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Distribution of Terminal Cost Functional in Discrete-Time Controlled System with Noise-Corrupted State Information

Valery Y. Glizer, Vladimir Turetsky, and Josef Shinar

***Abstract*—Recursive formulae for the terminal cost distribution in a scalar linear discrete-time system with noise corrupted measurements are obtained. The system is subject to a linear saturated control strategy. The initial state and the estimator error distributions are assumed to be known. An example, inspired by an interception problem, is presented.**

***Index Terms*—linear discrete-time system, robust transferring strategy, noisy measurements, terminal state distribution.**

I. Introduction

VARIOUS real life control problems (including interception) can be formulated as a problem of transferring a controlled system to a prescribed hyperplane in the state space at a prescribed time in the presence of noise corrupted measurements and unknown bounded disturbance by bounded control [1], [2], [3], [4]. This problem can be reduced to a scalar one, where the new state variable z is the distance between a current point on the trajectory of the uncontrolled system motion and the hyperplane. By this scalarization, the problem of transferring to a prescribed hyperplane becomes a problem of robust transferring to zero.

Several classes of deterministic feedback strategies $u = u(t, z(t))$ that robustly transfer a scalar system from some domain of initial positions to zero, are known if perfect state information is available. Among such robust transferring strategies are differential game based bang-bang strategies [1], [2], as well as various linear, saturated linear and weakly nonlinear strategies (see e.g. the works of the authors [3], [5], [6], [7], [8], [9]).

In real life applications, the state information is corrupted by measurement noise and only part of the state variables can be directly measured. This fact impedes significantly the practical implementation of theoretically robust transferring strategies. Moreover, an estimator, restoring and filtering the state variables, becomes an indispensable component of the control loop. Due to the noisy measurements and the uncertain disturbance the control function $u(t, z(t))$ receives, instead of the exact value of $z(t)$, a random estimator output $\hat{z}(t) = z(t) + \eta(t)$, where $\eta(t)$ is the estimation error. As the consequence, the terminal value of z becomes a random variable with an a-priori unknown distribution. In order to appreciate the extent of performance deterioration of a deterministic robust transferring strategy by using such a stochastic data, the distribution of the terminal value of z has be found.

V.Y. Glizer and V. Turetsky (corresponding author, e-mail: turetsky1@braude.ac.il) are with Department of Mathematics, Ort Braude College, P.O.B. 78, Karmiel 21982, Israel.
J. Shinar is with Faculty of Aerospace Engineering, Technion - Israel Institute of Technology, Haifa, 32000, Israel

In the current practice, such a distribution is obtained, for any given estimator/strategy combination and specified disturbance and noise models, by a large set of Monte Carlo simulations (see e.g [10], [11]). Such a-posteriori test is absolutely necessary for validation purpose, but cannot be applied for an insightful control system design. There is an obvious need for an analytical a-priori estimate of the strategy performance as a part of the integrated control system design.

State estimates in the presence of deterministic information errors were obtained in [12] and [13]. In [13], such estimates are used to construct a robust control of a dynamic system with inexact state information. In interception problems, the scalar state variable z is the zero-effort miss distance and its terminal value is actual miss distance itself. Under some general linear assumptions without taking into account the system dynamics, the miss distance distribution was investigated in [14]. In the case of linear interceptor strategies, the dependence of the miss distance on the measurement noises was analyzed, by means of the adjoint approach in [15] and [16]. Unfortunately, this approach can be applied only in the case of non saturated linear strategies.

In this paper, the system dynamics is modeled by a discrete-time scalar linear equation controlled by a *saturated* linear transferring strategy. For the sake of simplicity, it is assumed that the system is disturbance free. Assuming that the distributions of z_0 and the estimation error η_n are known, a recurrence formula for the distribution of z_{n+1} is obtained. The random variable z_{n+1} is the linear combination of two dependent random variables - the state z_n at the previous time step and the control variable u_n, nonlinearly depending on z_n via the saturation function. This makes the problem to be mathematically nontrivial.

