Multiobjective Programming Problem Involving B-(p,r)-Type I Functions

Anurag Jayswal^{1, *}, Ioan Stancu-Minasian², I. Ahmad³

Abstract—In this paper we establish sufficient optimality conditions for multiobjective programming problems involving new classes of generalized B-(p, r)type I functions. Furthermore, appropriate duality theorems in the setting of Mond-Weir type dual are also presented in order to relate the efficient solutions of primal and dual problems.

Keywords: Multiobjective programming problem, B-(p,r)-type I functions, Efficiency, Sufficient optimality conditions, Duality

1 Introduction

In this paper, we consider the following multiobjective programming problem:

(VP)
$$\begin{cases} \text{Minimize } f(x) = (f_1(x), f_2(x), \dots, f_p(x)) \\ \text{subject to } x \in D = \{x \in X : g(x) \leq 0\}, \end{cases}$$

where X is an open subset of \mathbb{R}^n and $f: X \to \mathbb{R}^p$, $g: X \to \mathbb{R}^m$ are differentiable functions on X. Let $P = \{1, 2, \cdots, p\}$ and $M = \{1, 2, \cdots, m\}$.

Definition 1.1 A point $\bar{x} \in D$ is said to be *efficient point* to (VP) if and only if there does not exists $x \in D$ such that

$$f(x) \le f(\bar{x}).$$

Definition 1.2 A point $\bar{x} \in D$ is said to be *weak efficient* point to (VP) if and only if there does not exists $x \in D$ such that

$$f(x) < f(\bar{x}).$$

Convex functions have important properties that make them an ideal tool for solving practical problems. Generalizations of convexity related to optimality conditions and duality for nonlinear singleobjective or multiobjective optimization problems have been of much interest in the recent past and many contributions have been made to this development (see [1, 2, 3, 4, 9, 10, 11, 14] and the references therein).

Hanson [11] introduced the concept of invexity in constrained optimization as a generalization of convexity. These functions were named invex by Craven [8]. Hanson and Mond [12] introduced two new classes of functions called Type I and Type II functions, which were further generalized to pseudo Type I and quasi Type I functions by Rueda and Hanson [14]. Later, Aghezzaf and Hachimi [1] introduced generalized type I vector-valued functions and established appropriate duality theorems in the setting of Mond-Weir and general Mond-Weir type dual problems related to multiobjective programming problem.

Another generalization of convex functions, namely *B*-vex functions which satisfy many of basic properties of convex functions, was introduced by Bector and Singh [6]. Mishra *et al.* [13] focus his study on multiobjective programming problems and established sufficient optimality conditions and duality theorem involving generalized type I univex functions. Very recently, M. Soleimani-Damaneh [15] pointed out some omissions and inconsistency in the definitions and obtained results of Mishra *et al.* [13] and gave improved definitions of type I univex functions and an improved dual problem related to (VP). Furthermore, they proved sufficient optimality conditions and duality results involving improved definitions of type I univex functions.

Antczak [3] introduced the concept of B-(p, r)-invex functions by combining the concepts of B-invex functions [7] and (p, r)-invex functions [2]. In [4], Antczak extended the concept of B-(p, r)-invex functions to B-(p, r)pseudo-invex and B-(p, r)-quasi-invex functions and established sufficient optimality conditions and duality theorems for nonlinear mathematical programming problems under the assumptions of introduced functions.

In the present paper, inspired from the works of Antczak [3, 4] and M. Soleimani-Damaneh [15], we introduce new classes of generalized convex functions, namely B-(p, r)-type I, strong pseudo-quasi-B-(p, r)-type I, weak strictly-

^{*}Corresponding author. ¹Department of Applied Mathematics, Indian School of Mines, Dhanbad 826004, Jharkhand, India, Email: anurag_jais123@yahoo.com. ²Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, 13 Septembrie Street, No.13, 050711 Bucharest, Romania, Email: stancu_minasian@yahoo.com. ³Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran-31261, Saudi Arabia, Email: drizhar@kfupm.edu.sa. The research of the first author partially supported by the Indian School of Mines, Dhanbad, India under FRS (17)/2010-2011/AM.

pseudo-quasi-B-(p, r)- type I and weak strictly-pseudo-B-(p, r)-type I. A few Karush-Kuhn-Tucker type of sufficient optimality conditions are derived for an efficient solution to the problem involving the new classes of generalized B-(p, r)-type I functions. Furthermore, duality results for Mond-Weir type are proved under generalized B-(p, r)-type I assumptions on the functions involved.

