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Some Results on Fuzzy Metric Spaces

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***Abstract*—In the present paper we are proving a common fixed point theorem for random fuzzy 3- metric spaces for weakly commuting mappings.**

***Index Terms*—Fuzzy metric spaces, fuzzy 3- metric spaces, common fixed point, fuzzy random 3-metric spaces.**

I. Introduction

The study of common fixed points of mappings in a fuzzy metric space satisfying certain contractive conditions has been at the center of vigorous research activity. In 1965, the concept of fuzzy sets was introduced by Zadeh [21]. With the concept of fuzzy sets, the fuzzy metric space was introduced by Kramosil and J. Michalek [11] in 1975. Helpern [7] in 1981first proved a fixed point theorem for fuzzy mappings. Also M.Grabiec [6] in 1988 proved the contraction principle in the setting of the fuzzy metric spaces. Moreover, A. George and P. Veeramani [5] in1994 modified the notion of fuzzy metric spaces with the help of t-norm, by generalizing the concept of probabilistic metric space to fuzzy situation. Consequently in due course of time some metric fixed point results were generalized to fuzzy metric spaces by various athours.

Gahler in a series of papers [2, 3, and 4] investigated 2-metric spaces. Sharma, Sharma and Iseki [15] studied for the first time contraction type mappings in 2-metic space. We know that that 2-metric space is a real valued function of a point triples on a set X, which abstract properties were suggested by the area function in Euclidean spaces. Now it is natural to expect 3-Metric space, which is suggested by the volume function.

The concept of Fuzzy-random-variable was introduced as an analogous notion to random- variable in order to extend statistical analysis to situations when the outcomes of some random experiment are fuzzy sets. But in contrary to the classical statistical methods no unique definition has been established before the work of Volker, Volker [20]. He presented set theoretical concept of fuzzy-random-variables using the method of general topology and drawing on results from topological measure theory and the theory of analytic spaces. Puri and Ralescu [13] studied fuzzy random variables as a generalization of random sets. The purpose of this generalization was the introduction of statistical techniques. Stojakovic,M.[19] introduce a new distance on the set of integrably bounded fuzzy random variables No results in fixed point are introduced in fuzzy random 3- metric spaces till now . In the present paper we are introducing the fuzzy random 3-metric spaces and proving a common fixed point theorem for weakly commuting mappings.

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II. Procedure for Paper Submission

A. Definitions

Definition (2.1): A binary operation *: $[0, 1]^4 \to [0, 1]$ is called a continuous t-norm if

([0, 1], *) is an abelian topological monoide with unit 1 such that

$a_1 * b_1 * c_1 * d_1 \geq a_2 * b_2 * c_2 * d_2$ Whenever $a_1 \geq a_2$, $b_1 \geq b_2$, $c_1 \geq c_2$ and $d_1 \geq d_2$ for all a_1, a_2, b_1, b_2, c_1, c_2 and d_1, d_2 are in [0, 1].

Definition (2.1.A): Throughout this paper, (Ω, Σ) denotes a measurable space. $\xi : \Omega \to X$ is a measurable selector. X is any non empty set.* is continuous t-norm, M is a fuzzy set in $X^4 \times [0, \infty)$, then (X, Ω ,M,*) is said to be randomized fuzzy 3- metric spaces (R.F.M.) if followings are true.

For all ξ x, ξ y, ξ z $\in$ X and s, t > 0,

B. Final Stage

Definition (2.1.B): The (X, Ω ,M,*) is called a fuzzy 3-metric space if X is an arbitrary set, * is a continuous t-norm monoid and M is a fuzzy set in X^4 x $[0, \infty]$ satisfying the following conductions:

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$(RFM^{"}-1): M(\xi x, \xi y, \xi z, \xi w, 0) = 0$

$(RFM^{"}-2): M(\xi x, \xi y, \xi z, \xi w, t) = 1, \forall t \succ 0,$

Only when the three simplex $\langle x, y, z, w\rangle$ *degenerate*

$(RFM^{"}-3): M(\xi x, \xi y, \xi z, \xi w, t) = M(\xi x, \xi w, \xi z, \xi y, t)$

$= M(\xi z, \xi w, \xi x, \xi y, t) = ----$

$(RFM^{"}-4): M(\xi x, \xi y, \xi z, \xi w, t + t_2 + t_3)$

A. $\geq M(\xi x, \xi y, \xi z, \xi u, t_1) *$

$M(\xi x, \xi y, \xi u, \xi w, t_2)$

$* M(\xi x, \xi u, \xi z, \xi w, t_3)$

$* M(\xi u, \xi y, \xi z, \xi w, t_4)$

$(RFM^{"}-5): M(\xi x, \xi y, \xi z, \xi w): [0,1) \to [0,1]$

is left continuous

$\forall \xi x, \xi y, \xi z, \xi u, \xi w \,\varepsilon\, X, t_1, t_2, t_3, t_4 \succ 0$

