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Generalization of the Takacs' Formula for *GI*/*M*/*n*/0 Queuing System with Heterogeneous Servers

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Abstract — This study is mainly concerned with the finite-capacity queuing system with recurrent input, *n* heterogeneous servers, and no waiting line. In the system customers choose only one server from the empty servers with equal probability. When all servers are busy, customers depart from the system without taking any service. These customers are called "lost customers". In this study, the transition probabilities (p_{ij}) of the embedded Markov chain are calculated using the generalization of the Takacs' formula. The steady-state probabilities can be obtained from $\pi_j = \sum_{i=0}^{\infty} \pi_i p_{ij}$ with $\sum_{j=0}^{\infty} \pi_j = 1$. Since there is no waiting line, the loss probability (P_L) is equal to the probability that all servers are busy.

Index Terms— Embedded Markov chain, heterogeneous servers, Laplace-Stieltjes transform, loss probability

I. Introduction

THE fundamental queuing models with no waiting lines but identical servers have been examined and analyzed so far. The most efficient measurement of the system in such models is the loss probability of customers. Computation of the loss probability and minimization of this probability constitute a serious problem. Among the queuing models with no waiting lines, the most classical model known is the *M*/*M*/*n*/0 queuing system and this system was first examined by Erlang (1917). Erlang [6] obtained the state probabilities for the *M*/*M*/*n*/0 model as follows:

$$P_k = \frac{\rho^k / k!}{\sum_{k=0}^{n} (\rho^k / k!)} \quad , \; 0 \le k \le n \tag{1}$$

where $\rho = \lambda / \mu$ is the offered load, λ^{-1} and μ^{-1} are the means of interarrival times and service time, respectively and both interarrival times and service times are exponentially distributed. Formula (1) is known as Erlang's loss formula for $k = n$. Communication is of great importance in the mathematical modeling of systems and this formula has been a source of inspiration to analyze more complex systems. Palm (1943) examined the *GI*/*M*/*n*/0 queuing model and further extended the model, suggested by Erlang, for the state of having independent interarrival times with a general distribution. Palm [13] analyzed the stream of the lost customers in the *GI*/*M*/*n*/0 queuing model and computed the loss probability of customers in the system as follows:

$$\frac{1}{P_n} = \sum_{k=0}^{n} \binom{n}{k} c_k , \tag{2}$$

where, with φ being the Laplace-Stieltjes transform of the interarrival time, c_k are

$$c_0 = 1, \quad c_k = \prod_{k=1}^{n} \frac{1-\varphi(k\mu)}{\varphi(k\mu)} \; (1 \le k \le n) .$$

Takacs [16] defined that arrival times $\tau_n \; (n = 1,2,\ldots,n)$ are a recurrent process, that the random variables $\chi_n \; (n = 1,2,\ldots,n)$ are exponentially distributed and that $\eta(t)$ is the number of the busy servers at instant *t*. He showed that the sequence of random variables $\{\eta_n\} \; (n = 1,2,\ldots,n)$ forms a Markov chain and obtained its one-step transition probabilities $p_{jk} = P[\eta_{n+1} = k \,|\, \eta_n = j]$ as follows:

$$p_{jk} = \binom{j+1}{k} \int_0^{\infty} e^{-k\mu x} (1 - e^{-\mu x})^{j+1-k} \, dF(x) \tag{3}$$

for $k = 1, 2, \ldots, n-1$, and $p_{n,k} = p_{n-1,k}$.

Konig and Matthes [10] generalized Erlang's formula for dependent service times. Cinlar and Disney [5] obtained the Laplace-Stieltjes (LS) transforms of the independent interoverflow times with an identical distribution for the single-server queuing model with recurrent entries and finite capacity. By means of the discrete-parameter stochastic process (the Markov chain) generated considering the arrival times of the customer in the system and his time of departure from the system, Takacs [17] analyzed the model, suggested by Erlang, using a different method. Brumelle [4] generalized Erlang's formula for dependent arrivals and dependent service rates and obtained the mean waiting time of the customer in the system. In the $GI / G / 1$ queuing model with no waiting lines, Halfin [9] obtained the distribution function of the interoverflow times of customers. In addition, he performed an implementation of the results,

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which were obtained in the study concerned, for the $M/G/1$ model, where interarrival times were exponentially distributed, and for the $GI/E_k/1$ model where the service time was Erlang distributed. By making the discrete-time analysis of the *GI*/*G*/2 loss system, Atkinson [1] presented an alternative to Erlang's loss model when the arrival process did not well approximate the Poisson process. Again in another study by Atkinson (1999), the $C_2/G/1$ queue model and the $C_2/G/1$ loss system model were examined. Atkinson [2] showed that with c_X being the coefficient of variation of interarrival time, when $c_X^2 < 1$, the probability of delay and the probability of loss are both increasing in $\beta(s)$ for the above-mentioned models, respectively. $\beta(s)$ is the Laplace-Stieltjes transform of the service time distribution.

