Treatment of Lane-Emden Type Equations via Second Derivative Backward Differentiation Formula using Boundary Value Technique

S. A. Okunuga*, J. O. Ehigie† and A. B. Sofoluwe‡

Abstract—**In this paper, second order non-linear ordinary differential equations of Lane-Emden type are solved using the boundary value technique. A class of second derivative backward differentiation formula is derived from some continuous multistep schemes using the multistep collocation technique. The technique transforms the numerical methods to a system of non-linear equations represented as a tridiagonal matrix, thereby obtaining numerical solutions concurrently on the the entire range of integration. General properties of the numerical method are presented as well as the stability properties. Some equations of Lane-Emden type are solved to demonstrate the efficiency of the method.**

Keywords: Continuous Schemes, Multistep Collocation, Lane-Emden type equation, Second Derivative Backward Differentiation Formula

1 Introduction

In mathematical physics and astrophysics, the modeling of the temperature variations of a spherical gas cloud under the natural attraction of its molecules and subject to the law of classical thermodynamics have been the Lane-Emden type equation, which is generally expressed as,

$$y'' + \tfrac{k}{x}y' + f(x)g(y) = h(x) \qquad x > 0, k > 0 \qquad y(0) = a, y'(0) = b \tag{1}$$

A steady case, for $k = 2$, $h(x) = 0$ is known as the generalized Emden-Fowler equation given by,

$$y'' + \frac{2}{x}y' + f(x)g(y) = 0 \tag{2}$$

The derivation of equations (1) and (2) can be found in the literatures [9],[17]. Several second order non-linear ordinary differential equations of Lane-Emden type are derived as special cases of (2). Examples of such are:

when $g(y) = y^m$ and $f(x) = 1$,

(2) is known as the standard Lane-Emden equation, while when,

$$g(y) = \left(y^2 - C\right)^{\frac{3}{2}} \text{ and } f(x) = 1$$

gives the white dwarf equation introduced in [4], in the study of gravitational potential of the degenerate white dwarf stars.

Many researchers have attempted the solution of second order non-linear ordinary differential equations of Lane-Emden type, they include [8], [9], [10], [13], [14], [17], [18], [19], [20], [21]. But the results in Horedt [9], "Polytropes Application in Astrophysics and related fields" is used as the model for this study.

Because of the singularity at the initial value of the problem, we propose a class of Second Derivative Backward Differentiation Formulae (SDBDF) which shall be implemented using the boundary value technique as in [1], [2], [3], [11]. These formulae derived from multistep collocation technique allows the generation of the complementary method that shall be used together with the main method as a boundary value method.

The article is organized as follows: The theoretical procedure is presented in section 2 which involves the framework for the derivation of the second derivative backward differentiation formulae for special cases of $k = 3$ and $k = 4$. Some properties such as the error constants and the region of absolute stability of the second derivative backward differentiation formula is presented in section 3. The implementation strategy is given in section 4. Finally, some experimental illustration are solved to show the efficiency and accuracy of the methods in section 5.

2 Theoretical Procedure

The Lane-Emden type equation (1) is transformed to a system of first order ODEs,

$$\begin{aligned} y' &= z & y(0) &= a \\ z' &= -\left(\tfrac{k}{x}z + f(x)g(y) - h(x)\right) & z(0) &= b \end{aligned} \tag{3}$$

*Department of Mathematics, University of Lagos, Nigeria, sokunuga@unilag.edu.ng
†Corresponding Author: Department of Mathematics, University of Lagos, Nigeria, jehigie@unilag.edu.ng
‡Department of Computer Sciences, University of Lagos, Nigeria, absofoluwe@yahoo.com

