Analytical of ARL for Trend Stationary First Order of Autoregressive Observations on CUSUM Procedure

Jaruchat Busaba, Saowanit Sukparungsee, and Yupaporn Areepong

***Abstract*— We derive explicit formulas for the average run length (the first exit times) and the average of delay time on CUSUM procedure when observations are trend stationary first order of autoregressive (trend AR(1)) model with white noise exponential distribution by using an integral equation approach. Comparisons are made for the accuracy of results of the explicit formulas and the numerical approximations, which are in excellent agreement results. We also show that the computational times obtained from the explicit formulas are much less than the computational time obtained from the numerical approximations.**

***Index Terms*— Trend Stationary First Order Autoregressive Observation, Cumulative Sum, Average Run Length, Average of Delay Time**

I. Introduction

Traditionally, the observation of the Cumulative Sum (CUSUM) control chart, which was first introduced by Page in 1954, are normally and identically independent distributed random variables. In practice, this assumption is not always happen in real application such as in the continuous manufacturing processes where most observations are sequentially autocorrelated. Recently, many authors proposed new methods to investigate the CUSUM procedure when observation processes are autocorrelations for both case of stationary and non-stationary processes. In 1974, Johnson and Bagshaw [1] discussed the effect of autocorrelations on the performance of the cumulative sum (CUSUM) chart. In 1991, Harris and Ross [2] discussed the impact of autocorrelation on CUSUM and EWMA charts and pointed out that the average and the median run lengths of these charts were sensitive to presence of autocorrelation. Woodall and Faltin [3] have been discussed on the effect of autocorrelation on the performance of control charts and on how to deal with autocorrelation. Rao et al. [4] focused on the integral equation approach for computing the ARL for CUSUM control charts for AR(1) process. Karaoglan et al. [5] have discussed the performance comparison of residual control chart for trend stationary first order autoregressive processes. They applied the Shewhart, EWMA, CUSUM or GMA chart to the uncorrelated residuals. Busaba et al.[6] analyzed the average run length for AR(1) on CUSUM procedure by using Fredholm integral equation technique. Busaba et al.[7] have shown numerical approximations of ARL for AR(1) on Exponential CUSUM by using Gauss-Legendre rule. They obtained the results from the numerical integration method compared with results obtained from explicit formula which were in excellent agreement.

There are many characteristics to show the performance of procedure; such as Average Run Length (ARL) and the Average of Delay Time (AD). Both of them are frequently method used in procedure for evaluating the detection performance of various control charts. The expressions of them for the CUSUM and EWMA charts in detecting of mean shift in process, have been studied by [8], [9], [10], [11] and [12]. Various methods used to measure this performance of procedure; Monte Carlo Simulation (MC), Markov Chain Approach (MCA) (see [13]), Martingale Approaches (see [14, 15]) and Integral Equations (IE) (see [16], [17] and [18]). The first three methods give only closed-form formulas, while the last method gives the explicit formulae for the ARL and AD. Many authors have been derived the explicit form of ARL for the CUSUM (see [7], [19], [20], [21], [22], [23] and [24]). The proposed explicit expression is simple and easy to implement.

In the present paper, the explicit formulae of ARL and AD on CUSUM procedure when observation processes are trend stationary first order autoregressive with exponential distributed white noise are proposed by using an integral equation approach, Fredholm second type integral equation. Next section describes the properties of CUSUM procedure. In section 3, the uniqueness of solution by using Banach's fixed point theorem is described (see [25]). The solution for integral equation on CUSUM procedure for trend stationary first order autoregressive observations with exponential distributed white noise, based on the discussion of [19], [20] and [21, 22], is given in section 4. Comparison results are in section 5. Conclusions are pointed out in Section 6.

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II. The CUSUM Procedure

A. The Characteristics of CUSUM Procedure

The CUSUM chart, which we consider is under the assumption that sequential observations $\xi_1, \xi_2, \ldots$ are sequentially observed identically independent random variables with an exponential distribution function $F(x, \lambda)$, where the parameter λ has the value λ_0 in the in-control state (before a change-point time $(\theta \le \infty)$), and this parameter λ_0 changes to λ (where $\lambda \ne \lambda_0$) for out-of-control state. We assume that the parameters of the in-control and out-of-control states are known.