In many studies in the literature, queuing models were examined under the assumption that the mean service times of servers were equal. However, this assumption is mostly invalid in real life. Gumbel [8] obtained the limit distribution of the number of customers in the system for the $M/M/n$ model with infinite waiting lines and heterogeneous servers. Singh [15] examined the Markovian $M/M_i/2/(\beta)$ queuing system with two heterogeneous servers. Singh computed the mean number of customers in the system, the mean queue length, and the mean waiting time of the customer in the system and compared these results with the homogeneous $M/M/2/(\beta)$ model. Again, Singh [16] obtained the steady-state probabilities, the mean number of customers waiting in the queue, and the mean waiting time of the customer in the system for the $M/M_i/3$ queuing model with infinite waiting lines and 3 heterogeneous servers. Fakinos [7] gave a generalization of the Erlang Loss formula for the case of non-identical servers. Nath and Enns [12] proved that the loss probability was minimum under fast service condition in the $M/M/n/0$ queuing model with no waiting lines but heterogeneous servers. Kumar, Madheswari and Venkatakrishnan [11] examined the $M/M/2$ queuing model with heterogeneous servers and infinite waiting lines also considering that catastrophes fitting the Poisson distribution with rate γ might occur.

In this study, the $GI/M/n/0$ queuing model with exponentially distributed service times but different mean service times (with heterogeneous servers) is examined and the one-step transition probabilities of the embedded Markov chain presented are computed. On the other hand, the obtained results are implemented for the $GI/M/2/0$ queuing model with heterogeneous servers and the state probabilities of the system concerned and the probability of lost customers in the system are calculated.

II. $GI/M/n/0$ Model with Heterogeneous Servers and Its Assumptions

"The $GI/M/n/0$ queuing system with finite capacity and heterogeneous servers" is analyzed in this study. In this model, interarrival times are independent of each other and have distribution function $F(t)$ and their expected value is finite $(a = \int_0^\infty [1-F(t)]\,dt < \infty)$. There are n non-identical servers in the system. That is, their mean service times are different from each other. The service time of each customer in server k is the random variable η_k and has an exponential distribution with parameter μ_k $(k = 1,2,\ldots,n)$, i.e. $P(\eta_k \le t) = 1 - e^{-\mu_k t}$, $t \ge 0$. The service time is independent of the arrival process.

The service discipline takes place with the "Random" principle. That is, the customer, who arrives in the system, starts the service in any of the free servers with probability $1/l$, $l = 1,2,\ldots,n$. Here l is the number of free servers at the arrival time of the customer. If all servers are busy, the customer, who arrives in the system, leaves the system without receiving any service. Such customers are called "lost customers". The basic problem herein is the computation of the probability of lost customers (P_L) in the system.

III. Semi-Markov Prosees Representing The system

Let the arrival times of customers in the system be $t_0, t_1, t_2, \ldots$ where $0 = t_0 < t_1 < \cdots$. Let $T_n = t_n - t_{n-1}$ for $n \ge 1$, and $T_0 = 0$. Let $S(t)$ be the number of customers in the system at time *t* and $S_n = S(t_n - 0)$, $n \ge 0$, where S_n is the number of customers in the system immediately before *n*th customer reaches the system. Let the semi-Markov process representing the system be defined as $\{X(t),\, t \ge 0\}$, $X(t) = S_n$ if and only if $t_n \le t < t_{n+1}$. Let $Q(x)$ be the square matrix consisting of the elements of probabilities $Q_{ij}(x)$ and let $q(s) = \int_0^\infty e^{-st} dQ(x)$, $\mathrm{Re}\{s\} \ge 0$. Probabilities $Q_{ij}(x)$ are defined as follows for $i, j = 0,1,\ldots,n$ and $x \ge 0$:

$$Q_{ij}(x) = P[(S_{n+1} = j,\, T_n \le x) \,|\, S_n = i] \tag{4}$$

According to the semi-Markov process and the total probability formula, functions (4) are calculated as follows:

$$Q_{ij}(x) = \frac{1}{\binom{n}{i+1}} \int_0^x \sum p_{k_1} p_{k_2} \cdots p_{k_u} q_{l_1} q_{l_2} \cdots q_{l_v}\, dF(t) \tag{5}$$

for $i = 1,2,\ldots,n-1$, and $Q_{n,j}(x) = Q_{n-1,j}(x)$ where $p_{k_u} = e^{-\mu_{k_u} t}$, $q_{l_v} = 1 - e^{-\mu_{l_v} t}$. In addition, under probability (5), the summation extends over all k_u's and l_v's such that $1 \le k_1 < k_2 < \cdots < k_u \le n$ and $1 \le l_1 < l_2 < \cdots < l_v \le n$ with $k_u \ne l_v$ where (u,v) pair takes the values $(j, i+1-j)$ for $0 \le j \le i+1$. Note that an empty product of probabilities denotes 1. Furthermore, because only one customer arrives in the system within any interarrival time, $Q_{ij}(x) = 0$ for $j > i+1$.