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Hence for simplicity, we consider the scalar first order ordinary differential equation,

$$y' = f(x, y), \qquad y(a) = y_0, \qquad x \in [a, b] \tag{4}$$

The proposed second derivative backward differentiation formula will be of the form,

$$y_{n+k} = \sum_{j=0}^{k-1} \alpha_j y_{n+j} + h\beta_k f_{n+k} + h^2 \delta_k g_{n+k} \tag{5}$$

where $y_{n+j} = y(x_n + jh)$, $f_{n+k} \equiv f(x_n + jh, y(x_n + jh), y'(x_n + jh))$ and

$$g_{n+k} \equiv \frac{df}{dx}\Big|_{\substack{x=x_{n+k}\\ y=y_{n+k}}}$$

x_n is a node point and α_j, β_j and δ_j are parameters to be obtained from the multistep collocation technique.[5], [11], [15], [16]. To derive this method, we use the basis function,

$$y(x) = \sum_{j=0}^{p} a_j \left(\frac{x - x_n}{h}\right)^j \tag{6}$$

Equation (6) is then interpolated at points x_{n+j}, $j = 0, 1, 2, \cdots, k-1$, while $y'(x)$ and $y''(x)$ are collocated at point x_{n+k}. The system of equations obtained is solved for variables $a_0, a_1, a_2, \cdots, a_{k+1}$ which is substituted back in (6) to obtain the continuous second derivative backward differentiation formula of the form,

$$y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + h\beta_k(x) f_{n+k} + h^2 \delta_k(x) g_{n+k} \tag{7}$$

The main method is obtained at evaluation of (7) at $x = x_{n+k}$ and complementary methods are obtained on differentiating (7) and evaluating at x_{n+j}, $j = 1, 2, \cdots, k-1$.

2.1 Derivation of the Second Derivative Backward Differentiation Formula for $k = 3$

Using the multistep collocation method to derive the continuous Second Derivative Backward Differentiation Formula, In the basis function $y(x)$ in (6), we set $p = k + 1$, Hence,

$$y(x) = \sum_{j=0}^{4} a_j \left(\frac{x - x_n}{h}\right)^j \tag{8}$$

interpolating (8) at point $x = x_{n+j}, j = 0, 1, 2$, collocating $y'(x)$ and $y''(x)$ at $x = x_{n+3}$. We obtain a system of equations represented in the matrix form,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \\ 0 & 1 & 6 & 27 & 108 \\ 0 & 0 & 2 & 18 & 108 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ hf_{n+3} \\ h^2 g_{n+3} \end{pmatrix}$$

solving the system for a_i, $i = 0(1)4$, we get

$$\begin{aligned}
a_0 &= y_n \\
a_1 &= -\frac{351}{170} y_n + \frac{324}{85} y_{n+1} + \frac{58}{85} hf_{n+3} - \frac{39}{85} h^2 g_{n+3} - \frac{297}{170} y_{n+2} \\
a_2 &= -\frac{243}{170} y_n - \frac{342}{85} y_{n+1} - \frac{99}{85} hf_{n+3} + \frac{139}{170} h^2 g_{n+3} + \frac{441}{170} y_{n+2} \\
a_3 &= -\frac{69}{170} y_n + \frac{116}{85} y_{n+1} + \frac{47}{85} hf_{n+3} - \frac{36}{85} h^2 g_{n+3} - \frac{163}{170} y_{n+2} \\
a_4 &= \frac{7}{170} y_n - \frac{13}{85} y_{n+1} - \frac{6}{85} hf_{n+3} + \frac{11}{170} h^2 g_{n+3} + \frac{19}{170} y_{n+2}
\end{aligned}$$

substituting a_i, $i = 0(1)4$ in (8) yields the continuous second derivative backward differentiation formula,