II. Problem Statement

A. Original Control Problem

Consider the controlled system

$$\dot{X} = A(t)X + b(t)u + c(t)v + f(t), \tag{1}$$

where $X \in \mathbb{R}^n$ is the state vector; $t \in [t_0, t_f]$, $X(t_0) = X_0$, t_f is a fixed time instant, $t_0 \in [0, t_f)$; the matrix function $A(t)$ and the vector functions $b(t)$, $c(t)$, $f(t)$ are differentiable for a sufficient number of times on the interval $[0, t_f]$. The scalar control u and disturbance v are assumed to be measurable on $[t_0, t_f)$ and satisfying the constraints

$$|u(t)| \le 1, \quad |v(t)| \le 1, \quad t \in [t_0, t_f). \tag{2}$$

The target set is the hyperplane $\mathcal{D} = \{X \in \mathbb{R}^n \mid d^T X + d_0 = 0\}$, where $d \in \mathbb{R}^n$ is a prescribed non-zero vector, d_0 is a

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prescribed scalar, the superscript T denotes the transposition. The control objective is to guarantee $X(t_f) \in \mathcal{D}$ against any admissible disturbance function $v(t)$.

By the transformation of the state variable in (1),

$$z = z(t, X) =$$

$$d^T \Big(\Phi(t_f, t) X + \int_t^{t_f} \Phi(t_f, \tau) f(\tau) d\tau \Big) + d_0, \qquad (3)$$

the system (1) is reduced to the scalar one

$$\dot{z} = h_1(t) u + h_2(t) v, \quad z(t_0) = z_0, \qquad (4)$$

where $h_1(t) = d^T \Phi(t_f, t) b(t)$, $h_2(t) = d^T \Phi(t_f, t) c(t)$, $z_0 = z(t_0, X_0)$, $\Phi(t, t_0)$ is the fundamental matrix of the homogeneous system $\dot{X} = A(t) X$. The control objective becomes to guarantee $z(t_f) = 0$.

It is assumed that the control is given by a saturated linear strategy

$$u(t, z) = \mathrm{sat}(K(t) z), \qquad (5)$$

where

$$\mathrm{sat}(y) = \begin{cases} 1, & y > 1, \\ y, & |y| \le 1, \\ -1, & y < -1, \end{cases} \qquad (6)$$

the gain function $K(t)$ satisfies the conditions (given e.g. in [9]), guaranteeing that the linear strategy $u = K(t) z$ is robust transferring. For the sake of simplicity, in the sequel it is assumed that $v(t) \equiv 0$, $t \in [0, t_f]$.

B. Discrete-Time Estimation Problem

Define the division of the interval $[0, t_f]$: $0 = t_0 < t_1 < \ldots < t_N = t_f$, where $t_{n+1} - t_n = \Delta t$, $n = 0, \ldots, N-1$. As a simplified model of system (4), consider the discrete-time equation without the disturbance:

$$z_{n+1} = z_n + b_n u_n, \qquad (7)$$

where for the simplest Euler approximation of the differential equation (4), $b_n = \Delta t h_1(t_n)$. The control is

$$u_n = \mathrm{sat}(k_n (z_n + \eta_n)), \qquad (8)$$

where $k_n = K(t_n)$ is the control gain and η_n is the estimation error. The probability density functions $f_{z_0}(x)$ of z_0 and $f_{\eta_n}(x)$ of η_n, $n = 0, 1, \ldots, N-1$, are assumed to be known. The problem is to obtain the probability density function $f_{z_N}(x)$. Note that since the random variables z_n and u_n are dependent, the distribution function of z_{n+1} cannot be calculated by using the convolution formula.