According to the assumptions, $F(x, \lambda)$ is absolute continuous distribution with respect to $F(x, \lambda_0)$. The alarm times for type of procedure for a statistic X_n typically defined as in equation (1) is

$$\tau_h = \inf\{n \ge 0; X_n \ge h\}, \qquad (1)$$

where h is a control limit on the value of X_n.

The statistical process control are required to measure of the average run length (ARL) which is the expectation of an alarm time (τ) is taken to signal (wrongly) about a possible change. Ideally, an acceptable ARL of in-control process should be enough large and a small ARL when the process is out-of-control, so-called Average of Delay Time (AD) - the expectation of delay for true alarm time. Let $\mathbb{E}_\infty(\cdot)$ denote the expectation under distribution $F(x, \lambda_0)$ that the change-point occurs at point θ. Typical measures for alarm times τ are

$$ARL \approx \mathbb{E}_\infty \tau_h \ge T \qquad (2)$$

where T is given (usually large) and

$$AD \approx \mathbb{E}_\infty \tau_h \le (\tau | \tau \ge 1). \qquad (3)$$

The ARL and AD are two conflicting criteria that must be balanced in control charts.

B. The trend AR(1) on CUSUM Procedure

The CUSUM procedure is designed to detect an increase in the mean of an independent and identically distributed (i.i.d.) observed sequence of random variables $\xi_1, \xi_2, \ldots$ the recursive equation for CUSUM charts is defined as

$$X_n = (X_{n-1} + Z_n - a)^+, \; n = 1, 2, \ldots, \; X_0 = x \qquad (4)$$

where X_n is the CUSUM value of a statistic after n observations, x is an initial value for X_n, $y^+ = \max(0, y)$ and a is a constant. Mazalov and Zhuravlev [20] and George et. al [26] discussed many cases which lead to this recursive representation.

If the observations are trend stationary first order autoregressive (trend AR(1)) model with exponential distributed white noise as,

$$Z_n = \alpha + \delta n + \rho Z_{n-1} + \xi_n, \qquad (5)$$

where n is the time of sampling, Z_n is the sample value at time n, α is the constant, δ is the trend slope in terms of n, ρ is the autoregressive coefficient $(-1 < \rho < 1)$, and ξ_n is the autoregressive white noise at time n following $\xi_n \sim \exp(\lambda)$.

Substitute Z_n from (5) into (4) then the CUSUM procedure can be written as

$$X_n = X_{n-1} + (\alpha + \delta n + \rho Z_{n-1} + \xi_n) - a, \quad n = 1, 2, \ldots, X_0 = x. \qquad (6)$$

III. Uniqueness of Solution to Average Run Length Integral Equation

Let $\mathbb{P}_X$ and $\mathbb{E}_X$ be the probability measure and the induced expectation corresponding to the initial value $X_0 = x$. Then it can be shown that the ARL for CUSUM at a given level (see [21] and [25]), defined as $j(x) = ARL = \mathbb{E}_x \tau_h < \infty$, is a solution of the following integral equation

$$j(x) = 1 + \mathbb{E}_X\left[I\{0 < X_1 < h\}\, j(X_1)\right] + \mathbb{P}_X\{X_1 = 0\}\, j(0). \qquad (7)$$

For the case, ξ_n are exponential distributed observations have been proposed in [21, 22] and [23]. In this paper, we define ξ_n are exponential distributed white noise with trend AR(1) observations by $Z_n = \alpha + \delta n + \rho Z_{n-1} + \xi_n$ where $-1 < \rho < 1$ so (7) can be written as

$$j(x) = 1 + \lambda e^{-\lambda(a - x - \alpha - \delta - \rho Z_0)} \int_0^h j(y) e^{-\lambda y} dy + \left(1 - e^{-\lambda(a - x - \alpha - \delta - \rho Z_0)}\right) j(0), \quad x \in [0, a). \qquad (8)$$

It is clear that solutions of the integral equation (8) are continuous functions, because the right hand side of (8) contains only continuous functions.