$$\begin{gathered}
y(x) = \\
\left(1 - \tfrac{351}{170}\left(\tfrac{x-x_n}{h}\right) + \tfrac{243}{170}\left(\tfrac{x-x_n}{h}\right)^2 - \tfrac{69}{170}\left(\tfrac{x-x_n}{h}\right)^3 + \tfrac{7}{170}\left(\tfrac{x-x_n}{h}\right)^4\right) y_n \\
+\left(\tfrac{324}{85}\left(\tfrac{x-x_n}{h}\right) - \tfrac{342}{85}\left(\tfrac{x-x_n}{h}\right)^2 + \tfrac{116}{85}\left(\tfrac{x-x_n}{h}\right)^3 - \tfrac{13}{85}\left(\tfrac{x-x_n}{h}\right)^4\right) y_{n+1} \\
+\left(-\tfrac{297}{170}\left(\tfrac{x-x_n}{h}\right) + \tfrac{441}{170}\left(\tfrac{x-x_n}{h}\right)^2 - \tfrac{163}{170}\left(\tfrac{x-x_n}{h}\right)^3 + \tfrac{19}{170}\left(\tfrac{x-x_n}{h}\right)^4\right) y_{n+2} \\
+h\left(\tfrac{58}{85}\left(\tfrac{x-x_n}{h}\right) - \tfrac{99}{85}\left(\tfrac{x-x_n}{h}\right)^2 + \tfrac{47}{85}\left(\tfrac{x-x_n}{h}\right)^3 - \tfrac{6}{85}\left(\tfrac{x-x_n}{h}\right)^4\right) f_{n+3} \\
+h^2\left(-\tfrac{39}{85}\left(\tfrac{x-x_n}{h}\right) + \tfrac{139}{170}\left(\tfrac{x-x_n}{h}\right)^2 - \tfrac{36}{85}\left(\tfrac{x-x_n}{h}\right)^3 + \tfrac{11}{170}\left(\tfrac{x-x_n}{h}\right)^4\right) g_{n+3}
\end{gathered} \tag{9}$$

evaluating (9) at $x = x_{n+3}$ yields the main method given in (10) below, while differentiating (9) and evaluating at points $x = x_{n+1}$ and $x = x_{n+2}$ $\{x_{n+1}, x_{n+2}\}$ yields the complementary methods (11) and (12).

$$y_{n+3} = \frac{4}{85} y_n - \frac{27}{85} y_{n+1} + \frac{108}{85} y_{n+2} + \frac{66}{85} hf_{n+3} - \frac{18}{85} h^2 g_{n+3} \tag{10}$$

$$hf_{n+1} = -\frac{22}{85} y_n - \frac{64}{85} y_{n+1} + \frac{86}{85} y_{n+2} - \frac{23}{85} hf_{n+3} + \frac{14}{85} h^2 g_{n+3} \tag{11}$$

$$hf_{n+2} = \frac{1}{10} y_n - \frac{4}{5} y_{n+1} + \frac{7}{10} y_{n+2} + \frac{2}{5} hf_{n+3} - \frac{1}{5} h^2 g_{n+3} \tag{12}$$

2.2 Derivation of Second Derivative Backward Differentiation Formula for $k = 4$

The same technique follows for $k = 4$ and a continuous second derivative backward differentiation formula is,

