Proceedings of the World Congress on Engineering 2011 Vol I
WCE 2011, July 6 - 8, 2011, London, U.K.

Pricing of Volatility Derivatives using 3/2-Stochastic Models

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***Abstract*—Analytic solutions are found for prices of both variance and volatility swaps and VIX options under new 3/2-stochastic models for the dynamics of the underlying assets. The main features of the new stochastic differential equations are an empirically validated $c\nu^{3/2}$ diffusion term, a nonlinear drift providing a balancing effect of a stronger mean reversion with high volatility, and for the case of the variance and volatility swaps, a free function of time as a moving target in the reversion term, allowing additional flexibility for model calibration against market data.**

***Index Terms*—variance swaps, volatility swaps, stochastic variance, VIX options**

I. Introduction

The aim of this paper is to price volatility contracts using the 3/2- model

$$dw = (aw + bw^2)dt + kw^{\frac{3}{2}}dZ, \quad b < 0, \tag{1}$$

where w is the price of the underlying asset (either variance for volatility and variance swaps, or the VIX for VIX options), b and k are constant and dZ, here and elsewhere in the paper denotes an increment in a Wiener process Z with probability measure P. For the VIX we assume a is constant, while for variance, we allow a to be a free function of time. The novel features of model (1) are i) the specification for the diffusion having a high power law of 1.5 which can reduce the heteroskedasticity of volatility and ii) a nonlinear drift so that it exhibits substantial nonlinear mean-reverting behaviour when the underlying w is above its long-run mean. Hence after a large price spike, the underlying price can potentially quickly decrease while after a low price period it can be slow to increase.

A. Volatility and Variance Swaps

Volatility and variance swaps are forward contacts whose payoff is based on a realised annualised volatility σ_R. Their payoffs are

$$\begin{aligned}\text{volatility swap payoff} &= (\sigma_R - K_{vol}) \times B \\ \text{variance swap payoff} &= (\sigma_R^2 - K_{var}) \times B\end{aligned}$$

where K_{vol} and K_{var} are the annualised volatility and variance delivery prices respectively and B is the notional amount of the swap in dollars per annualised volatility (variance) point. Hence an investor who holds a long position in a variance swap, receives $\$\sigma_R^2 \times B$ and pays $\$K_{var} \times B$ at expiry. We will scale the payoff so that we take $B \equiv 1$.

The measure of realised variance is defined at the start of the contract, and is typically taken to be $\frac{1}{T}\sum_{i=1}^{M}\left(\frac{S_i - S_{i-1}}{S_{i-1}}\right)^2$ where S_i is the stock price taken at time t_i. In continuous time this can be approximated by $\sigma_R^2 = \frac{1}{T}\int_0^T \sigma_t^2 dt$. The corresponding payoff then for the variance swap is $\sigma_R^2 - K_{var}$, and for the volatility swap $\sigma_R - K_{vol}$.

As variance is fundamentally easier to analyse than volatility, over the past 10 years, models have been introduced in the literature to address the valuation and hedging of variance products (see eg [3], [11]). In particular, Demeterfi *et al* [6] show that a variance swap can be theoretically replicated by a hedged portfolio of standard call and put options with suitably chosen stock and thus its value is the cost of the replication.

Most of the analytic formulae for variance swaps are based on the assumption that the underlying asset evolves continuously and that the variance ν, follows a particular form of stochastic differential equation (SDE), such as

$$d\nu = K(\theta - \nu)dt + c\nu^{\gamma}\ dZ \tag{2}$$

for $0 \leq \gamma \leq 1$ as in [13] ($\gamma = 1$), or the model by Heston [10] ($\gamma = \frac{1}{2}$).

For volatility swaps, to the author's knowledge, there are no known analytic formulae for their valuation, but Brockhaus and Long [2] provide an approximate volatility convexity-correction relating variance and volatility products. This adjustment has been used by many authors such as Swishchuk [16] and Javaheri *et al* [13] to find approximate solutions for volatility swaps based on their underlying variance models.

While models for variance dynamics such as (2) with $0 \leq \gamma \leq 1$ are able to capture the mean-reverting nature of variance, they may not necessarily capture the actual behaviour of instantaneous variance for particular stock prices. In order to test their performance Chacko *et al* [4] performed a comprehensive empirical analysis on variance models of the form (2). Using the estimation technique of spectral GMM they found that the best value of γ was between 1 and 2, with the standard errors indicating that the differences between the values found for γ and one half (as in the Heston model) were statistically significant.