The paper is organized as follows. In Section 2, we introduce some notation and definitions. Sufficient optimality conditions under the introduced definitions are established in Section 3. Duality theorems in the setting of Mond-Weir type dual are proved in Section 4. Conclusion are given in Section 5.

2 Notation and Preliminaries

Let \mathbb{R}^n be the *n*-dimensional Euclidean space and \mathbb{R}^n_+ be its non-negative orthant. Let us introduce some notation. If $x, y \in \mathbb{R}^n$ then $x < y \Leftrightarrow x_i < y_i, i = 1, 2, ..., n$; $x \leq y \Leftrightarrow x_i \leq y_i, i = 1, 2, ..., n$ and $x \leq y \Leftrightarrow x_i \leq y_i, i = 1, 2, ..., n$ and $x \neq y$.

To impose the convexity assumption in the above problem (VP), we introduce the following generalized B-(p, r)-type I functions. Let $f: X \to R^p$, $g: X \to R^m$ are differentiable functions on a nonempty open subset $X \subseteq R^n$ and $b_0, b_1: D \times X \to R_+$ such that $b_0 > 0$ and $b_1 \ge 0$. Let p and r be arbitrary real numbers.

Definition 2.1 The pair (f,g) is said to be B-(p,r)-type I with respect to b_0, b_1, p, r and η at u on X if, there exist functions $\eta: D \times X \to R^n, b_0, b_1: D \times X \to R_+$ and real numbers p and r such that, for all $x \in D$, the following inequalities

$$\begin{cases} \frac{1}{r}b_{0}(x,u)(e^{r(f(x)-f(u))}-1) & \geqq \frac{1}{p}\nabla f(u)(e^{p\eta(x,u)}-1), \\ \text{for } p \neq 0, r \neq 0, \\ \frac{1}{r}b_{1}(x,u)(e^{rg(u)}-1) & \geqq \frac{1}{p}\nabla g(u)(e^{p\eta(x,u)}-1), \\ \text{for } p \neq 0, r \neq 0, \end{cases} \\\begin{cases} \frac{1}{r}b_{0}(x,u)(e^{r(f(x)-f(u))}-1) & \geqq \nabla f(u)\eta(x,u), \\ \text{for } p = 0, r \neq 0, \\ \frac{1}{r}b_{1}(x,u)(e^{rg(u)}-1) & \geqq \nabla g(u)\eta(x,u), \\ \text{for } p = 0, r \neq 0, \end{cases} \\\begin{cases} b_{0}(x,u)(f(x)-f(u)) & \geqq \frac{1}{p}\nabla f(u)(e^{p\eta(x,u)}-1), \\ \text{for } p \neq 0, r = 0, \\ b_{1}(x,u)g(u) & \geqq \frac{1}{p}\nabla g(u)(e^{p\eta(x,u)}-1), \\ \text{for } p \neq 0, r = 0, \end{cases} \\\begin{cases} b_{0}(x,u)(f(x)-f(u)) & \geqq \nabla f(u)\eta(x,u), \\ \text{for } p = 0, r = 0, \\ b_{1}(x,u)g(u) & \geqq \nabla f(u)\eta(x,u), \\ \text{for } p = 0, r = 0, \\ b_{1}(x,u)g(u) & \geqq \nabla g(u)\eta(x,u), \\ \text{for } p = 0, r = 0, \end{cases} \end{cases}$$

hold.