Recall that on the metric space of all continuous functions $(\mathbb{C}(\mathbb{I}), \|\ \|_1)$, where $\mathbb{I}$ is a compact interval, and the norm defined as $\|j\| = \sup_{x \in \mathbb{I}} |j(x)|$, the operator T is named a contraction if there exist a number $0 \le q < 1$ such that $\|T(j_1) - T(j_2)\| \le q \|j_1 - j_2\|$ for all $j_1, j_2 \in X$. Now, define the operators T as

$$T(j(x)) = 1 + \lambda e^{-\lambda(a - x - \alpha - \delta - \rho Z_0)} \int_0^h j(y) e^{-\lambda y} dy + \left(1 - e^{-\lambda(a - x - \alpha - \delta - \rho Z_0)}\right) j(0) \quad \text{if } x \in \mathbb{I}_1. \qquad (9)$$

Then the integral equations in (8) can be written as $T(j(x)) = j(x)$. According to Banach's Fixed Point Theorem if the operator T is contraction, then the fixed point equation $T\left(j\left(x\right)\right)=j\left(x\right)$ has a unique solution (see [23]). To show the uniqueness of the solution of (8), we will prove in Theorem 3.1 that T is a contraction. Define the norms $\|j\|_1 = \underset{x\in\mathbb{I}_1}{Sup}\left|j\left(x\right)\right|.$

Theorem 3.1 On the metric spaces $\left(\mathbb{C}\left(\mathbb{I}_1\right),\|\ \|_1\right)$ the operator T is a contraction. First, to show T is contraction we may check that for any $x\in\mathbb{I}_1$, and $j_1, j_2 \in\mathbb{C}\left(\mathbb{I}_1\right)$, we have the inequality $\left\|T\left(j_1\right)-T\left(j_2\right)\right\|_1 \le q\left\|j_1-j_2\right\|_1$, where q is a positive constant, $0\le q<1$. According to (9) we have that

$$\left\|T\left(j_1\right)-T\left(j_2\right)\right\| = Sup\left|j\left(x\right)\right|$$

$$= \underset{x\in[0,a)}{Sup}\left|\left(j_1\left(0\right)-j_2\left(0\right)\right)\left(1-e^{-\lambda\left(a-x-\alpha-\delta-\rho Z_0\right)}\right)\right.$$

$$\left.+\lambda e^{-\lambda\left(a-x-\alpha-\delta-\rho Z_0\right)}\int_0^h\left(j_1\left(y\right)-j_2\left(y\right)\right)e^{-\lambda y}dy\right|$$

$$\le \underset{x\in[0,a)}{Sup}\left|\left\|j_1\left(0\right)-j_2\left(0\right)\right\|_1\left(1-e^{-\lambda\left(a-x-\alpha-\delta-\rho Z_0\right)}\right)\right.$$

$$\left.+\left\|j_1-j_2\right\|_1\lambda e^{-\lambda\left(a-x-\alpha-\delta-\rho Z_0\right)}\int_0^h e^{-\lambda y}dy\right|$$

$$= \left\|j_1-j_2\right\|_1 \underset{x\in[0,a)}{Sup}\left[1-e^{-\lambda\left(a-x-\alpha-\delta-\rho Z_0\right)-\lambda h}\right]$$

$$= \left[1-e^{-\lambda\left(-\alpha-\delta-\rho Z_0\right)-\lambda h}\right]\left\|j_1-j_2\right\|_1$$

$$= q_1\left\|j_1-j_2\right\|, \text{ where } q_1 = \left[1-e^{\lambda\left(\alpha+\delta+\rho Z_0-h\right)}\right]<1. \square$$

We have used the triangular inequality and the fact that,

$$\left|j_1\left(0\right)-j_2\left(0\right)\right| \le \underset{x\in[0,a)}{Sup}\left|j_1\left(x\right)-j_2\left(x\right)\right| = \left\|j_1-j_2\right\|.$$

IV. APPROACH FOR INTEGRAL EQUATION OF TREND AR(1) OBSERVATIONS ON CUSUM PROCEDURE

A. *The explicit formulae*

In Theorem 4.1, we derive explicit solutions of the integral equations (8). The uniqueness of solutions is guaranteed by Theorem 3.1.