$$\begin{gathered}
y(x) = \\
\begin{pmatrix} 1 - \tfrac{2842}{1245}\left(\tfrac{x-x_n}{h}\right) + \tfrac{4669}{2490}\left(\tfrac{x-x_n}{h}\right)^2 - \tfrac{2357}{3320}\left(\tfrac{x-x_n}{h}\right)^3 \\ + \tfrac{157}{1245}\left(\tfrac{x-x_n}{h}\right)^4 - \tfrac{17}{1992}\left(\tfrac{x-x_n}{h}\right)^5 \end{pmatrix} y_n \\
+\begin{pmatrix} \tfrac{2064}{415}\left(\tfrac{x-x_n}{h}\right) - \tfrac{2644}{415}\left(\tfrac{x-x_n}{h}\right)^2 + \tfrac{1219}{415}\left(\tfrac{x-x_n}{h}\right)^3 \\ - \tfrac{483}{830}\left(\tfrac{x-x_n}{h}\right)^4 + \tfrac{7}{166}\left(\tfrac{x-x_n}{h}\right)^5 \end{pmatrix} y_{n+1} \\
+\begin{pmatrix} -\tfrac{1986}{415}\left(\tfrac{x-x_n}{h}\right) + \tfrac{7057}{830}\left(\tfrac{x-x_n}{h}\right)^2 - \tfrac{15483}{3320}\left(\tfrac{x-x_n}{h}\right)^3 \\ + \tfrac{426}{415}\left(\tfrac{x-x_n}{h}\right)^4 - \tfrac{53}{664}\left(\tfrac{x-x_n}{h}\right)^5 \end{pmatrix} y_{n+2} \\
+\begin{pmatrix} \tfrac{2608}{1245}\left(\tfrac{x-x_n}{h}\right) - \tfrac{4988}{1245}\left(\tfrac{x-x_n}{h}\right)^2 + \tfrac{1011}{415}\left(\tfrac{x-x_n}{h}\right)^3 \\ - \tfrac{1421}{2490}\left(\tfrac{x-x_n}{h}\right)^4 + \tfrac{23}{498}\left(\tfrac{x-x_n}{h}\right)^5 \end{pmatrix} y_{n+3} \\
+h\begin{pmatrix} -\tfrac{57}{83}\left(\tfrac{x-x_n}{h}\right) + \tfrac{115}{83}\left(\tfrac{x-x_n}{h}\right)^2 - \tfrac{305}{332}\left(\tfrac{x-x_n}{h}\right)^3 \\ + \tfrac{20}{83}\left(\tfrac{x-x_n}{h}\right)^4 - \tfrac{7}{332}\left(\tfrac{x-x_n}{h}\right)^5 \end{pmatrix} f_{n+4} \\
+h^2\begin{pmatrix} \tfrac{168}{415}\left(\tfrac{x-x_n}{h}\right) - \tfrac{691}{830}\left(\tfrac{x-x_n}{h}\right)^2 + \tfrac{947}{1660}\left(\tfrac{x-x_n}{h}\right)^3 \\ - \tfrac{131}{830}\left(\tfrac{x-x_n}{h}\right)^4 + \tfrac{5}{332}\left(\tfrac{x-x_n}{h}\right)^5 \end{pmatrix} g_{n+4}
\end{gathered} \tag{13}$$

Again, evaluating (13) at $x = x_{n+4}$ yields the main method (14), while differentiating (13) and evaluating x at the 3 points $\{x_{n+1}, x_{n+2}, x_{n+3}\}$ yields the complementary methods (15) - (17) as given below:

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$$y_{n+4} = -\frac{9}{415}y_n + \frac{64}{415}y_{n+1} - \frac{216}{415}y_{n+2} + \frac{576}{415}y_{n+3} + \frac{60}{83}hf_{n+4} - \frac{72}{415}h^2g_{n+4} \quad (14)$$

$$hf_{n+1} = -\frac{333}{1660}y_n - \frac{891}{830}y_{n+1} + \frac{3213}{1660}y_{n+2} - \frac{549}{830}y_{n+3} + \frac{31}{166}hf_{n+4} - \frac{87}{830}h^2g_{n+4} \quad (15)$$

$$hf_{n+2} = \frac{127}{2490}y_n - \frac{212}{415}y_{n+1} - \frac{229}{830}y_{n+2} + \frac{916}{1245}y_{n+3} - \frac{12}{83}hf_{n+4} + \frac{31}{415}h^2g_{n+4} \quad (16)$$

$$hf_{n+3} = -\frac{187}{4980}y_n + \frac{237}{830}y_{n+1} - \frac{1911}{1660}y_{n+2} + \frac{2249}{2490}y_{n+3} + \frac{51}{166}hf_{n+4} - \frac{111}{830}h^2g_{n+4} \quad (17)$$

