,x)\right],\quad n=1,\,2,\,3,\,...,$$

where the zero-order iteration $V_0(t,\,x)$ is an arbitrary element of the space $H(Q_T)$. Moreover an approximate solution $V_n(t,\,x)$ satisfies

$$\left\|V(t,\,x)-V_n(t,\,x)\right\|_{H(Q_T)}\le \le\frac{(T\lambda_0\varphi_0)^n}{1-T\lambda_0\varphi_0}\left\|K[V_0]-V_0\right\|_{H(Q_T)} \tag{1.14}$$

Note, that between elements of the control space $H(0,T)$ and the state space $H(Q_T)$ there is one-to-one mapping only in if

$$\frac{\partial f\left[t,\,u\right]}{\partial u}\ne0,\ \ t\in[0,\,T] \tag{1.15}$$

Therefore, the mapping $u(t)\to V(t,x)$ is one-to-one mapping if conditions (1.11) and (1.15) are satisfied.

III. OPTIMAL CONTROL PROBLEM AND OPTIMUM CONDITIONS

Let the controlled process $V(t,x)$ be described by boundary problem (1.1) – (1.4).

Definition 2.1. *If* $V(t,x)\in H(Q_T)$ *is the unique solution of the boundary value problem* (1.1) – (1.4) *corresponding to control* $u(t)\in H(0,T)$, *then the pair* $\left(u(t),V(t,x)\right)\in H(0,T)H(Q_T)$ *is said to be admissible.*

Nonlinear optimization problem. Let us consider a nonlinear optimal control problem. Suppose that functions $S[t,x,V],P[t,u],\Phi\left[V(T,x)\right]$ are defined and have the following properties:

I. Function $S[t,x,V]$ is continues and differentiable $\dfrac{\partial S}{\partial V}\in H(Q_T)$;

ISBN: 978-988-18210-6-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

II. Function $P\left[t,u(t)\right]$ is integrable on $\left[0,T\right]$ and differentiable $\frac{\partial P}{\partial u}\in H\left(0,T\right)$;

III. Function $\Phi[V]$ is integrable on the range Q and differentiable $\frac{\partial \Phi}{\partial V}\in H\left(Q\right)$.

It is required to find an admissible pair $\left(u^0\left(t\right),V^0\left(t,x\right)\in H\left(0,T\right)\times H\left(Q_T\right)\right)$ on which the functional

$$J[u(t)]=\int_0^T\int_Q S\left[t,x,V\left(t,x\right)\right]dxdt+ \quad (2.1)$$
$$+\int_0^T P\left[t,u\left(t\right)\right]dt+\int_Q \Phi\left[V\left(T,x\right)\right]dx$$

is minimized.

Definition 2.2. *The admissible pair* $\left(u^0\left(t\right),V^0\left(t,x\right)\in H\left(0,T\right)\times H\left(Q_T\right)\right)$, *that gives minimum to functional* (2.1) *is said to be an optimal pair. The control* $u^0\left(t\right)$ *is said to be the optimal control,,* $V^0\left(t,x\right)$ *is said to be the optimal process.*

Theorem 2.1. *Let functions* $P\left[t,u\right]$ *and* $f\left[t,u\right]$ *be differentiable* $\frac{\partial P\left(t,u\right)}{\partial u}\neq 0$, $\frac{\partial^2 P\left(t,u\right)}{\partial u^2}\in H\left(0,T\right)$ *and* $\frac{\partial f\left(t,x\right)}{\partial u}\neq 0$, $\frac{\partial^2 f\left(t,u\right)}{\partial u^2}\in H\left(0,T\right)$. *Then the admissible pair* $\left(u\left(t\right),V\left(t,x\right)\right)\in H\left(0,T\right)\times H\left(Q_T\right)$ *is optimal if the function*