Theorem 4.1 The solution of (8) is

$$j\left(x\right)=\left(1+e^{\lambda\left(a-\alpha-\delta-\rho Z_0\right)}-\lambda h\right)e^{\lambda h}-e^{\lambda x}, \quad x\ge 0. \qquad (10)$$

Proof.

$$j\left(x\right)=1+\lambda e^{-\lambda\left(a-x-\alpha-\delta-\rho Z_0\right)}\int_0^h j\left(y\right)e^{-\lambda y}dy$$
$$+\left(1-e^{-\lambda\left(a-x-\alpha-\delta-\rho Z_0\right)}\right)j\left(0\right), \quad x\in[0,a)$$

Define $d=\int_0^h j\left(y\right)e^{-\lambda y}dy.$

Now, we have

$$j\left(x\right)=1+\lambda e^{-\lambda\left(a-x-\alpha-\delta-\rho Z_0\right)}\cdot d$$
$$+\left(1-e^{-\lambda\left(a-x-\alpha-\delta-\rho Z_0\right)}\right)j\left(0\right). \qquad (11)$$

If $x=0$ then $j(0)=e^{\lambda\left(a-\alpha-\delta-\rho Z_0\right)}+\lambda d.$

Substitute $j\left(0\right)$ into (10), we found that

$$j\left(x\right)=1+\lambda d+e^{\lambda\left(a-\alpha-\delta-\rho Z_0\right)}-e^{\lambda x}. \qquad (12)$$

Now the constant d can be found as

$$d=\int_0^h j\left(1+\lambda d+e^{\lambda\left(a-\alpha-\delta-\rho Z_0\right)}-e^{\lambda y}\right)e^{-\lambda y}dy$$
$$=\frac{e^{\lambda h}}{\lambda}\left(1-e^{-\lambda h}\right)\left(1+e^{\lambda\left(a-\alpha-\delta-\rho Z_0\right)}\right)-he^{\lambda h}.$$

Substitute the constant d into (12), we have

$$j\left(x\right)=\left(1+e^{\lambda\left(a-\alpha-\delta-\rho Z_0\right)}-\lambda h\right)e^{\lambda h}-e^{\lambda x}, \quad x\ge 0. \quad \square$$

The explicit formula for the ARL and AD are presented as following

$$ARL=j_0\left(x\right)=\left(1+e^{\left(a-\alpha-\delta-\rho Z_0\right)}-h\right)e^{h}-e^{x}, x\ge 0 \qquad (13)$$

and

$$AD=j_1\left(x\right)=\left(1+e^{\lambda\left(a-\alpha-\delta-\rho Z_0\right)}-\lambda h\right)e^{\lambda h}-e^{\lambda x}, x\ge 0. \qquad (14)$$

B. *The numerical integral equation approach*

The numerical scheme to evaluate solutions of the integral equations from section 4.1 is shown in the section. Firstly, the integral equation (8) can be written as follows:

$$j\left(x\right)=1+j\left(0\right)F\left(a-x-\alpha-\delta-\rho Z_0\right)$$
$$+\int_0^h j\left(y\right)f\left(a-x-\alpha-\delta-\rho Z_0+y\right)dy, \qquad (15)$$

where $F\left(x\right)=1-e^{-\lambda x}$ and $f\left(x\right)=\frac{dF\left(x\right)}{dx}=\lambda e^{-\lambda x}.$

By Gauss-Legendre rule (See [16], [21], [22] and [27]) approximated the function $j\left(x\right)$ as

$$j^{IE}\left(x\right)\approx 1+j\left(a_1\right)F\left(a-x-\alpha-\delta-\rho Z_0\right)$$
$$+\sum_{k=1}^{m}w_k j\left(a_k\right)f\left(a_k+a-a_i-\alpha-\delta-\rho Z_0\right) \qquad (16)$$

with $w_k=\frac{h}{m}$ and $a_k=\frac{h}{m}\left(k-\frac{1}{2}\right).$

V. COMPARISON RESULTS

All tables give a comparison of the approximated solutions $j^{IE}(x)$, the exact solutions $j(x)$, the absolute percentage difference

$$Diff\,(\%)=\frac{\left|j(x)-j^{IE}(x)\right|}{j(x)}\times 100\%$$

for several levels of a, h and the number of divisions m.