Definition 2.2 The pair (f,g) is said to be *strong* pseudo-quasi-B-(p,r)-type I with respect to b_0, b_1, p, r and η at u on X if, there exist functions $\eta : D \times X \to R^n$, $b_0, b_1 : D \times X \to R_+$ and real numbers p and r such that, for all $x \in D$, the following inequalities

$$\begin{cases} \frac{1}{r}b_0(x,u)(e^{r(f(x)-f(u))}-1) \leq 0 \\ \Rightarrow \frac{1}{p}\nabla f(u)(e^{p\eta(x,u)}-1) \leq 0, \text{ for } p \neq 0, r \neq 0, \\ \frac{1}{r}b_1(x,u)(e^{rg(u)}-1) \leq 0 \\ \Rightarrow \frac{1}{p}\nabla g(u)(e^{p\eta(x,u)}-1) \leq 0, \text{ for } p \neq 0, r \neq 0, \\ \begin{cases} \frac{1}{r}b_0(x,u)(e^{r(f(x)-f(u))}-1) \leq 0 \\ \Rightarrow \nabla f(u)\eta(x,u) \leq 0, \text{ for } p = 0, r \neq 0, \\ \frac{1}{r}b_1(x,u)(e^{rg(u)}-1) \leq 0 \\ \Rightarrow \nabla g(u)\eta(x,u) \leq 0, \text{ for } p = 0, r \neq 0, \end{cases} \end{cases}$$

$$\begin{cases} b_0(x,u)(f(x)-f(u)) \leq 0 \\ \Rightarrow \frac{1}{p}\nabla f(u)(e^{p\eta(x,u)}-1) \leq 0, \text{ for } p \neq 0, r = 0, \\ b_1(x,u)g(u) \leq 0 \\ \Rightarrow \frac{1}{p}\nabla g(u)(e^{p\eta(x,u)}-1) \leq 0, \text{ for } p \neq 0, r = 0, \end{cases}$$

$$\begin{cases} b_0(x,u)(f(x)-f(u)) \leq 0 \\ \Rightarrow \nabla f(u)\eta(x,u) \leq 0, \text{ for } p = 0, r = 0, \\ b_1(x,u)g(u) \leq 0 \\ \Rightarrow \nabla f(u)\eta(x,u) \leq 0, \text{ for } p = 0, r = 0, \\ b_1(x,u)g(u) \leq 0 \\ \Rightarrow \nabla f(u)\eta(x,u) \leq 0, \text{ for } p = 0, r = 0, \end{cases}$$

hold.

Definition 2.3 The pair (f, g) is said to be *weak strictly*pseudo-quasi- B-(p, r)- type I with respect to b_0, b_1, p, r and η at u on X if, there exist functions $\eta : D \times X \to R^n$, $b_0, b_1 : D \times X \to R_+$ and real numbers p and r such that, for all $x \in D$, the following inequalities

$$\begin{cases} \frac{1}{r}b_0(x,u)(e^{r(f(x)-f(u))}-1) \leq 0\\ \Rightarrow \frac{1}{p}\nabla f(u)(e^{p\eta(x,u)}-1) < 0, \text{ for } p \neq 0, r \neq 0,\\ \frac{1}{r}b_1(x,u)(e^{rg(u)}-1) \leq 0\\ \Rightarrow \frac{1}{p}\nabla g(u)(e^{p\eta(x,u)}-1) \leq 0, \text{ for } p \neq 0, r \neq 0,\\ \begin{cases} \frac{1}{r}b_0(x,u)(e^{r(f(x)-f(u))}-1) \leq 0\\ \Rightarrow \nabla f(u)\eta(x,u) < 0, \text{ for } p = 0, r \neq 0,\\ \frac{1}{r}b_1(x,u)(e^{rg(u)}-1) \leq 0\\ \Rightarrow \nabla g(u)\eta(x,u) \leq 0, \text{ for } p = 0, r \neq 0, \end{cases}$$

$$\begin{cases} b_0(x,u)(f(x)-f(u)) \leq 0\\ \Rightarrow \frac{1}{p}\nabla f(u)(e^{p\eta(x,u)}-1) < 0, \text{ for } p \neq 0, r = 0,\\ b_1(x,u)g(u) \leq 0\\ \Rightarrow \frac{1}{p}\nabla g(u)(e^{p\eta(x,u)}-1) \leq 0, \text{ for } p \neq 0, r = 0,\\ \end{cases}$$

$$\begin{cases} b_0(x,u)(f(x)-f(u)) \leq 0\\ \Rightarrow \nabla f(u)\eta(x,u) < 0, \text{ for } p = 0, r = 0,\\ b_1(x,u)g(u) \leq 0\\ \Rightarrow \nabla f(u)\eta(x,u) < 0, \text{ for } p = 0, r = 0,\\ b_1(x,u)g(u) \leq 0\\ \Rightarrow \nabla g(u)\eta(x,u) \leq 0, \text{ for } p = 0, r = 0,\\ \end{cases}$$

hold.