$$\prod\left[t,x,V\left(t,x\right),\omega\left(t,x\right),u\left(t\right)\right]= \quad (2.2)$$
$$=\int_Q g\left(x\right)\omega\left(t,x\right)dxf\left[t,u\left(t\right)\right]-P\left[t,u\left(t\right)\right]$$

where $\omega\left(t,x\right)$ *is a weak generalized solution of the boundary value problem*

$$\omega_t+A\omega+\frac{\partial\varphi}{\partial V}\omega=\frac{\partial S}{\partial V},x\in Q,0\le t\le T, \quad (2.3)$$

$$\omega\left(T,x\right)-\frac{\partial\Phi[V]}{\partial V}=0,x\in Q, \quad (2.4)$$

$$\Gamma\omega(t,x)=0,x\in\gamma,0\le t\le T, \quad (2.5)$$

almost everywhere satisfy the relation

$$\Pi[t,x,V^0(t,x),\omega(t,x),u^0(t)]= \quad (2.6)$$
$$=\sup_{U\in D}[t,x,V^0(t,x),\omega(t,x),u],$$

for $[0,T]$, *where* D *is the open set of admissible values of* u.

We do not give the proof of this theorem since it follows from the famous theorems on the maximum principle for systems with distributed parameters of [1,3-6,9,15].

According to (2.2) and (2.6), we get the relations:

$$\int_Q g(x)\omega(t,x)dx\frac{\partial f(t,u)}{\partial u}-\frac{\partial P(t,u)}{\partial u}=0, \quad (2.7)$$

$$\int_Q g(x)\omega(t,x)dx\frac{\partial^2 f(t,u)}{\partial u^2}-\frac{\partial^2 P(t,u)}{\partial u^2}<0, \quad (2.8)$$

which are satisfied only for the optimal control $u^0(t)$. Taking into account (1.15) equality (2.7) can be rewritten in the form $\frac{\partial P[t,u]}{\partial u}\left(\frac{\partial f(t,u)}{\partial u}\right)^{-1}=\int_Q g(x)\omega(t,x)dx.$ (2.9)

On the basis of this equality, condition (2.8) can be rewritten in the form

$$\frac{\partial f[t,u]}{\partial u}\frac{\partial}{\partial u}\left(\frac{\partial P(t,u)}{\partial u}\left(\frac{\partial f(t,u)}{\partial u}\right)^{-1}\right)>0. \quad (2.10)$$

Therefore the optimal control $u^0(t)$ is found from the system of relations (2.9) and (2.10). They are known as optimality conditions.

IV. ADJOINT BOUNDARY PROBLEM AND ITS SOLUTION

Consider boundary value problem (2.3)-(2.5), adjointed to the boundary value problem (1.1)-(1.4).

Definition 3.1. *A function* $\omega(t,x)\in H\left(Q_T\right)$, *for an admissible pair* $\left(u(t),V(t,x)\right)\in H(0,T)\times H(Q_T)$ *satisfying the linear integral equation*

$$\omega(t,x)=$$
$$=-\sum_{n=1}^{\infty}\int_t^T e^{-\lambda_n(\tau-t)}\int_Q\frac{\partial\varphi[\tau,\xi,V(\tau,\xi)]}{\partial V}\omega(\tau,\xi)z_n\left(\xi\right)d\xi d\tau z_n\left(x\right)+$$
$$+\sum_{n=1}^{\infty}\int_t^T e^{-\lambda_n(\tau-t)}\int_Q\frac{\partial S[\tau,\xi,V(\tau,\xi)]}{\partial V}z_n\left(\xi\right)d\xi d\tau z_n\left(x\right)-$$
$$-\sum_{n=1}^{\infty}\left(e^{-\lambda_n(T-t)}\int_Q\frac{\partial\Phi[V]}{\partial V}z_n\left(\xi\right)d\xi\right)z_n\left(x\right), \quad (3.1)$$

is said to be the weak generalized solution of boundary value problem (2.3)-(2.5).

Let us investigate the uniqueness of solutions of linear integral equation (3.1).