In this paper we present a new model for variance, ν, namely

$$d\nu = (c(t)\nu + c_3\nu^2)dt + k\nu^{\frac{3}{2}}dZ, \tag{3}$$

where k and c_3 are constants and $c(t)$ is an arbitrary function of time and provide analytic solutions for both variance and volatility swaps under this model. In addition to the advantages of this model as listed at the start of this section, the free function of time lends itself to calibration so that theoretical and current market prices can be matched for all maturities.

Manuscript received January 7, 2011.

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ISBN: 978-988-18210-6-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Volatility and variance swaps can be priced as path-dependent quantities by solving an appropriate partial differential equation (PDE). For a variance model

$$d\nu = f(\nu,t)dt + g(\nu,t)dZ$$

(so that the corresponding risk-neutral model is $d\nu = [f(\nu,t) - \lambda(\nu,I,t)g(\nu,t)]dt + g(\nu,t)d\tilde{Z}$, where $\lambda(\nu,I,t)$ is the market price of variance risk and $\tilde{Z}$ is a Wiener process under a risk-neutral measure Q, under which ν becomes a martingale), this PDE is given by (see [18])

$$V_t + \frac{g(\nu,t)^2}{2}V_{\nu\nu} + (f(\nu,t) - \lambda(\nu,I,t)g(\nu,t))V_\nu + \nu V_I - rV = 0, \tag{4}$$

for $V(\nu,I,t)$ the value of the swap and where $I = \int_0^t \nu_s ds$.

The payoff of the variance swap can then be written as $V(\nu,I,T) = \frac{I}{T} - K_{var}$ and for the volatility swap $V(\nu,I,T) = \sqrt{\frac{I}{T}} - K_{vol}$, where T is the expiry date.

In Section II we show how with variance model (3), a reduction of order can be achieved for the pricing equation. This enables us to provide an exact solution for the variance swap as well as a series solution which produces accurate answers to all orders and demonstrates the accuracy of the method. The series solution method is then used to price the volatility swap. We also derive an asymptotic expansion solution valid at times near expiry.

B. VIX Options

Currently the most popular indicator of overall market volatility in the US is the CBOE volatility index, VIX, which provides a measure of the **implied** volatility of options with a maturity of 30 days on the S&P500 index from eight different SPX option series. It thus presents a measure of the market's expectation of volatility over the next 30 days.

The VIX was introduced by Whaley [17] and has become of particular interest in recent years with the introduction of VIX futures contracts in 2004 and of options on the VIX in 2006. These offered investors new instruments for speculating and hedging volatility risk directly on the S&P500 index and so valuations on VIX derivatives became necessary. Whaley himself used Black's (1976) formula (see eg [12]) to price volatility options under the assumptions of a lognormal volatility process and the existence of a futures contract on volatility with futures price equal to the current index level.

Detemple and Osakwe [7] provide valuations for European and American volatility options under the volatility, V, models

$$dV = (\alpha - \beta V)dt + kV^\gamma dZ \tag{5}$$

with $\alpha = 0, \gamma = 1$ (Geometric Brownian Motion), $\gamma = 0$, (Mean-Reverting Gaussian), $\gamma = 0.5,\ k^2 = 4\alpha$ (an example of a Mean-Reverting Square Root process) as well as a mean-reverting log process, $d(\ln V) = (\alpha - \lambda \ln V)dt + \sigma dZ$.

Many continuous-time stochastic volatility models have been proposed in the literature in order to price volatility contracts. This includes models of the form (5) as well as the popular Heston [10] model discussed in the previous section.

One of the main aims of this paper is to present results of an empirical study of many of these models, as well as the 3/2- model (1), in their ability to capture the dynamics of the VIX. Achieving an empirically validated model is important as the ability of the stochastic model to capture the dynamics of the VIX ultimately affects the valuations for which it is used. What we found was that the power of the diffusion term is an important feature differentiating the volatility models and its unconstrained estimate is 1.5. The 3/2-model (1) is shown to outperform the current popular models in capturing the behaviour of the VIX. Further, an analytic solution is found for the value of a call option on the VIX under the 3/2- volatility model.