If the first inequality in the above definition is satisfied as

$$\frac{1}{p} \nabla f(u)(e^{p\eta(x,u)} - \mathbf{1}) \ge 0 \Rightarrow \frac{1}{r} b_0(x,u)(e^{r(f(x) - f(u))} - 1) > 0 \text{ for } p \ne 0, \ r \ne 0,$$

then we say that (f,g) is strictly-pseudo-quasi-B-(p,r)-type I with respect to b_0, b_1, p, r and η at $u \in X$. Similarly for others cases.

Definition 2.4 The pair (f,g) is said to be *weak strictly-pseudo-B*-(p,r)-type I with respect to b_0, b_1, p, r and η at u on X if, there exist functions $\eta : D \times X \to R^n$, $b_0, b_1 : D \times X \to R_+$ and real numbers p and r such that, for all $x \in D$, the following inequalities

$$\begin{cases} \frac{1}{r}b_{0}(x,u)(e^{r(f(x)-f(u))}-1) \leq 0\\ \Rightarrow \frac{1}{p}\nabla f(u)(e^{p\eta(x,u)}-1) < 0, \text{ for } p \neq 0, r \neq 0, \\ \frac{1}{r}b_{1}(x,u)(e^{rg(u)}-1) \leq 0\\ \Rightarrow \frac{1}{p}\nabla g(u)(e^{p\eta(x,u)}-1) < 0, \text{ for } p \neq 0, r \neq 0, \\ \begin{cases} \frac{1}{r}b_{0}(x,u)(e^{r(f(x)-f(u))}-1) \leq 0\\ \Rightarrow \nabla f(u)\eta(x,u) < 0, \text{ for } p = 0, r \neq 0, \\ \frac{1}{r}b_{1}(x,u)(e^{rg(u)}-1) \leq 0\\ \Rightarrow \nabla g(u)\eta(x,u) < 0, \text{ for } p = 0, r \neq 0, \end{cases} \end{cases}$$

$$\begin{cases} b_{0}(x,u)(f(x)-f(u)) \leq 0\\ \Rightarrow \frac{1}{p}\nabla f(u)(e^{p\eta(x,u)}-1) < 0, \text{ for } p \neq 0, r = 0, \\ b_{1}(x,u)g(u) \leq 0\\ \Rightarrow \frac{1}{p}\nabla g(u)(e^{p\eta(x,u)}-1) < 0, \text{ for } p \neq 0, r = 0, \end{cases}$$

$$\begin{cases} b_{0}(x,u)(f(x)-f(u)) \leq 0\\ \Rightarrow \frac{1}{p}\nabla g(u)(e^{p\eta(x,u)}-1) < 0, \text{ for } p \neq 0, r = 0, \\ b_{1}(x,u)g(u) \leq 0\\ \Rightarrow \nabla f(u)\eta(x,u) < 0, \text{ for } p = 0, r = 0, \\ b_{1}(x,u)g(u) \leq 0\\ \Rightarrow \nabla f(u)\eta(x,u) < 0, \text{ for } p = 0, r = 0, \end{cases}$$

hold.

If the first inequality in the above definition is satisfied as

$$\frac{1}{p} \nabla f(u)(e^{p\eta(x,u)} - \mathbf{1}) > 0 \Rightarrow \frac{1}{r} b_0(x,u)(e^{r(f(x) - f(u))} - 1) > 0 \text{ for } p \neq 0, r \neq 0,$$

then we say that (f,g) is quasi strictly-pseudo-B-(p,r)-type I with respect to b_0, b_1, p, r and η at $u \in X$. Similarly for others cases.

Remark 2.1 In the Definitions 2.1, 2.2, 2.3 and 2.4, if the pair (f, g) is type I with respect to any $u \in X$, then it is called type I on X.

Remark 2.2 It should be noted that the exponentials appearing on the right-hand sides of inequalities above are understood to be taken componentwise and $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{R}^n$.