Lemma 3.1. *A function*

$$y_1\left(t,x\right)=\sum_{n=1}^{\infty}\left(e^{-\lambda_n(T-t)}\int_Q\frac{\partial\Phi[V]}{\partial V}z_n\left(\xi\right)d\xi\right)z_n\left(x\right) \quad (3.2)$$

is an element of the space $H\left(Q_T\right)$.

Lemma 3.2. *A function*

$$y_2\left(t,x\right)=$$
$$=\sum_{n=1}^{\infty}\int_t^T e^{-\lambda_n(\tau-t)}\int_Q\frac{\partial S[\tau,\xi,V(\tau,\xi)]}{\partial V}z_n\left(\xi\right)d\xi d\tau z_n\left(x\right) \quad (3.3)$$

is an element of the space $H\left(Q_T\right)$.

Lemma 3.3. *For a function* $\omega\left(t,x\right)\in H\left(Q_T\right)$ *the operator* $G_0\left[t,x,\omega\left(t,x\right)\right]$, *is defined by the formula*

$$G_0\left[t,x,\omega(t,x)\right]=$$
$$=\sum_{n=1}^{\infty}\int_0^T e^{-\lambda_n(\tau-T)}\int_Q\frac{\partial\varphi\left[\tau,\xi,V\right]}{\partial V}\omega\left(\tau,\xi\right)z_n\left(\xi\right)d\xi d\tau z_n\left(x\right) \quad (3.4)$$

is a mapping the space $H\left(Q_T\right)$ *into itself.*

ISBN: 978-988-18210-6-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Now integral equation (3.1), according to representations (3.2), (3.3) and (3.4), can be rewritten in the form:

$$\omega = G[\omega], \qquad (3.5)$$

where the operator

$$G[\omega] = G_0[\omega] - y_2 - y_1 \qquad (3.6)$$

according lemmas (3.1) - (3.3) is a mapping the space $H(Q_T)$ into itself.

Theorem 3.1. *If conditions* (1.11) *and* (1.12) *are fulfilled, then equation* (3.5) *has the unique solution in the space* $H(Q_T)$.

Proof. According to (3.6) we have an inequality:

$$\|G[\omega] - G[\tilde{\omega}]\|_{H(Q_T)} = \|G_0[\omega] - G_0[\tilde{\omega}]\|_{H(Q_T)} \le$$
$$\le T\lambda_0\varphi_0 \|\omega(t,x) - \tilde{\omega}(t,x)\|_{H(Q_T)},$$

from which it follows that under condition (1.12) the operator $G[\omega]$ is contracting operator. Therefore operator equation (3.5) in space $H(Q_T)$ has the unique solution [12]. It can be found by the method of fixed point iterations [12].

$$\omega_n(t,x) = G[\omega_{n-1}(t,x)], \; n = 1,2,3,..., \qquad (3.7)$$

where $\omega_0(t,x)$ is the arbitrary element of space $H(Q_T)$. The following estimations is valid for $\omega_n(t,x)$

$$\|\omega(t,x) - \omega_n(t,x)\|_{H(Q_T)} \le$$
$$\le \frac{(T\lambda_0\varphi_0)^n}{1 - T\lambda_0\varphi_0} \|G[\omega_0] - \omega_0\|_{H(Q_T)} \qquad (3.8)$$

V. NONLINEAR INTEGRAL EQUATION OF OPTIMAL CONTROL

Optimal control $u^0(t)$ can be found according to optimality condition (2.9) and (2.10). Taking into account representation (3.1), we rewrite the equality (2.9) in the form:

$$\frac{\partial P[t,u]}{\partial u}\left(\frac{\partial f[t,u]}{\partial u}\right)^{-1} = \sum_{n=1}^{\infty} g_n \int_t^T e^{-\lambda_n(\tau - t)} \int_Q \frac{\partial S[\tau,\xi,V]}{\partial V} z_n(\xi) d\xi d\tau -$$
$$-\sum_{n=1}^{\infty} g_n \int_t^T e^{-\lambda_n(\tau - t)} \int_Q \frac{\partial \varphi[\tau,\xi,V]}{\partial V} \omega(\tau,\xi) z_n(\xi) d\xi d\tau -$$
$$-\sum_{n=1}^{\infty} e^{-\lambda_n(T - t)} g_n \int_Q \frac{\partial \Phi[V(T,\xi)]}{\partial V} z_n(\xi) d\xi$$