Theorem. When assumed that the mean service times of servers are equal $(\mu_1 = \mu_2 = \cdots = \mu_n = \mu)$ and for $x \to +\infty$, formula (5) yields formula (3).

Proof. Depending on the above-mentioned assumption, formula (5) is written in the following way for $i = 1,2,\ldots,n-1$ and $0 \le j \le i+1$:

$$Q_{ij}(x) = \frac{1}{\binom{n}{i+1}} \int_0^x \binom{n}{j} e^{-j\mu t} \binom{n-j}{i+1-j} (1-e^{-\mu t})^{i+1-j} dF(t) .$$

After some algebraic operations and when $x \to +\infty$

$$Q_{ij}(+\infty) = \binom{i+1}{j} \int_0^\infty e^{-j\mu t} (1-e^{-\mu t})^{i+1-j} dF(t) .$$

The proof has been completed. In conclusion, formula (5) is a generalization of formula (3) for 'the $GI/M/n/0$ queuing system with heterogeneous servers' and "Random" service discipline.

IV. Steady-state

$\{S_n, n \geq 0\}$ is an embedded Markov chain with probabilities p_{ij} of the semi-Markov process $\{X(t), t \geq 0\}$ with the state space $E = (0,1,\ldots,n)$. This Markov chain is irreducible and aperiodic. In addition, when x adequately approximates infinity, $\lim_{x\to\infty} Q_{ij}(x) = p_{ij}$, and $P = [p_{ij}]_0^n$ is a stochastic matrix. On the other hand, according to the Tauberian theorem (see, Widder [18]), it is written as $P = q(0)$.

π_j satisfy the equations

$$\pi_j = \sum_i \pi_i p_{ij} \tag{6}$$

with $\sum_j \pi_j = 1$, and the distribution $\{\pi_j\}$ is uniquely determined (see, Bharucha-Reid [3]).

After one-step transition probabilities (p_{ij}, $0 \leq i, j \leq n$) are calculated for the *GI/M/n/0* model with heterogeneous servers using formula (5), the steady-state probabilities π_j are easily obtained by means of formula (6).

Example. Consider the *GI/M/2/0* queue model with heterogeneous servers. The assumptions of the system are as explained in the second section. Using formulae (5) and (6) and after some algebraic operations, the steady-state probabilities for the above-mentioned queuing model are obtained as follows:

$$\pi_0 = \frac{1 - f(\mu_1) - f(\mu_2) + f(\mu_1 + \mu_2)}{1 - [f(\mu_1) + f(\mu_2)]/2 + f(\mu_1 + \mu_2)}$$

$$\pi_1 = \frac{[f(\mu_1) + f(\mu_2)][1 - f(\mu_1 + \mu_2)]/2}{1 - [f(\mu_1) + f(\mu_2)]/2 + f(\mu_1 + \mu_2)}$$

$$\pi_2 = \frac{f(\mu_1 + \mu_2)[f(\mu_1) + f(\mu_2)]/2}{1 - [f(\mu_1) + f(\mu_2)]/2 + f(\mu_1 + \mu_2)}$$

where $F(t)$ is the distribution function of the interarrival times, and

$$f(\mu_1) = \lim_{s\to 0} \int_0^\infty e^{-st} e^{-\mu_1 t} dF(t) ,$$

$$f(\mu_2) = \lim_{s\to 0} \int_0^\infty e^{-st} e^{-\mu_2 t} dF(t) ,$$

$$f(\mu_1 + \mu_2) = \lim_{s\to 0} \int_0^\infty e^{-st} e^{-(\mu_1+\mu_2)t} dF(t) .$$

Probabilities π_0, π_1 and π_2 denote the probability that the system is free, the probability that only 1 server is busy in the system and the probability that all servers are busy, respectively. As no waiting lines are available in the system, the probability that all servers are busy is equivalent to the loss probability of customers in the system, i.e. $P_L = \pi_2$.

V. Conclusion

Formula (5) obtained in this study is a generalization of formula (3), which was obtained by Takacs [16], for the *GI/M/n/0* model with heterogeneous servers. An implementation of formula (5) was performed for the *GI/M/2/0* queuing model and the loss probability of customers in the system was computed

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