Definition (2.1.C): Let (X, Ω, M, *) be a random fuzzy 3-metric space:

(1)A sequence {Xn} in random fuzzy 3-metric space X is said to be convergent to a point

ξ x $\in$ X, if

$$\lim_{n\to\infty} M(\xi x_n, \xi x, a, b, t) = 1, \text{ for all } a, b \,\varepsilon\, X \text{ and } t \succ 0$$

(2)A sequence {ξx_n} in random fuzzy 3-metric space X is called a Cauchy sequence, if

$$\lim_{n\to\infty} M(\xi x_{n+p}, \xi x_n, a, b, t) = 1, \text{ for all } a, b \,\varepsilon\, X \text{ and } t, p \succ 0$$

(3)A random fuzzy 3-metric space in which every Cauchy sequence is convergent is said to be complete.

Definition (2.1.D) A function M is continuous in random fuzzy 3-metric space if

$\xi x_n \to \xi x, \xi y_n \to \xi y,$

then $\lim_{n\to\infty} M(\xi x_n, \xi y_n, a, b, t)$

$= M(\xi x, \xi y, a, t), \forall a, b \,\varepsilon\, X \text{ and } t \succ 0$

Definition (2.1.E): Two mappings A and S on random fuzzy 3-metric space X are weakly commuting iff,

$M(AS\xi u, SA\xi u, a, b, t)$

$\geq M(A\xi u, S\xi u, a, t)$

$\forall u, a, b \,\varepsilon\, X \text{ and } t \succ 0$

III. MAIN RESULTS

THEOREM 3.1 Let (X, Ω, M, *) be a complete random fuzzy 3-metric space and let S and T be continuous mappings of X in X, then S and T have a common fixed point in X if there exists continuous mapping A of X into S (X) $\cap$ T (X) which commute weakly with S and T

(3.1a)

$$M(A\xi x, A\xi y, a, b, qt) \geq \min \begin{Bmatrix} M(T\xi y, A\xi y, a, b, t), \\ M(S\xi x, A\xi x, a, b, t), \\ M(S\xi x, T\xi y, a, t), \\ \dfrac{M(S\xi x, T\xi y, a, b, t)}{M(A\xi x, T\xi y, a, b, t)} \end{Bmatrix}$$

for all ξ x, ξ y, a, b$\in$ X, t > 0, 0 < q < 1

(3.1 b)

$\lim_{n\to\infty}$ M (ξ x, ξ y, ξ z, ξ w, t) = 1 for all ξ x, ξ y, ξ z, ξ w in X.

Then S, T and A have a unique common fixed point.

PROOF: We define a sequence {x_n} such that

$A\xi x_{2n} = S\xi x_{2n-1}$ and $A\xi x_{2n-1} = T\xi x_{2n}$, n = 1, 2,--

We shall prove that {$A\xi x_n$} is a Cauchy sequence.

For this suppose $\xi x = \xi x_{2n}$ and $\xi y = \xi x_{2n+1}$ in (3.1 a), we write

$$M(A\xi x_{2n}, A\xi x_{2n+1}, a, b, qt) \geq \min \begin{Bmatrix} M(T\xi x_{2n+1}, A\xi x_{2n+1}, a, b, t), \\ M(S\xi x_{2n}, A\xi x_{2n}, a, b, t), \\ M(S\xi x_{2n}, T\xi x_{2n+1}, a, b, t), \\ \dfrac{M(S\xi x_{2n}, T\xi x_{2n+1}, a,, b, t)}{M(A\xi x_{2n}, T\xi x_{2n+1}, a, b, t)} \end{Bmatrix}$$

$$= \min \begin{Bmatrix} M(A\xi x_{2n}, A\xi x_{2n+1}, a, b, t), \\ M(A\xi x_{2n+1}, A\xi x_{2n}, a, b, t), \\ M(A\xi x_{2n+1}, A\xi x_{2n}, a, b, t), 1 \end{Bmatrix}$$

$$\geq \min \begin{Bmatrix} M(A\xi x_{2n-1}, A\xi x_{2n}, a, b, \frac{t}{q}), \\ M(A\xi x_{2n}, A\xi x_{2n-1}, a, b, \frac{t}{q}) \end{Bmatrix}$$

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Therefore

$$M\left(A\xi x_{2n}, A\xi x_{2n+1}, a, b, qt\right) \geq M\left(A\xi x_{2n-1}, A\xi x_{2n}, a, b, \frac{t}{q}\right)$$