III. Solution

Due to (7) – (8), the distribution function of z_{n+1} is

$$F_{z_{n+1}}(x) = P(z_{n+1} < x) =$$

$$p_1 P(k_n(z_n + \eta_n) > 1) + p_2 P(|k_n(z_n + \eta_n)| \le 1) +$$

$$p_3 P(k_n(z_n + \eta_n) < -1), \qquad (9)$$

where p_1, p_2 and p_3 are the conditional probabilities

$$p_1 = P(z_n + b_n < x \;\big|\; k_n(z_n + \eta_n) > 1), \qquad (10)$$

$$p_2 =$$

$$P((1 + b_n k_n) z_n + b_n k_n \eta_n < x \;\big|\; |k_n(z_n + \eta_n)| \le 1), \qquad (11)$$

$$p_3 = P(z_n - b_n < x \;\big|\; k_n(z_n + \eta_n) < -1). \qquad (12)$$

Thus, the problem is reduced to calculating the conditional probabilities (10) – (12).

A. Calculation of p_1 and p_3

By using (10) and the formula for the probability of the product of dependent events,

$$p_1 = \tilde{p}_1 P(z_n < x - b_n) \Big/ P(z_n + \eta_n > 1/k_n), \qquad (13)$$

where

$$\tilde{p}_1 = P\Big(z_n + \eta_n > 1/k_n \;\Big|\; z_n < x - b_n\Big). \qquad (14)$$

Let calculate the conditional probability $\tilde{p}_1$.

First, instead of the event $z_n < x - b_n$, consider the event $z_n \in (a, x - b_n)$, where a is a negative number with sufficiently large absolute value:

$$\tilde{p}_{1a} = P\Big(z_n + \eta_n > 1/k_n \;\Big|\; z_n \in (a, x - b_n)\Big). \qquad (15)$$

Note that

$$\tilde{p}_1 = \lim_{a \to -\infty} \tilde{p}_{1a}. \qquad (16)$$

Let divide the interval $(a, x - b_n)$ into M subintervals of equal length $\Delta x = (x - b_n - a)/M$: $x_j = a + j\Delta x$, $j = 0, 1, \ldots, M$. Then, since the events $z_n \in (x_j, x_{j+1})$, $j = 0, \ldots, M-1$, are mutually exclusive,

$$\tilde{p}_{1a} \approx \sum_{j=0}^{M-1} P\Big(z_n \in (x_j, x_{j+1}) \;\Big|\; z_n \in (a, x - b_n)\Big) \times$$

$$P\Big(z_n + \eta_n > 1/k_n \;\Big|\; z_n \in (x_j, x_{j+1})\Big). \qquad (17)$$

Let start with calculating the first conditional probability under the sum in (17). Note that

$$P\Big(\Big[z_n \in (x_j, x_{j+1})\Big] \& \Big[z_n \in (a, x - b_n)\Big]\Big) =$$

$$P\Big(z_n \in (x_j, x_{j+1}) \;\Big|\; z_n \in (a, x - b_n)\Big) P(z_n \in (a, x - b_n)). \qquad (18)$$

From the other hand,

$$P\Big(\Big[z_n \in (x_j, x_{j+1})\Big] \& \Big[z_n \in (a, x - b_n)\Big]\Big) =$$

$$\underbrace{P\Big(z_n \in (a, x - b_n) \;\Big|\; z_n \in (x_j, x_{j+1})\Big)}_{=1} P(z_n \in (x_j, x_{j+1}))$$

$$= P(z_n \in (x_j, x_{j+1})). \qquad (19)$$

From (18) – (19),

$$P\Big(z_n \in (x_j, x_{j+1}) \;\Big|\; z_n \in (a, x - b_n)\Big) =$$

$$P(z_n \in (x_j, x_{j+1})) \Big/ P(z_n \in (a, x - b_n)). \qquad (20)$$

For sufficiently small Δx, the second conditional probability under the sum in (17) can be approximated as

$$P\Big(z_n + \eta_n > 1/k_n \Big| z_n \in (x_j, x_{j+1})\Big) \approx$$

$$P\Big(z_n + \eta_n > 1/k_n \;\Big|\; z_n = \bar{x}_j\Big) = P\Big(\eta_n > 1/k_n - \bar{x}_j\Big), \qquad (21)$$

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where $\bar{x}_j = (x_j + x_{j+1})/2$.

