Analysis of Least Stable Mode of Buoyancy Assisted Mixed Convective Flow in Vertical Pipe Filled with Porous Medium

P. Bera, Ashok Kumar *

Abstract—**Least stable mode of convective flow, induced by external pressure gradient and buoyancy force in the vertical pipe filled with porous medium, is investigated. Non-Darcy Brinkman-extended model has been considered. To study least stable mode of the fully developed flow linear theory of stability analysis has been used for an wide range [0.01, 100] of Prandtl number (Pr). To this end, coupled ordinary differential equations obtained from linear theory of stability analysis, have been solved numerically using Spectral collocation method. Four different values 10^{-1}, 10^{-2}, 10^{-3}, and 10^{-4} of Darcy number (Da) have been considered to study the impact of permeability of the medium on the flow stability. Present study on the least stable mode analysis discloses that when fluid is gas (Pr = 0.7) or water (Pr = 7.0), for relatively large values of Darcy number (i.e. Da = 10^{-1} and 10^{-2}), first azimuthal mode is the least stable mode of the basic flow in the entire range of Reynold's number (Re) considered in this manuscript. However, when Pr = 70, based on the values of Da, there exist a minimum value of Re beyond it the least stability of the fluid is achieved for zero azimuthal number. Further, it was found that for Da equal to 10^{-3} or 10^{-4} the instability boundary curves in the (Re, Ra)-plane, for all above three fluids, are almost equal.**

Keywords: Porous media, mixed convection, linear stability.

1 Introduction

The research work in the area of convective heat transfer in fluid saturated porous media has substantially increased during recent years due to its many practical applications encountered in engineering and sciences. Most of the available studies are restricted to natural or forced convection, and are well documented in the book of Nield and Bejan [1]. The convection due to external pressure gradient and buoyancy forces (i.e mixed convection) in a porous media is a fundamental problem in fluid dynamics and still a field of active and ongoing research. For example, convection in permeable sediment due to hot spring/hydrothermal vent is a combination of forced and natural convection [2]. Understanding the dynamic behavior of fluid flow through porous media, especially flow-transition, is still a challenge in fluid dynamics research.

Few investigations in wall bounded mixed convection through vertical annuli and channel have been reported by different researchers, which are well documented in the works of Chen [3], as well as in the recent paper of Kumar and Bera [5].

It is well established [6, 7, 8, 9, 10] that stability of the fully developed one dimensional (1D) flow due to external pressure gradient and thermal buoyancy force in a vertical channel, is sensitive to wall temperature. Beyond a certain wall temperature, the 1D flow does not remain stable. Based on the nature of buoyancy force, in favor or against the flow, the basic flow profile may contain point of inflection or separation which accelerates/decelerates the flow instability. It is then natural to ask how these flow dynamics, especially flow transition, will be modulated when vertical channel is replaced by vertical pipe filled with a porous medium.

From the best knowledge of us, no one has attempted to answer the above question.

For fluid environments, however, there are some papers ([6], [11], [12], [13]) which deal with mixed convection. Among them, Su and Chung [13] have presented the numerical study on the linear stability of mixed convective flow in a vertical pipe in both the cases: buoyancy assisted and opposed and found that the most unstable flow pattern is double spiral i.e. the most unstable azimuthal wave number is unity.

In the present manuscript we have investigated the least stability mode of the fully developed flow in a vertical pipe filled with porous medium.

2 Mathematical Formulation

We consider a fully developed mixed convection flow caused by an external pressure gradient and a buoyancy force in a vertical pipe filled with porous medium.

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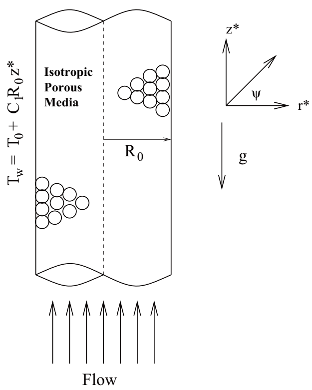

Figure 1: Sketch of physical model.

The wall temperature is linearly varying with z* as $T_w = T_0 + C_1 R_0 z^*$, where C_1 is a constant and T_0 is upstream reference wall temperature and R_0 is radius of the pipe. The gravitational force is aligned in the negative z^*-direction. As shown schematically in figure 1. The thermo-physical properties of the fluid are assumed to be constant except for density dependency of the buoyancy term in the momentum equations. The porous medium is saturated with a fluid that is in local thermodynamic equilibrium with the solid matrix. The medium is assumed to be isotropic in permeability.