II. Valuation of Variance and Volatility Swaps

A. Reduction of the Pricing PDE

From (4), for the risk-neutral variance process

$$d\nu = (c(t)\nu + c_3\nu^2)dt + k\nu^{\frac{3}{2}}d\tilde{Z} \tag{6}$$

the PDE for the value of swaps $V(\nu,I,t)$ is

$$V_t + \frac{k^2\nu^3}{2}V_{\nu\nu} + (c(t)\nu + c_3\nu^2)V_\nu + \nu V_I - rV = 0. \tag{7}$$

To solve (7), we use Lie's classical method to find a transformation to reduce the number of independent variables. In essence, a classical Lie point symmetry of a linear PDE such as (7) may be represented as a linear first-order operator

$$\begin{aligned}\Gamma &= \rho(\nu,I,t,V)\frac{\partial}{\partial\nu} + J(\nu,I,t,V)\frac{\partial}{\partial I} + \tau(\nu,I,t,V)\frac{\partial}{\partial t} \\ &+ \bar{V}(\nu,I,t,V)\frac{\partial}{\partial V}\end{aligned}$$

such that $\Gamma(V) = 0$ must be satisfied by every invariant solution (see e.g [1]).

The finite-dimensional symmetries for equation (7) are generated by $\frac{\partial}{\partial I}, V\frac{\partial}{\partial V}$ and

$$\begin{aligned}\Gamma &= \left[e^{\int_t^T c(x)dx}\int_t^T e^{-\int_y^T c(z)dz}dy\right]\frac{\partial}{\partial t} - \nu\tau'(t)\frac{\partial}{\partial\nu} \\ &+ V\left[\left(\frac{c'(t)}{\nu k^2} + r\right)\tau(t) + \frac{\tau''(t)}{\nu k^2} + \frac{c(t)\tau'(t)}{\nu k^2}\right]\frac{\partial}{\partial V}.\end{aligned} \tag{8}$$

Using (8) and solving the corresponding invariant surface condition ISC yields the functional form of the similarity solution as

$$V = \phi(I,\eta)e^{-r(T-t)} \tag{9a}$$

$$\text{with } \eta = \nu\tau(t), \tag{9b}$$

where $\tau(t)$ is the coefficient of $\frac{\partial}{\partial t}$ in (8). Substitution of (9a) into (7) implies that $\phi(I,\eta)$ needs to satisfy

$$k^2\eta^2\phi_{\eta,\eta} + 2\phi_\eta(c_3\eta - 1) + 2\phi_I = 0. \tag{10}$$

Hence equation (10) needs to be solved subject to

$$\phi(I,0) = \frac{I}{T} - K_{var} \tag{11}$$

for a variance swap and subject to

$$\phi(I,0) = \sqrt{\frac{I}{T}} - K_{vol} \tag{12}$$

for a volatility swap.

ISBN: 978-988-18210-6-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

B. Variance Swap Valuation

In this section we present the exact solution for K_{var}, the value of the delivery price, as well as the value of the variance swap at any time until expiry when the dynamics of the risk-neutral variance is described by equation (6).

We also provide a series solution which yields 'exact' solutions to all orders of a Taylor series expansion. This is done for two reasons: the series solution avoids the necessity of evaluating an integral as the exact solution does, but more importantly to validate the series method against the exact solution. This is important as the same method will be used to price volatility swaps for which an exact expression otherwise cannot be found.

1) Exact Solution:

Theorem 1: Under the risk-neutral variance model (6), the solution for the variance swap and the delivery price are given respectively by

$$V(\nu, I, t) = e^{-r(T-t)}\left\{\frac{I}{T} - K_{var} + \frac{2}{k^2 T}\int_0^{\eta} x^{\frac{2c_3}{k^2}-2} e^{\frac{2}{k^2 x}} P(x)dx + \frac{2}{k^2 T}P(\eta)\int_{\eta}^{\infty} x^{\frac{2c_3}{k^2}-2} e^{\frac{2}{k^2 x}} dx\right\}, \tag{13}$$

and

$$K_{var} = \frac{2}{k^2 T}\int_0^{\bar{\eta}} x^{(\frac{2c_3}{k^2}-2)} e^{\frac{2}{k^2 x}} P(x)dx + \frac{2}{k^2 T}P(\bar{\eta})\int_{\bar{\eta}}^{\infty} x^{(\frac{2c_3}{k^2}-2)} e^{\frac{2}{k^2 x}} dx \tag{14}$$

where

$$P(\eta) = \int_0^{\eta} x^{\frac{-2c_3}{k^2}} e^{\frac{-2}{k^2 x}} dx, \tag{15}$$

η is given in (9b) and

$$\bar{\eta} = \nu_0 \tau(0) = \nu_0\, e^{\int_0^T c(x)dx}\int_0^T e^{-\int_y^T c(z)dz} dy \tag{16}$$

with ν_0 the initial value of ν.

Proof: The solution for the variance swap is given by equation (9a) where ϕ satisfies equation (10) subject to condition (11). The solution for ϕ is of the simple form $\phi(I, \eta) = f(\eta)I + g(\eta)$ for which $f(0) = \frac{1}{T}$ and $g(0) = -K_{var}$. Substitution of ϕ into (10) and consideration of boundary conditions gives $f(\eta)$ and $g(\eta)$ and then using (9) gives the price of the variance swap as in (13).