(4.1)

where $V(t,x) = R(t,x,f[t,u(t)])$ (4.2)

is a solution of the nonlinear integral equation (1.6). In (4.1) we substitute $V(t,x)$ for $R(t,x,f[t,u(t)])$ to obtain the nonlinear integral equation

$$\frac{\partial P[t,u]}{\partial u}\left(\frac{\partial f[t,u]}{\partial u}\right)^{-1} = B(t, f[t,(u(t))]), \qquad (4.3)$$

where the operator $B[.]$ has complex structure. The solution of the equation (4.3), satisfying an additional condition (2.10) is the optimal control $u^0(t)$. Thus in the process of investigation of nonlinear thermal and diffusion processes optimal control problems, described by semi-linear parabolic equations, there occurs a peculiar new problem (4.3), (2.10) of theory of nonlinear integral equations. This problem is not investigated enough. There are only a few publications by author [8-10], where the methodology of the solution of similar problems of the simpler type is suggested. According to this methodology, in order to investigate equation (4.3), first of all it must be converted. Suppose

$$\frac{\partial P[t,u(t)]}{\partial u}\left(\frac{\partial f[t,u(t)]}{\partial u}\right)^{-1} = \theta(t) \qquad (4.4)$$

Due to condition (2.10) this equality uniquely solved with respect to control $u(t)$ and there exists function $\mu(\bullet)$ such that

$$u(t) = \mu[t,\theta(t)] \qquad (4.5)$$

According to (4.4) and (4.5) equation (4.3) can be written in the following form

$$\theta(t) = B(t, f[t, \mu[t,\theta(t)]]) \qquad (4.6)$$

This equation can be investigated by the methods of nonlinear analysis [12].

Let $\theta(t)$ is a solution of equation (4.6). Substitute this solution in to (4.5) and find control, which is the solution of equation (4.3). Complication, arising while optimal control $u^0(t)$ evaluation, is that, an additional clause (2.10) should be satisfied for this solution.

If the optimal control is found, optimal process $V^0(t,x)$ is found by formula (4.2). Then $u^0(t)$ and $V^0(t,x)$ put into (2.1) and calculate minimum functional value $J[u^0]$. Therefore, the founding of the set of triple $(u^0(t), V^0(t,x), J[u^0])$ is the solution of optimization problem.

VI. CASE OF APPLICATION OF THE FACTORIZATION METHOD

Till now we have considered the optimal control problem when the function $f[t,u(t)]$ is monotonic of the variable *u(t)*, $t \in [0,T]$, condition (1.15) holds and the imaging $u(t) \to V(t,x)$ is single-valued.

Let the function $f[t,u(t)]$ be not a monotonic function of a variable *u(t)*, $t \in [0,T]$. In this case the uniqueness of the mapping $u(t) \to V(t,x)$ is lost and according to (1.6) the same controlled process state $V(t,x)$ can be defined by several controls $u_k(t) \in H(0,T)$, *k=1,2,3,...,* number of which is not countable. This is due to the fact that when function $f[t,u(t)]$ is not monotonic, the control space *H(0,T)* factorizes into mutually disjoint classes

ISBN: 978-988-18210-6-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$$U^k = \left\{ u(t) \in H(0,T) \left| \begin{array}{l} f[t,u(t)] \equiv f^k - const, k \in [k_0, k_1] \\ k_0 \le f[t,u(t)] \le k_1, t \in [0,T] \end{array} \right. \right\}$$

elements of which according to (1.6), determine the unique controlled process state *V(t,x).*

Consider the following possible cases:

1st case.