Using non-dimensional quantities

$$r = \frac{r^*}{R_0{}^*}, \quad u = \frac{u^*}{W_c{}^*}, \quad v = \frac{v^*}{W_c{}^*}, \quad w = \frac{w^*}{W_c{}^*}, \quad z = \frac{z^*}{R_0{}^*},$$

$$p = \frac{p^*}{\rho_f W_c{}^{*2}}, \quad t = \frac{t^* W_c{}^*}{R_0{}^*}, \quad \theta = \frac{T_w{}^* - T^*}{C_1 R_0{}^* RePr}$$

the non-dimensional governing equations are given by

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \psi} + \frac{\partial w}{\partial z} = 0 \qquad (1)$$

$$\left[\frac{1}{\epsilon}\frac{\partial u}{\partial t} + \frac{1}{\epsilon^2}\left(\mathbf{J}u - \frac{v^2}{r}\right)\right] = -\frac{\partial p}{\partial r} + \frac{\Lambda}{Re}\left[\mathbf{D}^2 u - \frac{2}{r^2}\frac{\partial v}{\partial \psi} - \frac{u}{r^2}\right] - \frac{\Lambda}{ReDa}u \qquad (2)$$

$$\left[\frac{1}{\epsilon}\frac{\partial v}{\partial t} + \frac{1}{\epsilon^2}\left(\mathbf{J}v - \frac{uv}{r}\right)\right] = -\frac{1}{r}\frac{\partial p}{\partial \psi} + \frac{\Lambda}{Re}\left[\mathbf{D}^2 v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \psi}\right] - \frac{\Lambda}{ReDa}v \qquad (3)$$

$$\left[\frac{1}{\epsilon}\frac{\partial w}{\partial t} + \frac{1}{\epsilon^2}(\mathbf{J}w)\right] = -\frac{\partial p}{\partial z} + \frac{\Lambda}{Re}\left[\mathbf{D}^2 w\right] - \frac{\Lambda}{ReDa}w - \frac{Ra}{Re}\theta \qquad (4)$$

$$\left[\sigma\frac{\partial \theta}{\partial t} + (\mathbf{J}\theta)\right] = \frac{1}{RePr}w + \frac{1}{RePr}\left[\mathbf{D}^2\theta\right] \qquad (5)$$

where,

$$\mathbf{J} = u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \psi} + w\frac{\partial}{\partial z}, \text{ and } \mathbf{D}^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \psi^2} + \frac{\partial^2}{\partial z^2}$$

In the above equations the dimensionless parameters are the Rayleigh number, $Ra = \frac{\mathbf{g}\beta_T C_1{}^* R_0{}^{*4}}{\tilde{\nu}\alpha}$, and the Prandtl number, $Pr = \frac{\tilde{\nu}}{\alpha}$, $\Lambda = \frac{\mu_f}{\tilde{\mu}}$.

2.1 Basic Flow

The basic flow is a steady, unidirectional fully developed flow. Therefore, the above governing differential equations (1) - (5) are reduced into the following set of coupled differential equations.

$$\frac{d^2 W_0}{dr^2} + \frac{1}{r}\frac{dW_0}{dr} - \frac{\Lambda}{Da}W_0 - Ra\Theta_0 - Re\frac{dp}{dz} = 0\,, \qquad (6)$$

$$\frac{d^2\Theta_0}{dr^2} + \frac{1}{r}\frac{d\Theta_0}{dr} = -W_0\,, \qquad (7)$$

accompanied with boundary condition

$$\frac{dW_0}{dr} = \frac{d\Theta_0}{dr} = 0 \quad at \quad r = 0, \qquad (8)$$

$$W_0 = \Theta_0 = 0 \quad at \quad r = 1 \qquad (9)$$

The analytical solution of the above equations are given bellow.