Initially at time $t = 0$, the value of the swap must be zero. This then gives the value of the delivery price as given in (14). ♯♯♯

2) Series Solution:

Theorem 2: Under the risk-neutral variance model (6), a series solution for the value of a variance swap is

$$V(\nu, I, t) = e^{-r(T-t)}\Big(-K_{var} + \lim_{n\to\infty}\sum_{i=1}^{n}\frac{e^{\frac{-\lambda_i I}{2}}}{\eta^{\beta_i}} C_i M\left(\beta_i, 2\beta_i + 2 - \frac{2c_3}{k^2}, \frac{-2}{k^2\eta}\right)\Big) \tag{17}$$

where M is the Kummer M function, η is given in equation (9b),

$$\beta_i = \frac{1}{2}\left[\left(\frac{2c_3}{k^2} - 1\right) + \sqrt{\left(1 - \frac{2c_3}{k^2}\right)^2 + \frac{4\lambda_i}{k^2}}\right], \tag{18}$$

constants λ_i and D_i satisfy

$$\lim_{n\to\infty}\sum_{i=1}^{n} D_i\left[\sum_{j=0}^{\infty}\frac{(-\lambda_i I)^j}{2^j j!}\right] = \frac{I}{T}, \tag{19}$$

and

$$C_i = D_i \frac{\Gamma\left(\beta_i + 2 - \frac{2c_3}{k^2}\right)}{\Gamma\left(2\beta_i + 2 - \frac{2c_3}{k^2}\right)}\left(\frac{2}{k^2}\right)^{\beta_i}. \tag{20}$$

The delivery price is given by

$$K_{var} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{e^{\frac{-\lambda_i I}{2}}}{\bar{\eta}^{\beta_i}} C_i M\left(\beta_i, 2\beta_i + 2 - \frac{2c_3}{k^2}, \frac{-2}{k^2\bar{\eta}}\right) \tag{21}$$

where $\bar{\eta}$ is given in (16).

Proof: Equation (10) has a sum of separable solutions of the form

$$\phi(I, \eta) = \sum_{i=1}^{\infty}\frac{e^{\frac{-\lambda_i I}{2}}}{\eta^{\beta_i}}\left[C_i M\left(\beta_i, 2\beta_i + 2 - \frac{2c_3}{k^2}, \frac{-2}{k^2\eta}\right) + B_i U\left(\beta_i,\ 2\beta_i + 2 - \frac{2c_3}{k^2}\ ,\ \frac{-2}{k^2\eta}\right)\right] - K_{var} \tag{22}$$

where M and U are the Kummer M and Kummer U functions respectively, λ_i are separation constants and β_i are as given in (18). To satisfy the boundary condition at $\eta = 0$, we require $B_i = 0$ and

$$\sum_{i=1}^{\infty} e^{\frac{-\lambda_i I}{2}} C_i \frac{\Gamma\left(2\beta_i + 2 - \frac{2c_3}{k^2}\right)}{\Gamma\left(\beta_i + 2 - \frac{2c_3}{k^2}\right)}\left(\frac{k^2}{2}\right)^{\beta_i} = \frac{I}{T}. \tag{23}$$

Defining D_i in terms of C_i as in (20) and expanding the exponential term in (23) in a Taylor series about $I = 0$, implies we need equation (19) to be satisfied. (This can be done by equating coefficients of I and choosing values for λ_i with at least one negative value to ensure the solution for ϕ does not vanish for large η.) Initially at $t = 0$, we have $V = 0$. This gives K_{var} as in (21). ♯♯♯

3) Numerical Example: Using $k = 0.5$, $c_3 = -1$, $\nu = 0.04$, and both $c(t) = 0.05$ and $c(t) = 0.15\cos(t) + 0.27$, it was found that using equation (21) with $\lambda_1 = -\frac{1}{2}$, $\lambda_{i+1} = i$, $i = 1, ..\ (n-1)$ with $n = 10$, gave answers that agreed with the solution for K_{var} from equation (14), to 8 decimal places.

C. Volatility Swap Valuation

For the valuation of a volatility swap we need to solve equation (10) subject to condition (12). Finding such a closed-form solution is difficult due to the nature of the payoff (12). As such we find a series solution similar to the solution form in Section II.B. and which is capable of yielding 'exact answers' to any order of a Taylor series expansion, and an asymptotic expansion, yielding accurate answers near expiry. We note that as a consequence of Jensen's inequality (see e.g. [14]), $E(\sqrt{I}) \leq \sqrt{E(I)}$. Hence as $K_{var} = E\left(\frac{I}{T}\right)$ and $K_{vol} = E\left(\sqrt{\frac{I}{T}}\right)$ we expect $K_{vol} \leq \sqrt{K_{var}}$.