3 Some Properties of the Second Derivative Backward Differentiation Formula

The error constants C_{p+1} of the second derivative multistep methods (10), (11), (12), (14), (15), (16) and (17) recovered are obtained by associating the equations respectively with the equation,

$$LTE = L[y(x_n;h)] = \left[\sum_{j=0}^{k} y(x+jh) - h\beta_k y'(x+kh) - h^2\delta_k y''(x+kh)\right]$$

$$LTE = C_{p+1}h^{p+1}y^{(p+1)}(x_n) \quad (18)$$

On expansion and collecting in terms of the Taylor's series [12], the order and the error constants for the derived numerical methods in this paper are presented in the Table 1 below.

Table 1: Order and Error Constants for the Methods

Method	Order	Error Constant (C_{p+1})
(10)	4	$\frac{9}{425}$
(11)	4	$-\frac{127}{2550}$
(12)	4	$\frac{11}{300}$
(14)	5	$\frac{24}{2075}$
(15)	5	$\frac{531}{16600}$
(16)	5	$-\frac{601}{37350}$
(17)	5	$\frac{859}{49800}$

The stability of a linear multistep method determines the manner in which the error is propagated as the numerical computation proceeds. Hence, it would be necessary to investigate the stability properties of the main methods, that is the multistep methods methods recovered on evaluation of (9) and (13) at $x = x_{n+k}$

Definition

The stability region $\mathcal{S}$ of methods (10) and (14) is the set of all points $z \in \mathbb{C}$ such that all roots $\xi_k(z)$ of characteristic equation lie on the unit disc $|\xi_i| \leq 1$ and those with modulus one are simple.

Applying methods (10) and (14) to the test problem,

$$y' = \lambda y$$

with substitution $z = \lambda h$, we obtain the characteristic equation for method (10) as,

$$\left(1 - \frac{66}{85}z + \frac{18}{85}z^2\right)\xi^3 - \frac{108}{85}\xi^2 + \frac{27}{85}\xi - \frac{4}{85} = 0 \quad (19)$$

while the characteristic equation for method (14) is obtained as,

$$\left(1 - \frac{60}{83}z + \frac{72}{415}z^2\right)\xi^4 - \frac{576}{415}\xi^3 + \frac{216}{415}\xi^2 - \frac{64}{415}\xi + \frac{9}{415} = 0 \quad (20)$$

The region of absolute stability (RAS) of the methods (10) and (14) from their respective characteristics equations (19) and (20) are plotted using the maple software in figures 1 and 2 respectively: **Remark:** The RAS for

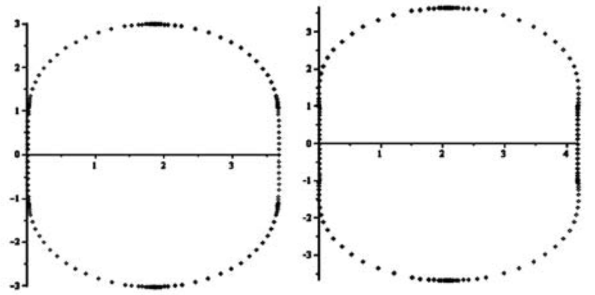

Figure 1: RAS for SDBDF for $k = 3$ and $k = 4$

the Second Derivative Backward Differentiation Formula presented shows that the main methods (10) and (14) are A-stable for order $p = 4$ and $p = 5$.