$$W_0(r) = a_1 J_0(P_1{}^{1/2} r) + a_2 I_0(P_2{}^{1/2} r) \qquad (10)$$

$$\Theta_0(r) = \frac{a_1}{P_1}\left[J_0(P_1{}^{1/2} r) - J_0(P_1{}^{1/2})\right] - \frac{a_2}{P_2} I_0(P_2{}^{1/2} r) - \frac{a_2}{P_2} I_0(P_2{}^{1/2}) \qquad (11)$$

where,

$$a_1 = \frac{\left[0.25 P_1{}^{1/2} P_2{}^{1/2} I_0(P_2{}^{1/2})\right]}{\left[P_2{}^{1/2} J_1(P_1{}^{1/2}) I_0(P_2{}^{1/2}) - P_1{}^{1/2} J_0(P_1{}^{1/2}) I_1(P_2{}^{1/2})\right]},$$

$$a_2 = -a_1\frac{J_0(P_1{}^{1/2})}{I_0(P_2{}^{1/2})}, \qquad \lambda^2 = Ra - \frac{1}{4Da^2}$$

$$P_1 = \begin{cases} i\left(\sqrt{Ra - \frac{1}{4Da^2}}\right) - \frac{1}{2Da} & for\ \lambda^2 > 0 \\ \left(\sqrt{\frac{1}{4Da^2} - Ra}\right) - \frac{1}{2Da} & for\ \lambda^2 < 0, \end{cases}$$

and

$$P_2 = \begin{cases} i\left(\sqrt{Ra - \frac{1}{4Da^2}}\right) + \frac{1}{2Da} & for\ \lambda^2 > 0 \\ \left(\sqrt{\frac{1}{4Da^2} - Ra}\right) + \frac{1}{2Da} & for\ \lambda^2 < 0. \end{cases}$$

J_0 and I_0 are zeroth order first and second kind of Bessel functions respectively.

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2.2 Disturbance

In linear stability analysis, infinitesimal disturbances are imposed on the base flow. Thus the velocity, pressure and temperature fields can be written as

$$(u, v, w, \theta, p) = \left(\tilde{u}, \tilde{v}, W_0(r) + \tilde{w}, \Theta_0(r) + \tilde{\theta}, P_0(z) + \tilde{p}\right) \quad (12)$$

where the tilde quantities denote the infinitesimal disturbances to the corresponding term. By using the normal-mode analysis, the disturbance can be expressed by

$$\left(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{\theta}, \tilde{p}\right) = \left(\hat{u}(r),\ \hat{v}(r),\ \hat{w}(r),\ \hat{\theta}(r),\ \hat{p}(r)\right) e^{[i\alpha(z-ct)+n\psi]}$$

where, α is the wave number, n is integer azimuthal wave number and c $= \hat{c_r} + i\hat{c_i}$ is complex wave speed. The growth and decay of the disturbances depend on $\hat{c_i}$. Depending on whether $\hat{c_i} < 0$, $\hat{c_i} = 0$ or $\hat{c_i} > 0$, three different possibilities stable, neutrally stable or unstable may be distinguished. Following the standard linear stability method [8], the governing linear equations for the infinitesimal disturbances can be written as

$$\frac{d\hat{u}}{dr} + \frac{\hat{u}}{r} + \frac{in\hat{v}}{r} + i\alpha\hat{w} = 0 \quad (13)$$

$$\frac{d^2\hat{u}}{dr^2} + \frac{1}{r}\frac{d\hat{u}}{dr} - \left[\left(\frac{n^2+1}{r^2}\right) + \alpha^2 + \frac{1}{Da} + i\alpha Re\left(\frac{W_0}{\epsilon^2} - \frac{c}{\epsilon}\right)\right]\hat{u}$$
$$-\frac{2in}{r^2}\hat{v} - Re\frac{d\hat{p}}{dr} = 0 \quad (14)$$

$$\frac{d^2\hat{v}}{dr^2} + \frac{1}{r}\frac{d\hat{v}}{dr} - \left[\left(\frac{n^2+1}{r^2}\right) + \alpha^2 + \frac{1}{Da} + i\alpha Re\left(\frac{W_0}{\epsilon^2} - \frac{c}{\epsilon}\right)\right]\hat{v}$$
$$+\frac{2in}{r^2}\hat{u} - \frac{inRe}{r}\hat{p} = 0 \quad (15)$$

$$\frac{d^2\hat{w}}{dr^2} + \frac{1}{r}\frac{d\hat{w}}{dr} - \left[\frac{n^2}{r^2} + \alpha^2 + \frac{1}{Da} + i\alpha Re\left(\frac{W_0}{\epsilon^2} - \frac{c}{\epsilon}\right)\right]\hat{w} + \frac{1}{\epsilon^2}\frac{dW_0}{dr}\hat{u}$$
$$-Ra\hat{\theta} - i\alpha Re\hat{p} = 0 \quad (16)$$