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Table 1 and 3 show the computational times of approximately 10-15 minutes, by intel® Core™ i5 CPU with M540@ 2.53GHz processor, RAM 4.00 GB and 32-bit operation system, while the results obtained from the explicit formula take less than 1 seconds which is much less than the former.

The analytical explicit solutions are in good agreement with the results obtained from the numerical approximation with an absolute percentage difference less than 1% for 500 iterations of numerical integral approximation.

Table 1: Comparisons of values ARL and AD of $j_0(x)$ and $j_1(x)$ from explicit formulae with numerical approximations $j^{IE}(x)$ for $\alpha = 0,\ \delta = 0.2$.

ρ	a	$\lambda=1, h=3$: $x=1$	$\lambda=1, h=3$: $x=3$	ρ	a	$\lambda=2, h=3$: $x=1$	$\lambda=2, h=3$: $x=3$
0.3	2	47.1278[1]	29.7605	-0.3	2	121.1330[2]	103.7650
		47.0453[3]	29.7301			120.8260	103.5110
		910.9680[4]	933.7130			937.6130	878.6290
		0.1751[5]	0.1021			0.2534	0.2448
	2.5	105.5240	88.1565		2.5	227.5370	210.1700
		105.2650	87.9496			226.9090	209.5940
		917.0990	909.4550			822.2340	796.6810
		0.2454	0.2347			0.2760	0.2741
	2.9	178.5170	161.1500		2.9	360.5390	343.1720
		178.2650	160.7220			359.5090	342.1940
		931.7630	879.1900			795.0280	796.5410
		0.1412	0.2656			0.2857	0.2850
0.5	2	30.8104	13.4432	-0.5	2	157.4470	140.0800
		30.7774	13.4621			157.0310	139.7160
		1027.5000	823.8100			807.6320	807.4770
		0.1071	0.1406			0.2642	0.2599
	2.5	78.6211	61.2538		2.5	287.4100	270.0430
		78.4434	61.1282			286.6010	269.2860
		807.1960	806.6970			804.0150	804.7620
		0.2260	0.2050			0.2815	0.2803
	2.9	138.3830	121.0160		2.9	449.8600	432.4920
		138.0240	120.7090			448.5590	431.2440
		805.9640	805.0120			805.5580	804.7310
		0.2594	0.2537			0.2892	0.2886
0.7	2	17.4509	0.1437	-0.7	2	201.8030	184.4350
		17.4552	0.1430			201.2520	183.9370
		838.3180	806.8840			810.7530	808.1940
		0.0246	0.4871			0.2730	0.2700
	2.5	56.5015	39.2277		2.5	360.5390	343.1720
		56.4839	39.1687			359.5090	342.0940
		805.4170	804.2020			805.5890	804.4340
		0.0311	0.1504			0.2857	0.3141
	2.9	105.5240	88.1565		2.9	558.9560	541.5880
		105.265	87.9496			557.3250	540.0100
		805.729	804.0130			804.1690	803.9670
		0.2454	0.2347			0.2918	0.2914

[1]The average run length is from explicit formulae as in (13).
[2]The average run length is from explicit formulae as in (14).
[3]The average run length is from numerical integral equation as in (16).
[4]The absolute percentage difference.
[5]CPU time used.

Table 2: Comparison of values $j_0(x)$ and $j_1(x)$ from explicit formulae with numerical approximations $j^{IE}(x)$ for $h=3, a=2, \delta=0.2, \rho=0.25$.