Due to (17) and (20) – (21),

$$\tilde{p}_{1a} \approx \frac{1}{P(z_n \in (a, x - b_n))} \sum_{j=0}^{M-1} P\Big(z_n \in (x_j, x_{j+1})\Big) \times$$

$$P\Big(\eta_n > 1/k_n - \bar{x}_j\Big) =$$

$$\frac{1}{\int_a^{x-b_n} f_{z_n}(y)dy} \sum_{j=0}^{M-1} \int_{x_j}^{x_{j+1}} f_{z_n}(y)dy \int_{1/k_n - \bar{x}_j}^{\infty} f_{\eta_n}(y)dy, \quad (22)$$

where $f_{z_n}(y)$ and $f_{\eta_n}(y)$ are the probability density functions of the random variables z_n and η_n, respectively. Since

$$\int_{x_j}^{x_{j+1}} f_{z_n}(y)dy \approx f_{z_n}(\bar{x}_j)\Delta x, \quad (23)$$

the equation (22) can be rewritten as

$$\tilde{p}_{1a} \approx \frac{1}{\int_a^{x-b_n} f_{z_n}(y)dy} \sum_{j=0}^{M-1} \left[f_{z_n}(\bar{x}_j) \int_{1/k_n - \bar{x}_j}^{\infty} f_{\eta_n}(y)dy \right] \Delta x, \quad (24)$$

Hence,

$$\tilde{p}_{1a} = \frac{\lim_{M\to\infty} \sum_{j=0}^{M-1} \left[f_{z_n}(\bar{x}_j) \int_{1/k_n - \bar{x}_j}^{\infty} f_{\eta_n}(y)dy \right] \Delta x}{\int_a^{x-b_n} f_{z_n}(y)dy} =$$

$$\frac{\int_a^{x-b_n} \left[f_{z_n}(s) \int_{1/k_n - s}^{\infty} f_{\eta_n}(y)dy \right] ds}{\int_a^{x-b_n} f_{z_n}(y)dy}. \quad (25)$$

By virtue of (16),

$$\tilde{p}_1 = \frac{\int_{-\infty}^{x-b_n} \left[f_{z_n}(s) \int_{1/k_n - s}^{\infty} f_{\eta_n}(y)dy \right] ds}{\int_{-\infty}^{x-b_n} f_{z_n}(y)dy}. \quad (26)$$

Due to (13) and (26),

$$p_1 = \frac{\int_{-\infty}^{x-b_n} \left[f_{z_n}(s) \int_{1/k_n - s}^{\infty} f_{\eta_n}(y)dy \right] ds}{\int_{1/k_n}^{\infty} f_{z_n+\eta_n}(y)dy}. \quad (27)$$

The random variables z_n and η_n are independent. Therefore,

$$f_{z_n+\eta_n}(y) = f_{z_n}(y) * f_{\eta_n}(y) = \int_{-\infty}^{\infty} f_{z_n}(y-s) f_{\eta_n}(s)ds. \quad (28)$$

Finally,

$$p_1 = \frac{\int_{-\infty}^{x-b_n} \left[f_{z_n}(s) \int_{1/k_n - s}^{\infty} f_{\eta_n}(y)dy \right] ds}{\int_{-1/k_n}^{\infty} \left[\int_{-\infty}^{\infty} f_{z_n}(y-s) f_{\eta_n}(s)ds \right] dy} \quad (29)$$

Calculation of p_3 is similar to the calculation of p_1, resulting in

$$p_3 = \frac{\int_{-\infty}^{x+b_n} \left[f_{z_n}(s) \int_{-\infty}^{-1/k_n - s} f_{\eta_n}(x)dx \right] ds}{\int_{-\infty}^{-1/k_n} \left[\int_{-\infty}^{\infty} f_{z_n}(x-s) f_{\eta_n}(s)ds \right] dx} \quad (30)$$