$$\frac{d^2\hat{\theta}}{dr^2} + \frac{1}{r}\frac{d\hat{\theta}}{dr} - \left[\frac{n^2}{r^2} + \alpha^2 + i\alpha RePr\,(W_0 - \sigma c)\right]\hat{\theta} + \hat{w}$$
$$-RePr\frac{d\Theta_0}{dr}\hat{u} = 0 \quad (17)$$

The required boundary conditions accompanying the above equations (13)-(17) are specified at the wall and center. At the wall all disturbances must vanish. This implies

$$\hat{u} = \hat{v} = \hat{w} = \hat{\theta} = \frac{d\hat{p}}{dr} = 0, \quad \text{at} \quad r = 1 \quad (18)$$

At the center of the pipe, the boundary conditions are as follows:

$$\begin{cases} \hat{u} = \hat{v} = \frac{d\hat{w}}{dr} = \frac{d\hat{\theta}}{dr} = \frac{d\hat{p}}{dr} = 0, & \text{at } r = 0\,,\ n = 0, \\ \hat{u} + i\hat{v} = 2\frac{d\hat{u}}{dr} + \frac{d\hat{v}}{dr} = \hat{w} = \hat{\theta} = \hat{p} = 0, & \text{at } r = 0\,,\ n = 1, \\ \hat{u} = \hat{v} = \hat{w} = \hat{\theta} = \hat{p} = 0, & \text{at } r = 0\,,\ n \geq 2. \end{cases} \quad (19)$$

The spectral collocation method has been used to solve above set of ordinary differential equations.

3 Results and Discussion

Linear stability analysis has been performed here to answer the question of the condition for the occurrence of the maximum stability of fluid in the vertical pipe filled with porous medium. Main emphasis is given on the dependency of the stability boundaries on the azimuthal wave numbers (n) for gas (Pr = 0.7), water (Pr = 7.0) and oil (Pr = 70). To give a thorough study in the highly porous media, four different Darcy numbers, i.e., 10^{-1}, 10^{-2}, 10^{-3} and 10^{-4} are used. Throughout this section, critical Ra and critical α have been represented by Ra and α respectively. Before discussing the effect of the azimuthal wave number on stability of the system a verification of the code is given via grid Independence and by comparison with published results. Table 1 shows the convergence of Spectral collocation method at Re = 1000, Ra = 100, $\alpha = 1$, Pr = 7, Da = 10^{-2}, F = 0, and azimuthal mode = 1. With an order of polynomial (N) of 19 and 25 already a 6-digit point accuracy can be obtained. As the number of terms in the approximations increases, the results remain consistent and accuracy improved too. Similar satisfactory results were obtained for other sets of input parameters. In all computations reported, it was found that accurate solutions could be reached by taking 50 terms of the Spectral approximation. A severe test (see Table 2) for linear stability calculation is provided by calculating the first eigen-mode for isothermal flow with Prandtl numbers, at Re = 100, $n = 1$, $\epsilon = 1$, F = 0, Da = 10^{12} and comparing the same with the results of Su and Chung [13] as particular case of our results. It has been found that the obtained eigenvalues are agreed well. The dependency of the stability boundaries on the azimuthal wave numbers, (n = 0,1) for gas, water and oil, is plotted in figures 2, 3 and 4 respectively for different Darcy number (Da). It can be observed from figure 2 (a)-(d) that, when Da equal to 10^{-1} and 10^{-2}, the first azimuthal mode i.e n = 1, is least stable mode in the entire range, [0,1000], of Re. But for Da equal to 10^{-3}, the first azimuthal mode will be the least stable mode provided the value of Re is ≥ 600. Of course, both curves (obtained for n = 0,1) almost coincide each other.