By induction

$$M\left(A\xi x_{2k}, A\xi x_{2m+1}, a, b, qt\right) \geq M\left(A\xi x_{2m}, A\xi x_{2k-1}, a, b, \frac{t}{q}\right)$$

For every k and m in N, Further if 2m +1 > 2k, then

$$M\left(A\xi x_{2k}, A\xi x_{2m+1}, a, b, qt\right) \geq M\left(A\xi x_{2k-1}, A\xi x_{2m}, a, b, \frac{t}{q}\right) \ldots\ldots\ldots\ldots\ldots \geq M\left(A\xi x_0, A\xi x_{2m+1-2k}, a, b, \frac{t}{q^{2k}}\right) \quad --(3.1\text{ b})$$

If 2k > 2m+1, then

$$M\left(A\xi x_{2k}, A\xi x_{2m+1}, a, b, qt\right) \geq M\left(A\xi x_{2k-1}, A\xi x_{2m}, a, b, \frac{t}{q}\right) \ldots\ldots\ldots\ldots\ldots \geq M\left(A\xi x_{2k-(2m+1)}, A\xi x_0, a, b, \frac{t}{q^{2m+1}}\right) \quad --(3.1\text{c})$$

By simple induction with (3.1 b) and (3.1 c) we have

$$M\left(A\xi x_n, A\xi x_{n+p}, a, b, qt\right) \geq M\left(A\xi x_0, A\xi x_p, a, b, \frac{t}{q^n}\right).$$

For n = 2k, p = 2m+1 or n = 2k+1, p = 2m +1 and by (RFM-4)

$$M\left(A\xi x_n, A\xi x_{n+p}, a, b, qt\right) \geq M\left(A\xi x_0, A\xi x_1, a, b, \frac{t}{2q^n}\right). * M\left(A\xi x_1, A\xi x_p, a, b, \frac{t}{q^n}\right). \quad ---(3.1d)$$

If n = 2k, p = 2m or n = 2k+1, p= 2m

for every positive integer p and n in N, by nothing that

$$M\left(A\xi x_0, A\xi x_p, a, b, \frac{t}{q^n}\right). \rightarrow 1 \, as \, n \rightarrow \infty$$

Thus $\{A\xi x_n\}$ is a Cauchy sequence.

Since the space X is complete there exists $\xi z \in X$, such that

$$\lim_{n\to\infty} A\xi x_n = \lim_{n\to\infty} S\xi x_{2n-1} = \lim_{n\to\infty} T\xi x_{2n} = \xi z$$

It follows that $A\xi z = S\xi z = T\xi z$ and

Therefore

$$M\left(A\xi z, AA\xi z, a, b, qt\right) \geq \min \left\{ \begin{array}{l} M\left(TA\xi z, AA\xi z, a, b, t\right), \\ M\left(S\xi z, A\xi z, a, b, t\right), \\ M\left(S\xi z, TA\xi z, a, b, t\right), \\ \dfrac{M\left(S\xi z, TA\xi z, a, b, t\right)}{M\left(A\xi z, TA\xi z, a, b, t\right)}, \end{array} \right\}$$

$$M(A\xi z, A^2\xi z, a, b, qt) \geq M(S\xi z, TA\xi z, a, b, t)$$
$$\geq M(S\xi z, AT\xi z, a, b, t)$$
$$\geq M(A\xi z, A^2\xi z, a, b, t)$$

Since, $\lim_{n\to\infty} M(A\xi z, A^2\xi z, a, b, \frac{t}{q^n}) = 1$

$$\Rightarrow A\xi z = A^2\xi z$$

Thus ξz is common fixed point of A, S and T.

For uniqueness, let ξv ($\xi v \neq \xi z$) be another common fixed point of S, T and A.

By (3.1 a) we write

$$M\left(A\xi z, A\xi v, a, b, qt\right) \geq \min \left\{ \begin{array}{l} M\left(T\xi v, A\xi v, a, b, t\right), \\ M\left(S\xi z, A\xi z, a, b, t\right), \\ M\left(S\xi z, T\xi v, a, b, t\right), \\ \dfrac{M\left(S\xi z, T\xi v, a, b, t\right)}{M\left(A\xi z, T\xi v, a, b, t\right)}, \end{array} \right\}$$

$$M\left(A\xi z, A\xi v, a, b, qt\right) \geq \min\left\{M\left(\xi z, \xi v, a, b, t\right)\right\}$$
$$M\left(\xi z, \xi v, a, b, qt\right) \geq \min\left\{M\left(\xi z, \xi v, a, b, t\right)\right\}$$
$$\geq M(A\xi z, A^2\xi z, a, b, \frac{t}{q^n}).$$

So $\xi z = \xi v$

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