4 Implementation Strategy of the Methods

In what follows, a general procedure for the implementation of the methods using a Boundary Value technique or method in matrix form as in Fatunla [6] is presented. To obtain the approximate solutions for y_{n+j}, $j = 0, 1, 2, \cdots, N = \frac{b-a}{h}$ points on the bound of integration $[a, b]$, with N-vector Y_N, F_N and G_N specified, we define the following,

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$$
\begin{aligned}
Y_N &= [y_{n+1}, y_{n+2}, y_{n+3}, ..., y_{n+N}]^T \\
Y_{N-1} &= [y_{n-N+1}, y_{n-N+2}, y_{n-N+3}, ..., y_n]^T \\
F_N &= [f_{n+1}, f_{n+2}, f_{n+3}, ..., f_{n+N}]^T \\
G_N &= [g_{n+1}, g_{n+2}, g_{n+3}, ..., g_{n+N}]^T
\end{aligned}
$$

where $y_{n+j} = y(x_n + jh)$, $f_{n+j} = f(x_n + jh, y(x_n + jh))$ and $g_{n+j} \equiv \frac{df(x,y(x))}{dx}|^{x=x_{n+j}}_{y=y_{n+j}}$. The integration on the entire block shall be written compactly as,

$$AY_N = BY_{N-1} + hCF_N + h^2DG_N \tag{21}$$

Which forms a non-linear equation because of the implicit nature, hence we employ the Newton-iteration for the evaluation of the approximate solutions. Hence, (21) can be written as,

$$AY_N - BY_{N-1} - hCF_N - h^2DG_N = 0 \tag{22}$$

Using the Newton's approach for the implementation of implicit schemes as given in [12], we have that the solutions of the block is given as,

$$Y_N^{(i+1)} = Y_N^{(i)} - J^{-1}(Y_N)\left(AY_N - BY_{N-1} - hCF_N - h^2DG_N\right) \tag{23}$$

where $J^{-1}(Y_N)$ the Jacobian matrix is,

$$J^{-1}(Y_N) = \left(A - hC\frac{\partial F_N}{\partial Y} - h^2D\frac{\partial G_N}{\partial Y}\right)^{-1}$$

Hence on transformation into matrix form, we have,

$$AY_N = BY_{N-1} + hCF_N + h^2DG_N \tag{24}$$

such that for $k = 2$,

$$A = \begin{pmatrix} \frac{64}{85} & -\frac{86}{85} & 0 & \cdots & 0 & 0 \\ \frac{4}{5} & -\frac{7}{10} & 0 & \cdots & 0 & 0 \\ \frac{27}{85} & -\frac{108}{85} & 1 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \frac{27}{85} & -\frac{108}{85} & 1 & 0 \\ 0 & 0 & 0 & \frac{27}{85} & -\frac{108}{85} & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -\frac{22}{85} \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{10} \\ 0 & 0 & 0 & \cdots & 0 & \frac{4}{85} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} -1 & 0 & -\frac{23}{85} & 0 & \cdots & 0 \\ 0 & -1 & \frac{2}{5} & 0 & \cdots & 0 \\ 0 & 0 & \frac{66}{85} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \frac{66}{85} & 0 & 0 \\ 0 & 0 & \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{66}{85} \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & \frac{14}{85} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{5} & 0 & 0 & 0 \\ 0 & 0 & -\frac{18}{85} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{18}{85} & 0 & 0 \\ \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{18}{85} \end{pmatrix}$$

and for $k = 4$, we have,

$$A = \begin{pmatrix} \frac{891}{830} & -\frac{3213}{1660} & \frac{549}{830} & 0 & \cdots & 0 & 0 \\ \frac{212}{415} & \frac{229}{830} & -\frac{916}{1245} & 0 & \cdots & 0 & 0 \\ -\frac{237}{830} & \frac{1911}{1660} & \frac{2249}{2490} & 0 & \cdots & 0 & 0 \\ -\frac{64}{415} & \frac{216}{415} & -\frac{576}{415} & 1 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & -\frac{64}{415} & \frac{216}{415} & -\frac{576}{415} & 1 & 0 \\ 0 & 0 & 0 & -\frac{64}{415} & \frac{216}{415} & -\frac{576}{415} & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{333}{1660} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{127}{2490} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{187}{4980} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{9}{415} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} -1 & 0 & 0 & \frac{31}{166} & 0 & \cdots & 0 \\ 0 & -1 & 0 & -\frac{12}{83} & 0 & \cdots & 0 \\ 0 & 0 & -1 & \frac{51}{166} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \frac{60}{83} & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \frac{60}{83} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{60}{83} \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 & -\frac{87}{830} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \frac{31}{415} & 0 & \cdots & 0 \\ 0 & 0 & 0 & -\frac{111}{830} & 0 & \cdots & 0 \\ 0 & 0 & 0 & -\frac{72}{415} & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & -\frac{72}{415} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{72}{415} \end{pmatrix}$$