λ	$x=1$: $j(x)$	$x=1$: $j^{IE}(x)$	*Diff* (%)	$x=3$: $j(x)$	$x=3$: $j^{IE}(x)$	*Diff* (%)
1.00	51.7431	51.6467	0.1867	34.3758	34.3314	0.1293
1.01	49.3561	49.2661	0.1827	32.5499	32.5098	0.1233
1.05	41.2700	41.2009	0.1677	26.4501	26.4234	0.1010
1.07	37.9503	37.8895	0.1605	23.9902	23.9685	0.0905
1.10	33.6809	33.6301	0.1511	20.8718	20.8559	0.0762
1.20	23.7555	23.7263	0.1231	13.8740	13.8695	0.0324
2	5.8384	5.8363	0.0360	3.0054	3.0076	0.0731
3	3.1614	3.1609	0.0158	1.8387	1.8396	0.0489

Table 3: Comparisons of values ARL and AD of $j_0(x)$ and $j_1(x)$ from explicit formulae with numerical approximations $j^{IE}(x)$ for $\alpha = 0,\ \delta = 0.2$.

ρ	a	$\lambda=1, h=4$: $x=1$	$\lambda=1, h=4$: $x=3$	ρ	a	$\lambda=2, h=4$: $x=1$	$\lambda=2, h=4$: $x=3$
0.3	2	78.1792	60.8119	-0.3	2	279.3450	261.9780
		78.0735	60.7756			278.4080	261.1100
		821.2050	834.7610			998.0320	837.7100
		0.1352	0.0597			0.3354	0.3313
	3	498.6290	481.2620		3	1045.4500	1028.0900
		496.7850	479.4870			1041.3500	1024.0500
		928.3780	1152.1000			816.7590	803.7020
		0.3698	0.3688			0.4013	0.4018
	3.5	930.1200	912.7530		3.5	1831.6800	1814.3200
		926.4930	909.1950			1824.3300	1807.0300
		1168.6000	1164.9200			801.7350	829.1300
		0.3899	0.3898			0.4013	0.4018
0.5	2	33.8241	16.4568	-0.5	2	378.0590	360.6920
		33.9018	16.6039			376.7140	359.4160
		841.7970	804.1070			830.9870	835.4940
		0.2297	0.8939			0.3558	0.3538
	3	378.0590	360.6920		3	1313.7900	1296.4200
		376.7140	359.4160			1308.5700	1291.2800
		803.3420	802.1110			801.5800	814.7150
		0.3558	0.3538			0.3973	0.3965
	3.5	731.3350	713.9630		3.5	2274.0900	2256.7200
		728.5290	711.2310			2264.9000	2247.6100
		801.7670	794.9660			798.8790	796.6030
		0.3837	0.3827			0.4041	0.4037
0.7	2	103.9140	86.5464	-0.7	2	930.1200	912.7530
		103.7020	86.4037			926.4930	909.1950
		856.2590	801.5960			803.0460	802.7500
		0.2040	0.1649			0.3899	0.3898
	3	279.4080	261.9780		3	1641.5300	1624.1600
		278.4080	261.1100			1634.9600	1617.6600
		803.8880	800.6290			803.9830	804.4350
		0.3579	0.3313			0.4002	0.4002
	3.5	568.5820	551.2150		3.5	2814.4500	2797.0800
		566.4490	549.1510			2803.0300	2785.7300
		799.3490	801.0340			802.4690	825.0730
		0.3751	0.3744			0.4058	0.4058

Table 4: Comparison of values $j_0(x)$ and $j_1(x)$ from explicit formulae with numerical approximations $j^{IE}(x)$ for $h=3, a=2, \delta=0.2, \rho=-0.25$.