ISBN: 978-988-18210-6-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

1) Series Solution leading to 'Exact' Solutions:

Theorem 3: Under the risk-neutral variance model (6), a series solution for the value of a volatility swap is

$$V(\nu, I, t) = e^{-r(T-t)}\Big[- K_{vol} + \lim_{n\to\infty}\sum_{i=1}^{n} \frac{\alpha_i}{\eta^{\beta_i}} e^{-\frac{\lambda_i(I-I_0)}{2}} M\left(\beta_i, 2\beta_i + 2 - \frac{2c_3}{k^2}, \frac{-2}{k^2\eta}\right)\Big] \quad (24)$$

and the delivery price is

$$K_{vol} = \lim_{n\to\infty}\sum_{i=1}^{n} \frac{\alpha_i}{\bar{\eta}^{\beta_i}} e^{-\frac{\lambda_i(I-I_0)}{2}} M\left(\beta_i, 2\beta_i + 2 - \frac{2c_3}{k^2}, \frac{-2}{k^2\bar{\eta}}\right) \quad (25)$$

where η is given in (9b), $\bar{\eta}$ is given in equation (16), β_i are given in (18), constants λ_i and D_i satisfy

$$\lim_{n\to\infty}\sum_{i=1}^{n} D_i \left[\sum_{j=0}^{\infty} \frac{(-\lambda_i(I-I_0))^j}{2^j j!}\right] = \frac{1}{\sqrt{T}}\left[\sqrt{I_0} + \frac{1}{2\sqrt{I_0}}(I-I_0) + \dots\right], \quad (26)$$

$$\alpha_i = D_i \frac{\Gamma\left(\beta_i + 2 - \frac{2c_3}{k^2}\right)}{\Gamma\left(2\beta_i + 2 - \frac{2c_3}{k^2}\right)}\left(\frac{2}{k^2}\right)^{\beta_i}, \quad (27)$$

and where I_0 is a constant value close to or equal to the expected value of I.

Proof: Following the approach of Section II.B., equation (10) has a sum of separable solutions of the form

$$\phi(I, \eta) = -K_{vol} + \sum_{i=1}^{\infty} \frac{e^{-\frac{\lambda_i I}{2}}}{\eta^{\beta_i}} C_i M\left(\beta_i, 2\beta_i + 2 - \frac{2c_3}{k^2}, \frac{-2}{k^2\eta}\right) \quad (28)$$

where β_i are given in (18) and which in order to satisfy condition (12), requires

$$\sum_{i=1}^{\infty} e^{\frac{-\lambda_i I}{2}} C_i \frac{\Gamma\left(2\beta_i + 2 - \frac{2c_3}{k^2}\right)}{\Gamma\left(\beta_i + 2 - \frac{2c_3}{k^2}\right)}\left(\frac{k^2}{2}\right)^{\beta_i} = \frac{\sqrt{I}}{\sqrt{T}}. \quad (29)$$

The right-hand side of equation (29) can be expanded using a Taylor series approximation of $\sqrt{I}$ about $I = I_0$ ie

$$\frac{\sqrt{I}}{\sqrt{T}} = \frac{1}{\sqrt{T}}\left[\sqrt{I_0} + \frac{1}{2\sqrt{I_0}}(I-I_0) - \frac{1}{8I_0^{3/2}}(I-I_0)^2 + \cdots\right] \quad (30)$$

where I_0 is a value close to I.

As the coefficients of I in (30) do not form a convergent series, we cannot equate powers of I in (29). Instead we rewrite (28) as

$$\phi(I, \eta) = \sum_{i=1}^{\infty} \frac{e^{\frac{-\lambda_i(I-I_0)}{2}}}{\eta^{\beta_i}} \alpha_i M\left(\beta_i, 2\beta_i + 2 - \frac{2c_3}{b^2}, \frac{-2}{b^2\eta}\right) - K_{vol} \quad (31)$$

where $\alpha_i = e^{\frac{-\lambda_i I_0}{2}} C_i$, so that condition (29) requires equation (26) to hold where D_i are defined as in (27). (This can be done by equating powers of $(I - I_0)$ and choosing values for λ_i with at least one negative value so that the solution for ϕ does not vanish for large η.) Using (31) and (9) then leads to the value (24). Initially at $t = 0$, we have $V = 0$. This then gives K_{vol} as in (25). ♯♯♯