Remark 2.3 All the theorems in the subsequent parts of this paper will be proved only in the the case when

 $p \neq 0, r \neq 0$. The proofs in the other cases are easier that in this one. Moreover, without loss of generality, we shall assume that p > 0, r > 0 (in the cases, the direction some of the inequalities in the proof of the theorems should be changed to the opposite one).

3 Sufficient Optimality Conditions

Theorem 3.1 (Sufficient optimality conditions). Let $\bar{x} \in D$ be a feasible solution to (VP). Assume that there exist vectors $\bar{\lambda} \in R^p$ and $\bar{\mu} \in R^m, \bar{\mu} \geq 0$ such that

$$\bar{\lambda}\nabla f(\bar{x}) + \bar{\mu}\nabla g(\bar{x}) = 0. \tag{1}$$

Furthermore, we assume that any one of the following conditions holds:

- (a) $\bar{\lambda} > 0$ and (f,g) is strong pseudo-quasi-B-(p,r)-type I with respect to b_0, b_1, p, r and η at \bar{x} on D,
- (b) $\bar{\lambda} \geq 0$ and (f,g) is weak strictly-pseudo-quasi-B-(p,r)-type I with respect to b_0, b_1, p, r and η at \bar{x} on D,
- (c) $\bar{\lambda} \geq 0$ and (f,g) is weak strictly-pseudo-B-(p,r)-type I with respect to b_0, b_1, p, r and η at \bar{x} on D,

Then \bar{x} is an efficient solution to (VP).

Proof. (a) Suppose to contrary that \bar{x} is not an efficient solution to (VP). Then there exists $\tilde{x} \in D$ such that

$$f(\tilde{x}) \le f(\bar{x}). \tag{2}$$

Since $b_0(\tilde{x}, \bar{x}) > 0$, the above inequality yields

$$\frac{1}{r}b_0(\tilde{x},\bar{x})(e^{r(f(\tilde{x})-f(\bar{x}))}-1) \le 0.$$
(3)

By the feasibility of \bar{x} , we have

$$g(\bar{x}) \leq 0. \tag{4}$$

Since $b_1(\tilde{x}, \bar{x}) \geq 0$, the above inequality yields

$$\frac{1}{r}b_1(\tilde{x},\bar{x})(e^{rg(\bar{x})}-1) \leq 0.$$
(5)

From (3) and (5) and the assumption (a), we have

$$\frac{1}{p}\nabla f(\bar{x})(e^{p\eta(\tilde{x},\bar{x})}-\mathbf{1}) \leq 0,$$

and

$$\frac{1}{p}\nabla g(\bar{x})(e^{p\eta(\tilde{x},\bar{x})}-\mathbf{1}) \leq 0.$$

As $\bar{\lambda} > 0$ and $\mu \geq 0$, the above inequalities together give

$$\frac{1}{p}(\bar{\lambda}\nabla f(\bar{x}) + \bar{\mu}\nabla g(\bar{x}))(e^{p\eta(\tilde{x},\bar{x})} - 1) < 0,$$

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which contradicts (1).

If conditions (b) hold. Similar to the proof of condition (a), we obtain inequality (3) and (5).

From (3) and (5), using the assumption that (f, g) is weak strictly pseudo-quasi-B-(p, r)-type I at \overline{x} , we have

$$\frac{1}{p}\nabla f(\bar{x})(e^{p\eta(\tilde{x},\bar{x})} - \mathbf{1}) < 0,$$

and

$$\frac{1}{p}\nabla g(\bar{x})(e^{p\eta(\tilde{x},\bar{x})}-\mathbf{1}) \leq 0.$$

As $\bar{\lambda} \ge 0$ and $\bar{\mu} \ge 0$, the above inequalities together give

$$\frac{1}{p}(\bar{\lambda}\nabla f(\bar{x}) + \bar{\mu}\nabla g(\bar{x}))(e^{p\eta(\tilde{x},\bar{x})} - 1) < 0,$$

which contradicts (1).

If conditions (c) hold. Similar to the proof of condition (a), we obtain inequality (3) and (5).