λ	$x=1$: $j(x)$	$x=1$: $j^{IE}(x)$	*Diff* (%)	$x=3$: $j(x)$	$x=3$: $j^{IE}(x)$	*Diff* (%)
1.00	113.1330	112.8510	0.2499	95.7659	95.5358	0.2409
1.01	107.3050	107.0420	0.2457	90.4992	90.2856	0.2366
1.05	87.7453	87.5426	0.2315	72.9255	72.7651	0.2204
1.07	79.8073	79.6284	0.2247	65.8472	65.7073	0.2129
1.10	69.6900	69.5406	0.2148	56.8809	56.7664	0.2017
1.20	46.6697	46.5830	0.1861	36.7881	36.7261	0.1688
2	8.6013	8.5951	0.0721	5.7684	5.7664	0.0347
3	3.9879	3.9866	0.0326	2.6652	2.6652	0.0000

The results are showed in Table 2 and 4, we compare the value obtained from explicit formulae and numerical approximations for varying levels of the parameter of white noise, λ. We assume that $h=3, a=2, \delta=0.2$ and the parameter of AR(1), $(\rho=-0.25, \rho=0.25)$, respectively.

From table 2 and 4, we found that the ARL and AD are decreasing as the formula's function and the results are in good agreement with the numerical approximation with an absolute percentage difference less than 1% for 500 iterations.

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Table 5: Comparison of values $j_0(x)$ and $j_1(x)$ from explicit formulae with numerical approximations $j^{IE}(x)$ for $\lambda=1, h=3, a=2, \alpha=0, \rho=0.25$.

δ	$x=1$		$Diff^{IE}(\%)$	$x=3$		$Diff^{IE}(\%)$
	$j(x)$	$j^{IE}(x)$		$j(x)$	$j^{IE}(x)$	
-1.5	475.1230	473.7470	0.2905	457.756	456.4320	0.2901
-1.0	271.3010	270.5410	0.2809	253.9340	253.2260	0.2796
-0.5	147.6770	147.2900	0.2627	130.3100	129.975	0.2577
0.0	72.6949	72.5352	0.2202	55.3277	55.2200	0.1950
0.1	61.6956	61.5691	0.2055	44.3284	44.2539	0.1683
0.2	51.7431	51.6467	0.1867	34.3758	34.3314	0.1293
0.5	27.2161	27.1938	0.0820	9.8488	9.8786	0.3017
0.6	20.5446	20.5426	0.0097	3.2374	3.2274	0.3098

Furthermore, we also compare these values for varying level of the parameter of trend slope (δ). We assume that $\lambda=1, h=3, a=2, \alpha=0$ and $\rho=0.25$ the results are showed in Table 5. The results are in good agreement with the numerical approximation with an absolute percentage difference less than 1% for 500 iterations.

VI. CONCLUSIONS

We derived analytically explicit formulas of ARL and AD on CUSUM procedure for the trend stationary first order autoregressive (trend AR(1)) observations with exponential distribution white noise. The accuracy of these explicit expressions are compared by numerical solutions of the integral equations based on using Gauss-Legendre integration rules. The numerical results and the values from the explicit formulas were in excellent agreement. The computation times required for the numerical computations were approximately 15 minutes compared with less than 1 second for the explicit formulas.