TABLE I
K_{vol} VALUES

	$c(t) = 0.05$		
	T=1	T=2	T=5
$\sqrt{K_{var}}$	0.200488	0.200952	0.202206
KumM	0.200406	0.200792	0.201835
	$c(t) = 0.15\cos(t) + 0.27$		
	T=1	T=2	T=5
$\sqrt{K_{var}}$	0.219521	0.236879	0.264242
KumM	0.219414	0.236624	0.263499

2) Asymptotic Expansion:

Theorem 4: Under the risk-neutral variance model (6) an asymptotic solution for the value of the volatility swap near expiry $t = T$ is $V = \phi(I, \eta)e^{-r(T-t)}$ with

$$\phi(I, \eta) = -K_{vol} + \frac{\sqrt{I}}{\sqrt{T}} + \sum_{p=1}^{\infty} \eta^p \psi_p(I) \quad (32)$$

where

$$\psi_1(I) = \frac{1}{2k_3\sqrt{I}\sqrt{T}}$$

$$\psi_{p+1}(I) = \frac{\{[b^2 p(p-1) + 2c_3 p]\psi_p(I) + \psi_p'(I)\}}{2k_3(p+1)}.$$

Proof: Substituting $\eta = 0$ into PDE (10) and using the final condition for the volatility swap (12) we can find $\phi_\eta(I, 0)$. Then successively differentiating (10) with respect to η and substituting $\eta = 0$ into the resultant equation, we find that the general form of the n^{th} derivative of ϕ with respect to η at $\eta = 0$ is

$$\phi\underbrace{\eta\ldots\eta}_{\text{n times}}(I, 0) = \sum_{i=1}^{2n-1} \frac{k_i}{I^{i-\frac{1}{2}}}$$

where the k_i are constants. This then suggests an asymptotic expansion of ϕ near $\eta = 0$ of the form (32) and upon substituting into (10) we find the recurrence relationship as given.

3) Numerical examples for Volatility Swaps: In this section we compare values for the delivery price K_{vol} and the value of the volatility swap $V_{volswap}$ using the solutions outlined in Theorems 3 and 4. These will be referred to respectively as KumM and Asym Exp solutions. Parameter values that are held constant in all examples are $c_3 = -1, k = 0.5, r = 0.05$ and the initial value of $\nu = 0.04$. For the KumM solution, $L_i (= \lambda_i I_0)$ values of $-\frac{1}{2}, 1, 2, \ldots, n-1$ were used, $I_0 = E(I)$ and n, the number of terms in the solution was such that the difference between successive partial sums was less than 1.0×10^{-5}.

K_{vol} values:

Delivery prices K_{vol} are listed in Table I for $c(t) = 0.05$ and $c(t) = 0.15\cos(t) + 0.27$. For the KumM solutions $I_0 = E(I) = K_{var} \times T$ was used. These results show that $\sqrt{K_{var}}$ values overestimate K_{vol} as expected.

Volatility swap values at different times:

Table II lists values of the volatility swap, $V_{volswap}$, at different times with $c(t) = 0.05$ and at specified values of I using the solutions found in Section II.C. The maturity date used was $T = 5$. Variance swap values, $V_{varswap}$, at the specified values of I and t are also given.

ISBN: 978-988-18210-6-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

TABLE II
COMPARISON OF VOLATILITY SWAP VALUES WITH $T = 5$ UNDER THE METHODS OUTLINED IN SECTION II.C.

				$V_{volswap}$	
t	η	I	$V_{varswap}$	KumM	Asym Exp
3	0.189308	0.15	0.021488	0.047308	0.054217
4	0.092288	0.15	0.0064326	0.015626	0.018095
4.5	0.045567	0.25	0.017579	0.0398699	0.040253
4.9	0.009023	0.25	0.010855	0.025625	0.025643
4.95	0.004506	0.25	0.0099870	0.023714	0.023719
4.99	0.00090023	0.25	0.0092883	0.022163	0.022163

It is known that results from the asymptotic expansion will be accurate for η values near 0. For small η then results in Table II confirm that the KumM results are correct to six decimal places. The more η increases from 0 the more error the asymptotic expansion results will accumulate until they reach the point where they become completely inaccurate. However, the Kummer M series give 'exact' results.

