# Solving Heat Equation by the Adomian Decomposition Method

A. Cheniguel and A.Ayadi

*Abstract*— **In this paper, a numerical algorithm, based on the Adomian decomposition method, is presented for solving heat equation with an initial condition and non local boundary conditions. This method provides an accurate and efficient technique in comparison with other classical methods. The numerical applications show that the obtained solution coincides with the exact one.**

*Index Terms*— **Adomian decomposition method, high-order, non local problem, numerical methods for partial differential equations.**

## I. Introduction

There has recently been a lot of attention to the search for better and more accurate solution methods for determining approximate or exact solution to one dimensional heat equation with non local boundary conditions.

Consider the heat equation:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+q(x,t)\ 0<x<1, 0<t\le T \quad (1)$$

Subject to the initial condition:

$$u(x,0)=f(x), 0\le x\le 1 \quad (2)$$

And the non local boundary conditions

$$u(0,t)=\int_0^1 \phi(x,t)u(x,t)dx+g_1(t),\ \ 0<t\le T \quad (3)$$
$$u(1,t)=\int_0^1 \psi(x,t)u(x,t)dx+g_2(t),\ \ 0<t\le T \quad (4)$$

Where $f, g_1, g_2, \phi, \psi$ are sufficiently smooth known functions and $T$ is a given constant. A number of authors as have suggested traditional techniques for solving this type of problems. For instance, a fourth-order numerical finite difference scheme was proposed by M.A. Rahman and M.S.A. Taj [13]. In this work, we present a new technique based on Adomian series solution method which yields the exact solution of problem (1) to (4).

## II. Adomian Decomposition Method

### A. Operator form

In this section, we outline the steps to obtain a solution of (1)-(4) using Adomian decomposition method, which was initiated by G. Adomian[9-11]. For this purpose, it is convenient to rewrite (1) in the standard form:

$$L_t(u)=L_{xx}(u)+q(x,t) \quad (5)$$

Where the differential operators are defined as :

$$L_t(.)=\frac{\partial}{\partial t}(.)\ \text{and}\ L_{xx}=\frac{\partial^2}{\partial x^2}$$

And the inverse operator $L_t^{-1}$, provided that it exists, is defined as:

$$L_t^{-1}=\int_0^t(.)dt \quad (6)$$

Applying the inverse operator on both the sides of (5) and using the initial condition, yields:

$$L_t^{-1}\big(L_t(u)\big)=L_t^{-1}\big(L_{xx}(u)\big)+L_t^{-1}\big(q(x,t)\big) \quad (7)$$

### B. Application to the solution of the problem

Developing (7), we obtain:

$$u(x,t)=f(x)+L_t^{-1}\big(L_{xx}(u)\big)+L_t^{-1}\big(q(x,t)\big) \quad (8)$$

Now, we decompose the unknown function *u(x,t)* as a sum of components defined by the series [12] :

$$u(x,t)=\sum_{k=0}^{\infty}u_k(x,t) \quad (9)$$

Where $u_0$ is identified as *u(x; 0)*. The components $u_k(x,t)$ are obtained by the recursive formula:

$$\sum_{k=0}^{\infty}u_k(x,t)=f(x)+L_t^{-1}\{L_{xx}(\sum_{k=0}^{\infty}u_k(x,t))\}+L_t^{-1}(q(x,t)) \quad (10)$$

Or:

$$u_0(x,t)=f(x)+L_t^{-1}(q(x,t)) \quad (11)$$

$$u_{k+1}(x,t)=L_t^{-1}\Big(L_{xx}\big(u_k(x,t)\big)\Big),\ k\ge 0 \quad (12)$$

We note that the recursive relationship is constructed on the basis that the component $u_0(x,t)$ is defined by all terms that arise from the initial condition and from integrating the source term. The remaining components $u_k(x,t), k\ge 1$, can be completely determined recursively.

Accordingly, considering the first few terms, equations (8)and (9) give:

$$u_0=f(x)+L_t^{-1}(x,t)$$
$$u_1=L_t^{-1}(L_{xx}(u_0))$$
$$u_2=L_t^{-1}(L_{xx}(u_1)) \quad (13)$$

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and so on. As a result, the components $u_0$ ,$u_1, u_2, \ldots$ are identified and the series solution is thus entirely determined. However, in many cases the exact solution in a closed form may be obtained as we can see in the following examples.

## III. EXAMPLES

### A. Example 1

We consider problem (1) to (4) with:

$$f(x) = x^2, \quad 0 < x < 1,$$
$$g_1(t) = \frac{1}{4(t+1)^2},\ g_2(t) = \frac{3}{4(t+1)^2},\ 0 < t \le 1,$$
$$\phi(x,t) = x\ ,\psi(x,t) = x, 0 < x < 1,$$

$$q(x,t) = \frac{-2(x^2+t+1)}{(t+1)^3}, 0 < x < 1, 0 < t \le 1$$

Which has the exact solution: $u(x,t) = (\frac{x}{t+1})^2$ .