As the value of Re is decreased below 600, the zero azimuthal mode becomes least stable mode. As can be seen from figure 2 (c) that the range of Re, in which zero azimuthal mode is least stable, is function of media permeability as well as fluid. For Da equal to 10^{-4}, zero azimuthal mode is least stable for the entire range of Re. Similar results can also be seen for fluid as water (see figure figure 3 (a)-(d)). In this case, zero azimuthal mode is least stable for Re ≤ 400 at Da = 10^{-3}. Judging from figures 2 (c) and 3 (c) it can be conclude that enhancement of Pr from 0.7 to 7 reduces the range of Re in which zero azimuthal mode is the least stable mode. In contrast to purely viscous fluid flow [13] in which the first azimuthal mode is the least stable mode, in porous media, zero az-

Table 1: Convergence of Chebyshev collocation methods, at Ra = 100, Re = 1000, $\alpha = 1$, Pr = 7.0, n=1 and Da = 10^{-2}.

N(terms)	Most unstable mode
15	0.5971472994381 - 0.0088745541020i
17	0.5971530076879 - 0.0088863797381i
19	0.5971510023715 - 0.0088843473069i
25	0.5971511828890 - 0.0088846738378i
31	0.5971511802630 - 0.0088846753046i
40	0.5971511795755 - 0.0088846760288i
50	0.5971511796095 - 0.0088846758491i
60	0.5971511796249 - 0.0088846757865i
70	0.5971511796327 - 0.0088846757598i

Table 2: Comparison of first eigen-mode with the published results [13] at Da = 10^{12}, Ra = 0, n = 1, Re = 100 and $\alpha = 1$.

Pr	Su and Chung	Present
0.01	0.57256-0.14713i	0.5725629 - 0.1471366i
0.1	0.57256-0.14713i	0.5725629 - 0.1471366i
1	0.57256-0.14713i	0.5725629 - 0.1471366i
10	0.91055-0.09044i	0.9105573 - 0.0904427i
100	0.97171-0.02838i	0.9717157 - 0.0283842i

imuthal mode is the least stable mode for fluid like heavy oil (Pr = 70) beyond a certain value of Re (which is also function of Da). As a result, two azimuthal modes: n = 0 and 1, in buoyancy assisted case, are used to find the least stable mode of basic flow.

To shed some more light on the dependency of the least stable mode as a function of Prandtl number instability boundary curves for n equal to 0 and 1 are plotted in (Pr, Ra)-plane for Re = 1000, which is shown in Fig. 5. As can be observed from above figure that the zero azimuthal mode is the least stable mode for Pr> 10.

4 Conclusions

We have attempted to gain an understanding of instability of pressure gradient driven buoyancy assisted mixed convection in a vertical pipe filled with fluid-saturated porous medium. To this end, we adopted Non-Darcy Brinkman-extended model. By means of linear theory, we were able to extract detailed information of transition of basic flow through a porous medium for different fluids. The Spectral Collocation Method is used to solve the set of linear ordinary differential equations. The main objective in this study was to investigate the dependency of the stability boundaries on the azimuthal wave numbers (n) for gas (Pr = 0.7), water (Pr = 7.0) and oil (Pr = 70). Four different values (10^{-1}, 10^{-2}, 10^{-3} and 10^{-4}) of Da were considered. Throughout the study, porosity (ϵ), viscosity ratio (Λ) and heat capacity ratio (σ) were given constant values of 0.9, 1 and 1, respectively. In

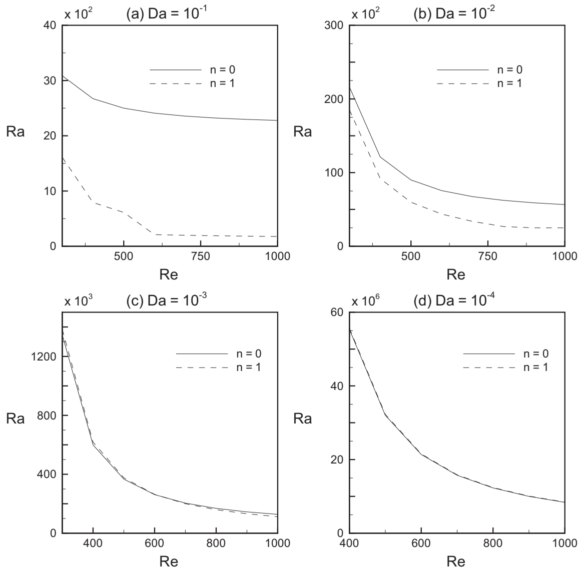

Figure 2: Effect of azimuthal wave number (n) on the stability map of assisted flow for (a) Da = 10^{-1}, (b) Da = 10^{-2}, (c) Da = 10^{-3} and (d) Da = 10^{-4} at Pr = 0.7.