From (3) and (5), using the assumption that (f, g) is weak strictly pseudo-*B*-(p, r)-type I at \overline{x} , we have

$$\frac{1}{p}\nabla f(\bar{x})(e^{p\eta(\tilde{x},\bar{x})}-1) < 0.$$

and

$$\frac{1}{p}\nabla g(\bar{x})(e^{p\eta(\tilde{x},\bar{x})} - \mathbf{1}) < 0.$$

As $\bar{\lambda} \geq 0$ and $\bar{\mu} \geq 0$, the above inequalities together give

$$\frac{1}{p}(\bar{\lambda}\nabla f(\bar{x}) + \bar{\mu}\nabla g(\bar{x}))(e^{p\eta(\tilde{x},\bar{x})} - 1) < 0,$$

which contradicts (1). This complete the proof.

4 Mond-Weir Type Duality

Now, in relation to (VP), we consider the following modified dual problem (MoDP), which is in the spirit of M. Soleimani-Damaneh [15]:

(MoDP)

Maximize f(y)

subject to
$$\lambda \nabla f(y) + \mu \nabla g(y) = 0,$$
 (6)

$$g(y) \le 0,\tag{7}$$

$$\lambda \geqq 0, \ \mu \geqq 0. \tag{8}$$

Let $W = \{(y, \lambda, \mu) \in X \times \mathbb{R}^p \times \mathbb{R}^m : \lambda \nabla f(y) + \mu \nabla g(y) = 0, g(y) \leq 0, \lambda \geq 0, \mu \geq 0\}$ denote the set of all feasible solutions of problem (MoDP). We denote by $pr_X W$ the projection of set W on X.

Theorem 4.1 (Weak duality). Let x and (y, λ, μ) be feasible solutions to (VP) and (MoDP), respectively. Furthermore, we assume that any one of the following conditions holds:

(a)
$$\lambda > 0$$
 and (f,g) is strong pseudo-quasi-B- (p,r) -type
I at y on $D \cup pr_X W$ with respect to b_0, b_1, p, r and η ,

- (b) $\lambda \geq 0$ and (f,g) is weak strictly-pseudo-quasi-B-(p,r)-type I at y on $D \cup pr_X W$ with respect to b_0, b_1, p, r and η ,
- (c) $\lambda \ge 0$ and (f,g) is weak strictly-pseudo-B-(p,r)-type I at y on $D \cup pr_X W$ with respect to b_0, b_1, p, r and η ,

Then

$$f(x) \not\leq f(y).$$

Proof. (a) Suppose contrary to the result, i.e.,

$$f(x) \le f(y).$$

Since $b_0(x, y) > 0$, the above inequality yields

$$\frac{1}{r}b_0(x,y)(e^{r(f(x)-f(y))}-1) \le 0.$$
(9)

By the feasibility of (y, λ, μ) for (MoDP), we have

$$g(y) \le 0. \tag{10}$$

Since $b_1(x, y) \ge 0$, the above inequality yields

$$\frac{1}{r}b_1(x,y)(e^{rg(y)}-1) \le 0.$$
(11)

From (9) and (11), and the assumption (a), we have

$$\frac{1}{p}\nabla f(y)(e^{p\eta(x,y)}-\mathbf{1}) \le 0,$$

$$\frac{1}{p}\nabla g(y)(e^{p\eta(x,y)}-\mathbf{1}) \leq 0.$$

As $\lambda > 0$ and $\mu \geq 0$, the above inequalities together give

$$\frac{1}{p}(\lambda \nabla f(y) + \mu \nabla g(y))(e^{p\eta(x,y)} - 1) < 0,$$

which contradicts (6).

If conditions (b) hold. Similar to the proof of condition (a), we obtain inequality (9) and (11).

From (9) and (11), using the assumption that (f,g) is weak strictly-pseudo-quasi-B-(p,r)-type I at y on $D \cup pr_X W$, we have

$$\frac{1}{p}\nabla f(y)(e^{p\eta(x,y)}-\mathbf{1})<0,$$

and

$$\frac{1}{p}\nabla g(y)(e^{p\eta(x,y)}-\mathbf{1}) \leq 0.$$

As $\lambda \geq 0$ and $\mu \geq 0$, the above inequalities together give

$$\frac{1}{p}(\lambda \nabla f(y) + \mu \nabla g(y))(e^{p\eta(x,y)} - 1) < 0,$$

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which contradicts (6).

If conditions (c) hold. Similar to the proof of condition (a), we obtain inequality (9) and (11).