III. VIX OPTIONS

A. Empirical Testing of Models

In this section, with V representing the value of the VIX index, we compare the performance of the models in Table III in capturing the dynamics of the VIX. We do this by firstly nesting them within the larger unrestricted model

$$dV = (c_1 + \frac{c_2}{V} + c_3 V \ln V + c_4 V + c_5 V^2)dt + kV^{\gamma} dZ \quad (33)$$

by placing certain restrictions on the parameters e.g for Model 1 $c_1 = 0, c_3 = 0, c_5 = 0, \gamma = 0$. Models 1-6 are known volatility models found in the literature while Models 7 and 8 are 3/2-models used for comparison.

TABLE III
THE MODELS TO BE INVESTIGATED EMPIRICALLY.

1	$dV = (\frac{c_2}{V} + c_4 V)dt + (k)dZ$
2	$dV = (c_1 + c_4 V)dt + (kV)dZ$
3	$dV = (c_1 + c_4 V)dt + (kV^{0.5})dZ$
4	$dV = (c_4 V)dt + (kV)dZ$
5	$dV = (c_1 + c_4 V)dt + (k)dZ$
6	$dV = (c_3 V \ln V + c_4 V)dt + (kV)dZ$
7	$dV = (c_4 V + c_5 V^2)dt + (kV^{1.5})dZ$
8	$dV = (c_1 + c_4 V)dt + (kV^{1.5})dZ$

The performances of the nested models 1-8 are benchmarked against the larger unrestricted model (33) using the estimation technique of Generalised Method of Moments (GMM). For each nested model a hypothesis test to test if the nested model was not imposing overidentifying restrictions was conducted using an appropriate test statistic. This statistic is asymptotically distributed χ^2 with degrees of freedom equal to the number of restrictions imposed on the unrestricted model to obtain the nested model. The data used in the analysis was the VIX index values between the years of 1990 and 2009 (collected using Bloomberg). The GMM results are presented in Table IV.

From Table IV, the χ^2 values for Models 1-6 imply that are all rejected at the 5% (and even 1%) level of significance. Hence these models are misspecified and place unreasonable restrictions on the unrestricted model.

However, Model 7 with a χ^2_4 of 5.82669 is accepted at all standard levels of significance with a p-value of 0.212. The model is thus not misspecified and the restrictions it imposes on the unrestricted model are reasonable. As well, all parameters in the model are statistically significantly different from zero. Model 8 as well, which has the same form for the diffusion term as Model 7, cannot be rejected at the standard levels of significance. However it did not perform as well as Model 7 with a smaller p-value of 0.129.

Hence, Models 7 and 8 are the only models from the models tested that are found to be acceptable models for describing the behaviour of the VIX, and of these, Model 7 performed the best. We now price VIX options based on this model.

B. VIX Option Price

Similar to many authors such as Stein and Stein [15] and Grünbichler and Longstaff [9], we assume that the market price of risk is such that the risk-neutral process for V is of the same form as the real process ie

$$dV = (\alpha V + \beta V^2)dt + kV^{\frac{3}{2}} d\tilde{Z}, \quad (34)$$

where $\tilde{Z}$ is a Wiener process under a risk-neutral measure Q, under which V is a martingale, and under this process find the fair value for a call option on the VIX.

Theorem 5: The value of a call option on the VIX when the VIX follows the risk-neutral process (34) with $\beta < 0$, is given by

$$\begin{aligned} C(V,t) = {} & \frac{2\alpha e^{-r(T-t)}}{k^2 p} e^{\left[\frac{-2\alpha e^{-\alpha(T-t)}}{k^2 V p}\right]} V^{-\frac{\beta}{k^2}+\frac{1}{2}} \\ & \times e^{\left[\alpha(T-t)\left(-\frac{\beta}{k^2}+\frac{1}{2}\right)\right]} \\ & \times \int_0^{\frac{1}{X}} \phi^{\frac{1}{2}-\frac{\beta}{k^2}} \left(\frac{1}{\phi} - X\right) e^{-\frac{2\alpha\phi}{k^2 p}} I_\nu \left(\frac{4\alpha\sqrt{\phi} e^{\frac{-\alpha(T-t)}{2}}}{k^2\sqrt{V}p}\right) d\phi \end{aligned} \quad (35)$$

where $\nu = 1 - \frac{2\beta}{k^2}$, $p = 1 - \exp(-\alpha(T-t))$ and $I_\nu(.)$ is the modified Bessel function of order ν.

Proof: Given that the VIX, V, follows the risk-neutral process (34), then by Itô's Lemma, $w = \frac{1}{V}$ follows the process

$$dw = ((k^2 - \beta) - \alpha w)dt - k\sqrt{w} dZ.$$

Hence w follows a mean-reverting square-root process such as that used by Cox *et al* [5] to model short interest rates. Further, as explained by Feller [8], for $k^2 \leq 2(k^2 - \beta)$, ie $\beta \leq \frac{k^2}{2}$, (and so for all negative β), w and hence V will remain positive.