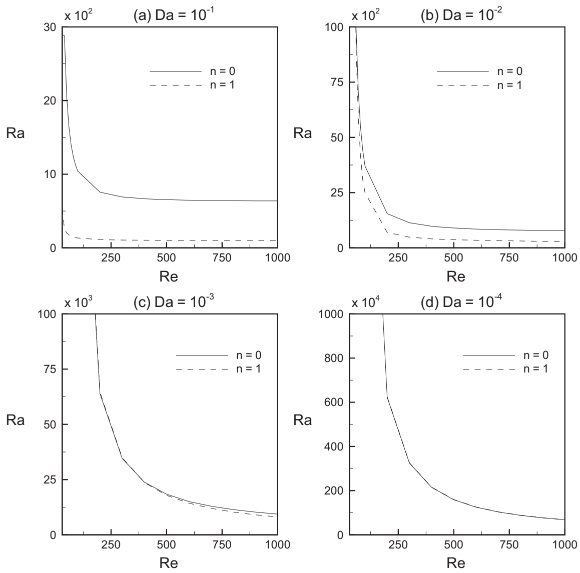

Figure 3: Effect of azimuthal wave number (n) on the stability map of assisted flow for (a) Da = 10^{-1}, (b) Da = 10^{-2}, (c) Da = 10^{-3} and (d) Da = 10^{-4} at Pr=7.

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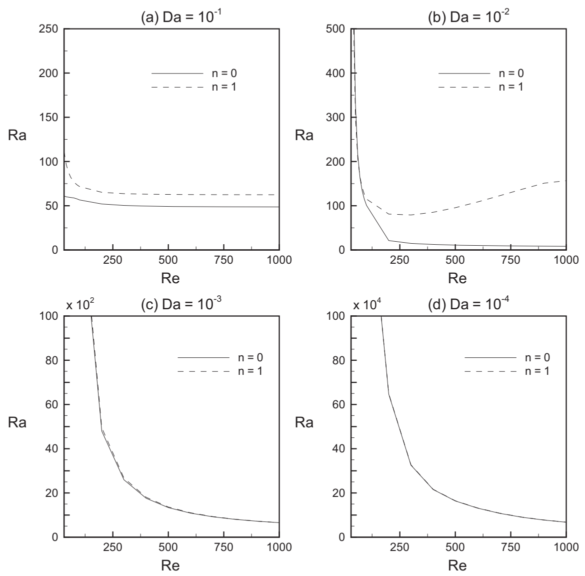

Figure 4: Effect of azimuthal wave number (n) on the stability map of assisted flow for (a) Da = 10^{-1}, (b) Da = 10^{-2}, (c) Da = 10^{-3} and (d) Da = 10^{-4} at Pr=70.

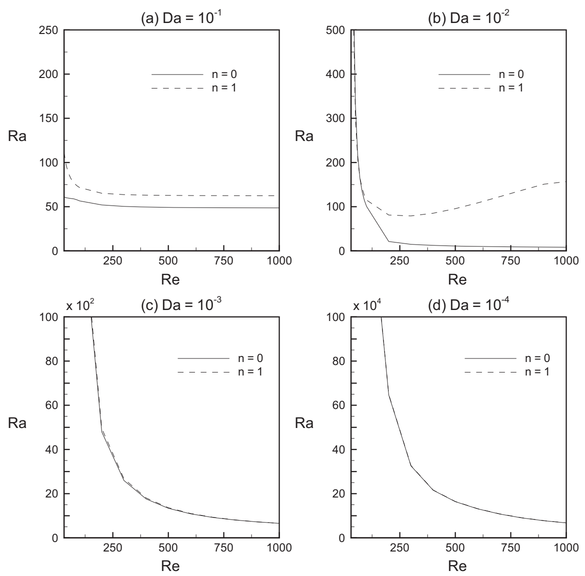

Figure 5: Influence of Prandtl number (Pr) on the stability map of basic flow for (a) Da = 10^{-1}, (b) Da = 10^{-2}, (c) Da = 10^{-3} and (d) Da = 10^{-4} for Re = 1000.

this study we have found that depending on the values of media permeability as well as Reynolds number, the flow will be least stable under either first azimuthal mode or zero azimuthal mode.

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