From (9) and (11), using the assumption that (f, g) is weak strictly-pseudo-B-(p, r)-type I at y on $D \cup pr_X W$, we have

$$\frac{1}{p}\nabla f(y)(e^{p\eta(x,y)}-\mathbf{1}) < 0$$

and

$$\frac{1}{p}\nabla g(y)(e^{p\eta(x,y)}-\mathbf{1})<0.$$

As $\lambda \geq 0$ and $\mu \geq 0$, the above inequalities together give

$$\frac{1}{p}(\lambda \nabla f(y) + \mu \nabla g(y))(e^{p\eta(x,y)} - 1) < 0,$$

which contradicts (6). This completes the proof. \Box

Theorem 4.2 (Strong duality). Let \overline{x} be an efficient solution to (VP) and \overline{x} satisfies a suitable constraint qualification for (VP) in Bazaraa et al. [5]. Then there exist $\overline{\lambda} \in \mathbb{R}^p$ and $\overline{\mu} \in \mathbb{R}^m$ such that $(\overline{x}, \overline{\lambda}, \overline{\mu})$ is feasible to (MoDP). Furthermore, if the hypotheses of the weak duality Theorems 4.1 also hold, then $(\overline{x}, \overline{\lambda}, \overline{\mu})$ is an efficient solution to (MoDP).

Proof. Since \bar{x} is an efficient solution to (VP) and satisfy the suitable constraints qualification for (VP), then by Karush-Kuhn-Tucker conditions, we obtain that there exist $\bar{\lambda} > 0$ and $\bar{\mu} \geq 0$ such that

$$\bar{\lambda}\nabla f(\bar{x}) + \bar{\mu}\nabla g(\bar{x}) = 0.$$

This, in turn, implies that the triplet $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is feasible to (MoDP), as $\bar{x} \in D$. The efficiency of $(\bar{x}, \bar{\lambda}, \bar{\mu})$ for (MoDP) follows from the weak duality theorems. This completes the proof.

Theorem 4.3 (Converse duality). Let $(\overline{y}, \overline{\lambda}, \overline{\mu})$ be an efficient solution to (MoDP). We assume that any one of the hypotheses of the Theorems 4.1 holds at \overline{y} on $D \cup pr_X W$. Then, \overline{y} is an efficient solution to (VP).

Proof. Suppose contrary to the result that \bar{y} is not an efficient solution to (VP). Then there exists $\bar{x} \in D$ such that

$$f(\bar{x}) \le f(\bar{y}).$$

Since $b_0(\bar{x}, \bar{y}) > 0$, the above inequality yields

$$\frac{1}{r}b_0(\bar{x},\bar{y})(e^{r(f(\bar{x})-f(\bar{y}))}-1) \le 0.$$
(12)

By the feasibility of $(\bar{y}, \bar{\lambda}, \bar{\mu})$ for (MoDP), we have

$$g(\bar{y}) \leqq 0. \tag{13}$$

Since $b_1(\bar{x}, \bar{y}) \geq 0$, the above inequality yields

$$\frac{1}{r}b_1(\bar{x},\bar{y})(e^{rg(\bar{y})}-1) \le 0.$$
(14)

From (12) and (14) and in light of condition (a) of Theorem 4.1, we have

$$\frac{1}{p}\nabla f(\bar{y})(e^{p\eta(\bar{x},\bar{y})}-\mathbf{1}) \le 0,$$

and

$$\frac{1}{p}\nabla g(\bar{y})(e^{p\eta(\bar{x},\bar{y})}-\mathbf{1}) \leq 0.$$

As $\bar{\lambda} > 0$ and $\bar{\mu} \geq 0$, the above inequalities together give

$$\frac{1}{p}(\bar{\lambda}\nabla f(\bar{y}) + \bar{\mu}\nabla g(\bar{y}))(e^{p\eta(\bar{x},\bar{y})} - 1) < 0, \qquad (15)$$

which contradicts (6).

By conditions (b) of Theorem 4.1, from (12) and (14), we have

$$\frac{1}{p}\nabla f(\bar{y})(e^{p\eta(\bar{x},\bar{y})}-\mathbf{1})<0,$$

and

and

$$\frac{1}{p}\nabla g(\bar{y})(e^{p\eta(\bar{x},\bar{y})}-\mathbf{1}) \leq 0.$$

Since $\bar{\lambda} \geq 0$, the above inequalities imply (15), again a contradiction to (6).