With $\beta < 0$, the probability density function of w at a future time T, conditional on its value at the current time t is given by

$$f(w_T | w_t) = ce^{-u-z} \left(\frac{z}{u}\right)^{\frac{q}{2}} I_q \left(2\sqrt{uz}\right) \quad (36)$$

where $c = \frac{2\alpha}{k^2(1-e^{-\alpha(T-t)})}$; $u = cw_t e^{-\alpha(T-t)}$; $z = cw_T$; $q = 1 - \frac{2\beta}{k^2}$ (see [5]). Using risk-neutral valuation, the value of the call option on the VIX can be found as

$$C(V,t) = e^{-r(T-t)} E^Q \left(B \max\left(\frac{1}{w_T} - X, 0\right)\right)$$

ISBN: 978-988-18210-6-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

where E^Q denotes the expectation under the risk-neutral measure Q, X is the exercise price and B is a notional amount of the option measured in currency units per volatility point. For brevity we scale the option value so that the notional amount B can be taken to be one. Using (36) then gives

$$C(V,t) = \frac{2\alpha e^{-r(T-t)}}{k^2(1-\exp(-\alpha(T-t)))} e^{\left[\frac{-2\alpha w_t e^{-\alpha(T-t)}}{k^2(1-\exp(-\alpha(T-t)))}\right]}$$
$$\times \int_0^\infty \left[e^{\frac{-2\alpha\phi}{k^2(1-\exp(-\alpha(T-t)))}} \left(\frac{\phi}{w_t} e^{\alpha(T-t)}\right)^{\frac{1}{2}-\frac{\beta}{k^2}} \times \right.$$
$$\left. \max\left(\frac{1}{\phi} - X, 0\right) I_\nu \left(\frac{4\alpha\sqrt{\phi}\sqrt{w_t} e^{\frac{-\alpha(T-t)}{2}}}{k^2(1-\exp(-\alpha(T-t)))} \right) d\phi \right],$$

where $w_t = \frac{1}{V_t}$. This can be simplified to give the form (35). ##

IV. Conclusion

In this paper, we have presented empirically validated models with a 3/2-power diffusion term, to price variance/volatility swaps and VIX options. Under the model for the variance, we have provided the exact solution for the variance swap, as well as a validated series solution which can in principle provide solutions to any degree of accuracy. As well, we have presented an asymptotic solution and a series solution for the volatility swap which also can provide in principle, solutions to any degree of accuracy. As exact volatility swap valuations are non-existent, such a valuation under a time-dependent variance model we feel is a significant step forward.

As well, an analytic solution to call option prices on the VIX under an empirically-proven 3/2-model has been provided. To the authors' knowledge, no other exactly solvable VIX option pricing model comes from using a stochastic volatility model that is statistically acceptable when compared with the data.

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TABLE IV
EMPIRICAL RESULTS FOR NESTING MODELS 1-8 IN TABLE III WITHIN EQUATION (33).

Model	c_1	c_2	c_3	c_4	c_5	k	γ	χ^2	DF
Unrestricted	66.0223 [.025]	-2.39462 [0.026]	269.118 [0.025]	228.220 [0.027]	328.78263 [0.036]	1.98898 <[.001]	1.47972 <[.001]	N/A	N/A
1	0	0.047608 <[.001]	0	-1.34495 0.001	0	0.180592 <[.001]	0	63.3919 <[.001]	4
2	.611602 <[.001]	0	0	-3.14312 <[.001]	0	-0.956694 <[.001]	1	24.4905 <[.001]	4
3	0.615456 <[.001]	0	0	-3.16371 <[.001]	0	0.426133 <[.001]	0.5	45.6118 <[.001]	4
4	0	0	0	0.639072 0.002	0	0.950413 <[.001]	1	57.0297 <[.001]	5
5	0.616113 <[.001]	0	0	-3.16372 <[.001]	0	0.180875 <[.001]	0	60.1120 <[.001]	4
6	0	0	-3.56258 <[.001]	-5.66506 <[.001]	0	0.958316 <[.001]	1	21.7566 <[.001]	4
7	0	0	0	3.71697 <[.001]	-16.990461 <[.001]	2.03420 <[.001]	1.5	5.92669 0.212	4
8	0.592560 <[.001]	0	0	-3.01339 <[.001]	0	2.02932 <[.001]	1.5	7.13872 0.129	4

ISBN: 978-988-18210-6-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)