We rewrite this problem in an operator form and apply the above developments, to obtain the first element as:

$$u_0 = x^2 + L_t^{-1}\{\frac{-2(x^2+t+1)}{(t+1)^3}\} \quad (14)$$

Now, using (12), we obtain:

$$u_0 = x^2 + \int_0^t \frac{-2(x^2+t+1)dt}{(t+1)^3} = \frac{x^2}{(t+1)^2} + \frac{2}{t+1} - 2 \quad (15)$$
$$u_1 = L_t^{-1}\big(L_{xx}(u_0)\big) = \int_0^t \frac{2dt}{(t+1)^2} = \frac{-2}{t+1} + 2 \quad (16)$$
$$u_k = 0, k \ge 2 \quad (17)$$

Applying (9), the solution in the series form is given by:

$$u(x,t) = \frac{x^2}{(t+1)^2} \quad (18)$$

which is the exact solution.

### B. Example 2

In this second example we consider:

$q(x,t) = 0;$

$u(x,0) = f(x) = 0.5x^2, 0 < x < 1$

$u_x$(1,t)=g(t)=1, 0<t<T

$$\int_0^b u(x,t)dx = m(t) = 0.75t + \frac{1}{6}(0.75)^3$$

Where $b$ belongs to ]0,1[.

We rewrite this problem in operator form as:

$$u(x,t) = u(x,0) + L_t^{-1}\left(L_{xx}\big(u(x,t)\big)\right) + L_t^{-1}\big(q(x,t)\big) \quad (19)$$

The recursive formula is then given by:

$$u_0(x,t) = f(x) + L_t^{-1}(0) = 0.5x^2 + L_t^{-1}(0) \quad (20)$$

And

$$u_{k+1}(x,t) = L_t^{-1}\left(L_{xx}\big(u_k(x,t)\big)\right), k > 0, \quad (21)$$

From (20), we can compute the initial component as:

$$u_0(x,t) = 0.5x^2 \quad (22)$$

And from (21), we obtain:

$$u_1(x,t) = L_t^{-1}(L_{xx}\big(u_0(x,t)\big) = \int_0^t dt = t \quad (23)$$

With the remaining elements given by:

$$u_k(x,t) = 0,\ k \ge 2 \quad (24)$$

Finally, the solution in series form is given by

$$u(x,t) = u_0(x,t) + u_1(x,t) = 0.5x^2 + t \quad (25)$$

### C. Example 3

Now we consider, the following second example:

$$f(x) = x^6, 0 < x < 1,$$
$$g_1(t) = \frac{4}{5}t^6 - \frac{1}{35},\ g_2(t) = \frac{3}{5}t^6 + \frac{33}{35},\ 0 < t < 1,$$
$$\phi(x,t) = 0.2, \quad \psi(x,t) = 0.4,$$
$$q(x,t) = -30x^4 + 6t^5, \quad 0 < x < 1, 0 < t \le 1$$

Applying our developments, we obtain the first element:

$$u_0(x,t) = x^6 + L_t^{-1}(-30x^4 + 6t^5) \quad (27)$$

And

$$u_0 = x^6 + \int_0^t(-30x^4 + 6t^5)dt = x^6 - 30x^4t + t^6 \quad (28)$$

And using (12), we obtain:

$$u_1 = L_t^{-1}\big(L_{xx}(u_0)\big) = L_t^{-1}(30x^4 - 360x^2t) = \int_0^t(30x^4 - 360x^2t)dt = 30x^4t - 180x^2t^2 \quad (29)$$

$$u_2 = L_t^{-1}\big(L_{xx}(u_1)\big) = L_t^{-1}(360x^2t - 360t^2) = \int_0^t(360x^2t - 360t^2)dt = 180x^2t^2 - 120t^3 \quad (30)$$

$$u_3 = L_t^{-1}\big(L_{xx}(u_2)\big) = L_t^{-1}(360t^2)$$
$$u_3 = \int_0^t 360t^2\, dt = 120t^3 \quad (31)$$
$$u_k = 0,\ k \ge 4 \quad (32)$$

Finally, using (9) we obtain the solution in series form:

$$u(x,t) = u_0 + u_1 + u_2 + u_3 \quad (33)$$

That is :

$$u(x,t) = x^6 + t^6 \quad (34)$$

This solution coincides with the exact one.

## IV. CONCLUSION

In this paper, Adomian decomposition method was proposed for solving the heat equation with nonlocal boundary conditions and initial condition. The results obtained show that the Adomian decomposition method gives the exact solution. On the other hand, the calculations are simpler and faster than in traditional techniques.

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