B. Calculation of p_2

Consider the case $b_n \geq 0$. By definition of the conditional probability,

$$p_2 = \frac{P\Big((z_n, \eta_n) \in R(x) \& (z_n, \eta_n) \in Q\Big)}{P\Big((z_n, \eta_n) \in Q\Big)} =$$

$$P\Big((z_n, \eta_n) \in S(x)\Big) \Big/ P\Big((z_n, \eta_n) \in Q\Big), \quad (31)$$

where (see Fig. 1)

$$R(x) \triangleq \{(z_n, \eta_n) : \quad \eta_n < -Az_n + B(x)\}, \quad (32)$$

$$A = 1 + \frac{1}{b_n k_n}, \; B(x) = \frac{x}{b_n k_n},$$

$$Q \triangleq \{(z_n, \eta_n) : \quad -z_n - 1/k_n \leq \eta_n \leq -z_n + 1/k_n\}, \quad (33)$$

$$S(x) \triangleq R(x) \bigcap Q. \quad (34)$$

The straight line $\eta_n = -Az_n + B(x)$ (the upper boundary of the set $R(x)$) intersects the straight lines $\eta_n = -z_n \pm 1/k_n$ (boundaries of the set Q) at $z_n = x \pm b_n$. Thus, the set $S(x)$ can be represented as

$$S(x) = S_1(x) \bigcup S_2(x), \quad S_1(x) \bigcap S_2(x) = \emptyset, \quad (35)$$

where

$$S_1(x) = \Big\{(z_n, \eta_n) : \quad z_n < x - b_n,$$

$$-z_n - 1/k_n \leq \eta_n \leq -z_n + 1/k_n\Big\}, \quad (36)$$

$$S_2(x) = \Big\{(z_n, \eta_n) : \quad z_n \in [x - b_n, x + b_n],$$

$$-z_n - 1/k_n \leq \eta_n \leq -Az_n + B(x)\Big\}. \quad (37)$$

Therefore,

$$P\Big((z_n, \eta_n) \in S(x)\Big) =$$

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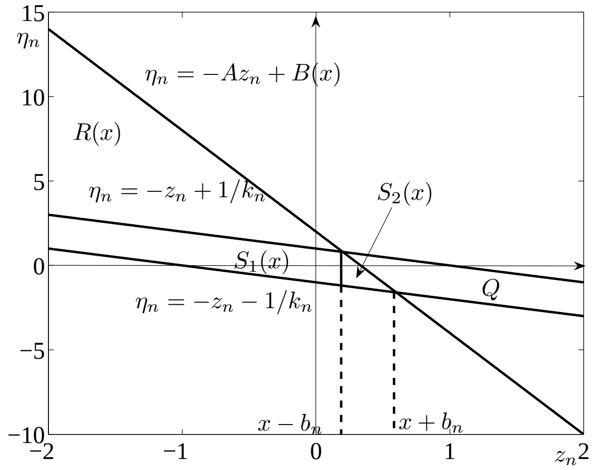

Fig. 1. Sets $R(x)$ and Q for $b_n > 0$

$$P\Big((z_n, \eta_n) \in S_1(x)\Big) + P\Big((z_n, \eta_n) \in S_2(x)\Big). \tag{38}$$

Similarly to the calculation of p_1, by discretization and limiting,

$$P\Big((z_n, \eta_n) \in S_1(x)\Big) = \int_{-\infty}^{x-b_n} \left[f_{z_n}(s) \int_{-s-1/k_n}^{-s+1/k_n} f_{\eta_n}(y)dy \right] ds, \tag{39}$$

$$P\Big((z_n, \eta_n) \in S_2(x)\Big) = \int_{x-b_n}^{x+b_n} \left[f_{z_n}(s) \int_{-s-1/k_n}^{-As+B(x)} f_{\eta_n}(y)dy \right] ds. \tag{40}$$