Let $\frac{\partial P[t,u(t)]}{\partial u} \neq 0, t \in [0,T]$,

i.e. $P[t,u(t)]$ is the monotonic function.

Represent the space *H(0,T)* in the form

$H(0,T) = \bigcup \overline{U}^k \bigcup U_0$

where $\overline{U}^k$ is set derived from U^k by the removal of elements, where

$\frac{\partial P[t,u(t)]}{\partial u} = 0, t \in [0,T]$

and U_0 is a set of all deleted elements. On the elements of the set $\overline{U}^k$ execute the correlation

$\frac{\partial P[t,u(t)]}{\partial u} \neq 0, t \in [0,T]$

According to optimality condition (2.9) and (2.10) in this case we can find the element

$u^*(t) = \mu[t, \theta(t)]$,

where $\theta(t)$ is a solution of the corresponding nonlinear integral equation (4.6). By found element $u^*(t)$ we make the class of control

$$U^* = \left\{ \begin{array}{l} u(t) \in H(0,T) | f[t,u(t)] = \\ \quad = f[t,u^*(t)] \end{array} \right\} = \{\tilde{u}(t)\}$$

and solve the problem of the functional minimization

$$J[\tilde{u}] = \int_0^T \int_Q S[t,x,V^*(t,x)]dxdt + \int_0^T P[t,\tilde{u}(t)]dt +$$

$$+ \int_Q \Phi[V^*(T,x)]dx$$

on the set U^*. Whereas on the set number U^*, the integral values

$\int_0^T \int_Q S[t,x,V^*(t,x)]dxdt$ and $\int_Q \Phi[V^*(T,x)]dx$

is constant, the problem $J[\tilde{u}(t)] \to \min, \tilde{u}(t) \in U^*$, is equivalent to the problem

$$\int_0^T P[t,\tilde{u}(t)]dt \to \min, \ \tilde{u}(t) \in U^* \qquad (5.1)$$

which can be solved by the optimization procedure from [17].

If $\tilde{u}^0(t)$ is a solution of (5.1) then it can exist a control pretending for "optimality". The optimal process $\tilde{V}^0(t,x)$ corresponding to the control $\tilde{u}^0(t)$, is established as a solution of integral equation (1.6), and the minimum value of $J[\tilde{u}^0(t)]$ calculated according to formula (2.1), and the solution of the control problem ($\tilde{u}^0(t)$, $\tilde{V}^0(t,x)$, $J[\tilde{u}^0(t)]$).

2nd case.

Let the $\frac{\partial P[t,u(t)]}{\partial u} = 0, t \in [0,T]$ (5.2)

Control set satisfying the relation (5.2), we denote by G_0, i.e.

$$G_0 = \left\{ u(t) \in H(0,T) | \frac{\partial P[t,u(t)]}{\partial u} = 0, \ t \in [0,T] \right\}$$

In this case optimality condition (2.9) has the meaning only on control set

$N = U_0 \cap G_0 \subset H(0,T)$,

and loose the meaning if it is empty. If $u^*(t)$ be an element of *N*. Denote by N^* the set of elements $u^*(t) \in N$ satisfying condition (2.10). Solving the problem

$J[u^*(t)] \to \min, u^*(t) \in N$,

by the optimization method [17] we find the control $\overline{u}^*(t)$ pretending for "optimality". The triplet ($\tilde{u}^*(t)$, $\overline{V}^*(t,x)$, $J[\overline{u}^*(t)]$) is also a solution of the control problem.

Comparing the values of $J[\tilde{u}(t)]$ and $J[\overline{u}(t)]$ we find an unknown optimal control $u^0(t)$, optimal process $V^0(t,x)$ and the minimum value of the functional

$J[u^0] = \min\{J[\tilde{u}(t)], J[\overline{u}^*(t)]\}$.

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ISBN: 978-988-18210-6-5
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