By conditions (c) of Theorem 4.1, from (12) and (14), we have

$$\frac{1}{p}\nabla f(\bar{y})(e^{p\eta(\bar{x},\bar{y})}-\mathbf{1})<0,$$

 $\frac{1}{n}\nabla g(\bar{y})(e^{p\eta(\bar{x},\bar{y})}-\mathbf{1})<0.$

Since $\bar{\lambda} \geq 0$, the above inequalities imply (15), again a contradiction to (6). This completes the proof.

Theorem 4.4 (Strict converse duality). Let $\bar{x} \in D$ and $(\bar{y}, \bar{\lambda}, \bar{\mu}) \in W$ such that

$$\frac{1}{r}b_0(\bar{x},\bar{y})(e^{r(f(\bar{x})-f(\bar{y}))}-1) = 0.$$
 (16)

Further, suppose that any one of the following conditions is satisfied:

(a) (f,g) is strictly-pseudo-quasi B-(p,r) type-I at \bar{y} on $D \cup pr_X W$ with respect to b_0, b_1, p, r and η ,

(b) (f,g) is quasi strictly-pseudo B-(p,r) type-I at \bar{y} on $D \cup pr_X W$ with respect to b_0, b_1, p, r and η . Then

 $\bar{x} = \bar{y}.$

Proof. (a) Assume that $\bar{x} \neq \bar{y}$ and exhibit a contradiction. If the hypothesis (a) holds, then by the assumption

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that (f,g) is strictly-pseudo-quasi B-(p,r) type-I at \bar{y} on $D\cup pr_XW,$ we have

$$\frac{1}{p}\nabla f(\bar{y})(e^{p\eta(\bar{x},\bar{y})} - \mathbf{1}) \ge 0 \Rightarrow \frac{1}{r}b_0(\bar{x},\bar{y})(e^{r(f(\bar{x}) - f(\bar{y}))} - \mathbf{1}) > 0$$
(17)

and

$$\frac{1}{r}b_1(\bar{x},\bar{y})(e^{rg(\bar{y})}-1) \leq 0 \Rightarrow \frac{1}{p}\nabla g(\bar{y})(e^{p\eta(\bar{x},\bar{y})}-1) \leq 0.$$
(18)

for $p \neq 0, r \neq 0$.

By the feasibility of $(\bar{y}, \bar{\lambda}, \bar{\mu})$ for (MoDP), we have

$$g(\bar{y}) \leq 0. \tag{19}$$

Since $b_1(\bar{x}, \bar{y}) \geq 0$, the above inequality yields

$$\frac{1}{r}b_1(\bar{x},\bar{y})(e^{rg(\bar{y})}-1) \le 0, \tag{20}$$

which with (18) gives

$$\frac{1}{p}\nabla g(\bar{y})(e^{p\eta(\bar{x},\bar{y})}-1) \leq 0.$$
(21)

Since $\bar{\mu} \geq 0$, inequality (21) imply that

$$\frac{1}{p}\bar{\mu}\nabla g(\bar{y})(e^{p\eta(\bar{x},\bar{y})}-\mathbf{1}) \leq 0.$$
(22)

Above inequality along with first dual constraint implies

$$\frac{1}{p}\bar{\lambda}\nabla f(\bar{y})(e^{p\eta(\bar{x},\bar{y})} - \mathbf{1}) \ge 0.$$
(23)

Since $\bar{\lambda} \geq 0$, inequality (23) imply that

$$\frac{1}{p}\nabla f(\bar{y})(e^{p\eta(\bar{x},\bar{y})} - \mathbf{1}) \ge 0, \qquad (24)$$

which together with (17) implies

$$\frac{1}{r}b_0(\bar{x},\bar{y})(e^{r(f(\bar{x})-f(\bar{y}))}-1) > 0.$$
(25)

which contradicts (16).

The proof of (b) runs on the same lines as that of the case (a) and hence omitted. \Box

5 Conclusion

The concept of generalized B-(p, r)-type I functions has been introduced. Sufficient optimality conditions and duality results are proved in order to relate efficient solutions of the primal and dual problems for a pair of multiobjective programming problems.

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