By virtue of (31), (33) and (39) – (40),

$$p_2 = \frac{1}{C_n} \left\{ \int_{-\infty}^{x-b_n} \left[f_{z_n}(s) \int_{-s-1/k_n}^{-s+1/k_n} f_{\eta_n}(y)dy \right] ds + \int_{x-b_n}^{x+b_n} \left[f_{z_n}(s) \int_{-s-1/k_n}^{-As+B(x)} f_{\eta_n}(y)dy \right] ds \right\}, \tag{41}$$

where

$$C_n = \int_{-1/k_n}^{1/k_n} \left[\int_{-\infty}^{\infty} f_{z_n}(y-s) f_{\eta_n}(s)ds \right] dy. \tag{42}$$

If $b_n < 0$, similarly to (41), it is obtained that

$$p_2 = \frac{1}{C_n} \left\{ \int_{-\infty}^{x+b_n} \left[f_{z_n}(s) \int_{-s-1/k_n}^{-s+1/k_n} f_{\eta_n}(y)dy \right] ds + \int_{x+b_n}^{x-b_n} \left[f_{z_n}(s) \int_{-As+B(x)}^{-s+1/k_n} f_{\eta_n}(y)dy \right] ds \right\}. \tag{43}$$

C. Calculation of $f_{z_{n+1}}(x)$

For $b_n \geq 0$, by substituting (27), (30) and (41) into (9) and by simplifying the obtained expression,

$$F_{z_{n+1}}(x) = \int_{-\infty}^{x-b_n} f_{z_n}(s)ds + \int_{x-b_n}^{x+b_n} \left[f_{z_n}(s) \int_{-\infty}^{-As+B(x)} f_{\eta_n}(y)dy \right] ds. \tag{44}$$

Similarly, for $b_n < 0$,

$$F_{z_{n+1}}(x) = \int_{-\infty}^{x+b_n} f_{z_n}(s)ds + \int_{x+b_n}^{x-b_n} \left[f_{z_n}(s) \int_{-As+B(x)}^{\infty} f_{\eta_n}(y)dy \right] ds. \tag{45}$$

By simple algebra it is shown that differentiating (44) and (45) with respect to x yields the same expression for the probability density function of z_{n+1}:

$$f_{z_{n+1}}(x) = f_{z_n}(x-b_n) + f_{z_n}(x+b_n) \int_{-\infty}^{-x-1/k_n-b_n} f_{\eta_n}(y)dy - f_{z_n}(x-b_n) \int_{-\infty}^{-x+1/k_n+b_n} f_{\eta_n}(y)dy + \frac{1}{b_n k_n} \int_{x-b_n}^{x+b_n} [f_{z_n}(s) f_{\eta_n}(-As + B(x))]ds. \tag{46}$$

The probability density function $f_{z_N}(x)$ is obtained by applying the recurrence formula (46) N times.

IV. Interception Example

In this example, a planar engagement between two objects (pursuer and evader) is considered. It is assumed that the dynamics of each object is expressed by a first-order transfer function with the time constants τ_p and τ_e, respectively. The velocities V_p and V_e and the bounds of the lateral acceleration commands $a_p^{\max}$ and $a_e^{\max}$ of the objects are constant. Subject to the assumption of small aspect angles φ_p and φ_e, the engagement is modeled by the system (1), where X_1 is the relative separation between the objects, normal to the initial line-of-sight; X_2 is the relative normal velocity; X_3 and X_4 are the lateral accelerations of the evader and the pursuer, respectively; $t_f = r_0/(V_p + V_e)$, where r_0 is the initial range between the objects;

$$A(t) \equiv \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1/\tau_e & 0 \\ 0 & 0 & 0 & -1/\tau_p \end{bmatrix}, \tag{47}$$

$$b(t) \equiv (0, 0, 0, a_p^{\max}/\tau_p)^T, \quad c(t) \equiv (0, 0, a_e^{\max}/\tau_e, 0)^T, \tag{48}$$

$$f(t) \equiv 0, \;\; X_0 = (0, X_{20}, 0, 0)^T, \;\; X_{20} = V_e\varphi_e(0) - V_p\varphi_p(0). \tag{49}$$

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The controls of the pursuer u and the evader v are the normalized lateral acceleration commands, satisfying the constraints (2). The objective of the pursuer is to nullify the miss distance $|X_1(t_f)|$, i.e. in the target hyperplane, $d = (1, 0, 0, 0)^T$, $d_0 = 0$.