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REFERENCES

[1] R.A. Johnson and M. Bagshaw, "The effect of serial correlation on the performance of CUSUM tests," Technometrics, Vol. 16, 1974, pp. 103-112.
[2] T. J. Harris and W. H. Ross, "Statistical Process control procedure for correlated observations," Canadian Journal of Chemical Engineering, Vol. 69, 1991, pp. 48-57.
[3] W.H. Woodal and F. Faltin, "Autocorrelated data and SPC," ASQC Statistics Division Newsletter, Vol. 13, 1993, pp. 18-21.
[4] B.V. Rao, R.L. Disney and J.J. Pignatiello, "Uniqueness and converges of solutions to average run length integral equations for cumulative sum and other control charts,"IEEE Transactions, Vol. 33, 2001, pp. 463-469.
[5] A.D Karaoglan and G.M. Bayhan, "Performance comparison of residual control charts for trend stationary first order autoregressive processes," Gazi university journal of science, Vol. 24, 2011, pp. 329-339.
[6] J. Busaba, S. Sukparungsee and Y. Areepong, "An analysis of average run length for first order of autoregressive observations on CUSUM procedure," (submitted for publication to Applied Mathematical Science Journal, 2012).
[7] J. Busaba, S. Sukparungsee and Y. Areepong, "Numerical Approximations of average run length for AR(1) on Exponential CUSUM," (accepted to the international multiconference of engineers and computer scientists 2012 (IMECS 2012) at Hong Kong, 2012).
[8] G. Lorden, "Procedures for reacting to a change in distribution," Annual Mathematics Statistics, Vol. 42, 1971, pp. 1897-1908.
[9] H. M. Taylor, "A stopped Brownian motion formula," Annual Probability, Vol. 3, 1975, pp. 234-246.
[10] M. Pallak and D. Siegmund, "A diffusion process and its application to detecting a change in the drift of Brownian motion," Biometrika, Vol. 72, 1985, pp. 267-280.
[11] M. Pallak, "Average run lengths of an optimal method of detecting a change in distribution," Annual statistics, Vol. 15, 1987, pp. 749-779.
[12] A.A. Novikov, "On the first passage time of an autoregressive process over a level and application to a disorder problem," Theory of Probability and Its Applications, Vol.35, 1990, pp.269-279.
[13] J.M. Lucus and M. S. Saccucci, "Exponential weighted moving average control schemes: properties and enhancements," Technometrics, Vol.32, 1990, pp.1-29.
[14] S. Sukparungsee and A.A. Novikov, "On EWMA procedure for detection of a change in observations via martingale approach," KMITL Science Journal, An International Journal of Science and Applied Science, Vol. 6, 2006, pp. 373-380.
[15] S. Sukparungsee and A.A. Novikov, "Analytical approximations for detection of a change-point in case of light-tailed distributions," Journal of Quality measurement and Analysis, Vol. 4, 2008, pp. 49-56.
[16] S.V. Crowder, "A simple method for studying run length distributions of exponentially weighted moving average charts,"Technometrics, Vol. 29, 1987, pp. 401-407.
[17] M. S. Srivastava and Y. Wu, "Evaluation of optimum weights and average run lengths in EWMA control schemes," Communications in Statistics: Theory and Methods, Vol. 26, 1997, pp. 1253-1267.
[18] S. Knoth, "Accurate ARL calculation for EWMA control charts monitoring normal mean and variance simultaneously," Sequential Analysis, Vol. 26, 2006, pp. 251-263.
[19] S. Vardeman and D. Ray, "Average Run Lengths for CUSUM schemes when observations are exponentially Distributed," Technometrics, Vol. 27, 1985, pp. 145-150.
[20] V.V. Mazalov and D.N. Ahuravlev, "A method of Cumulative Sums in the problem of detection of traffic in computer networks," Programming and Computer Software, Vol. 28, 2002, pp. 342-348.
[21] G. Mititelu, V.V. Mazalov and A. Novikov, "On CUSUM procedure for hyper-exponential distribution," (submitted for publication to Statistics and Probability Letters, 2011).
[22] G. Mititelu, Y. Areepong, S. Sukparungsee and A. Novikov, "Explicit Analytical Solutions for the Average Run Length of CUSUM and EWMA Charts,"

East-West Journal of Mathematics, Special Vol., 2010, pp. 253-265.

[23] J. Busaba, S. Sukparungsee, Y. Areepong and G. Mititelu, "On CUSUM Procedure for Negative Exponential Distribution," The 14th Conference of the ASMDA International Society, 2011, pp. 209-218.

[24] J. Busaba, S. Sukparungsee, Y. Areepong and G. Mititelu, "An analysis of average run length for CUSUM procedure with negative exponential data," (accepted for publication to Chaing Mai Journal of Science, 2011.

[25] B.R. Venkateshwara, L. D. Ralph and J. P. Joseph, "Uniqueness and convergence of solutions to average run length integral equations for cumulative sums and other control charts," IIE Transactions, Vol. 33, 2001, pp. 463-469.

[26] V. M. George, S. P. Aleksey and G. T. Alexander, "A Numerical Approach to Performance Analysis of Quickest Change-Point Detection Procedures," Statistica Sinica, 2009 (in print).

[27] J. Busaba, S. Sukparungsee and Y. Areepong, "Numerical approximations of average run length for trend AR(1) on exponential CUSUM," (submitted to The 14th international engineering conference 2012 (KKU-IENC 2012), 2012).

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