In the scalarized system (4), $z_0 = t_f X_{20}$,

$$h_1(t) = -h(t; \tau_p, a_p^{\max}), \quad h_2(t) = h(t; \tau_e, a_e^{\max}), \quad (50)$$

where $h(t; \tau, a^{\max}) = \tau a^{\max} \Psi((t_f - t)/\tau)$, $\Psi(\xi) = \exp(-\xi) + \xi - 1 > 0$, $\xi > 0$. The new state variable z is the well-known zero-effort miss distance [15]. The target point is $(t_f, 0)$.

The pursuer strategy is given by (5) with the gain function

$$K(t) = 2/(t_f - t)^3. \quad (51)$$

The scalar system was approximated by the discrete-time equation (7), where $b_n = \Delta t h_1(t_n)$, $\Delta t = 0.01$ s. In this example, $t_f = 4$ s, $N = 400$, $\tau_p = 0.2$ s, $a_p^{\max} = 30$ m/s^2.

It is assumed that the initial value z_0 and the estimation errors η_n are gaussian: $z_0 \sim Gauss(0.5, 0.1)$, $\eta_n \sim Gauss(\mu_n, \sigma_n)$, $n = 0, \ldots, N-1$. The values of μ_n and σ_n were obtained from a realistic Monte Carlo simulation with noisy line-of-sight measurements and an estimator in the control loop. The cumulative distribution function of the miss distance $|z_N|$ is calculated as

$$F_{|z_N|}(x) = F_{z_N}(x) - F_{z_N}(-x), \quad (52)$$

where $F_{z_N}(x) = \int_{-\infty}^{x} f_{z_N}(\xi) d\xi$.

In Fig. 2, the cumulative distribution of $|z_N|$, obtained by 2000 Monte Carlo runs of the discrete-time system (4) and by using (52) based on (46), applied 400 times. It is seen that two curves match very accurately.

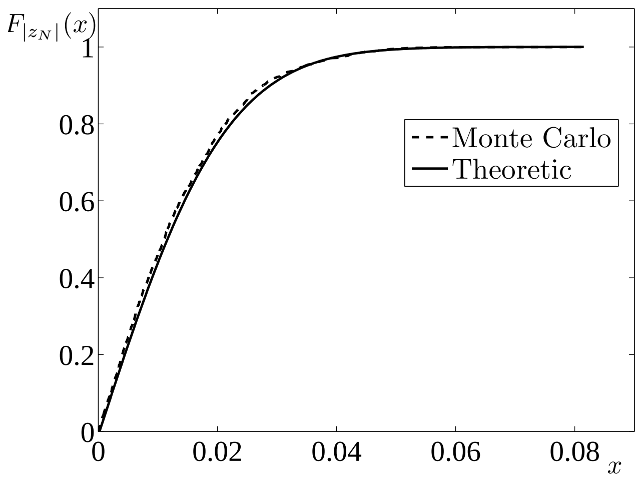

Fig. 2. Simulative and theoretic distribution functions of $|z_N|$

V. Conclusions

The problem of evaluating the probability distribution of the final state of a scalar discrete-time system is solved. In this problem, it is assumed that the state information is corrupted by an error with known distribution and the initial state distribution is also known. The control is realized by a saturated linear strategy. The formulation is motivated by various real-life control problems, and, especially, by the interception problem, where validating robust transferring deterministic strategies in realistic stochastic environment is of a high practical importance.

The problem is mathematically nontrivial, because the evaluation of the sum of two dependent random variables is required. The solution is based on proper discretization of some conditional probabilities. The resulting formula allows to evaluate the final state distribution without carrying out a great amount of Monte Carlo simulation runs.

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