Remark: The boundary value technique with (23) is made possible using the Newton's iteration features in the maple software.

5 Experimental Problems

In this section we consider some second order non-linear ordinary differential equations of Lane-Emden type (2) which are transformed to system of first order differential equations. The boundary value methods for $k = 3$ and $k = 4$ will be denoted as BVM3 and BVM4 respectively in the presentation of numerical results.

Problem 5.1 White-dwarf Equation: Consider the White dwarf equation,

$$y'' + \frac{2}{x}y'(x) + \left(y^2 - C\right)^{\frac{3}{2}} = 0 \tag{25}$$

introduced in [4] in the study of gravitational potential of the degenerate white-dwarf stars is solved with the BVMs for $C = 0.2, 0.4, 0.6$ and 0.8, where at $C = 0$, (25) becomes the standard Lane-Emden for $m = 3$. (25) can be transformed to the system,

$$\begin{array}{ll} y' = z & y(0) = 1 \\ z' = -\left(\frac{2}{x}z + \left(y^2 - C\right)^{\frac{3}{2}}\right) & z(0) = 0 \end{array} \quad (26)$$

Solving (26) using the BVMs for $k = 3$ and $k = 4$ is denoted as BVM3 and BVM4 respectively, the numerical results are compared with the results obtained in [9] and [10] and are presented in Tables 2 and 3, while graphical results is presented for $C = 0, 0.2, 0.4, 0.6$ and 0.8 in Figure 3 and 4.

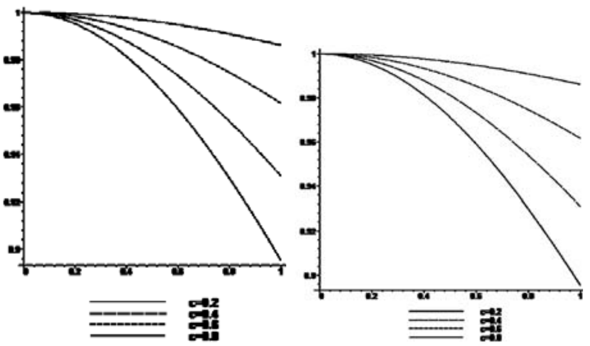

Figure 2: Graphical Result for White Dwarf Equation Using BVM3 and BVM4

Table 2: Numerical result for $C = 0$ using BVM3 for $h = 0.05$

x	BVM3	BVM4	Hojjati and Parand [9]	Horedt [10]
0.5	0.9598393004	0.9598390586	0.959839069883	0.959839
1.0	0.8550578750	0.8550575772	0.855057568546	0.855058
5.0	0.1108196525	0.1108198152	0.110819835160	0.110820
6.0	0.0437378675	0.0437379642	0.043737983910	0.043738

Table 3: Numerical result for $C = 0$ using BVM3 for $h = 0.01$

x	BVM3	BVM4	Hojjati and Parand [9]	Horedt [10]
0.5	0.9598390702	0.9598390699	0.959839069883	0.959839
1.0	0.8550575691	0.8550575686	0.855057568546	0.855058
5.0	0.1108198348	0.1108198351	0.110819835160	0.110820
6.0	0.0437379836	0.0437379839	0.043737983910	0.043738

Remark: From the Table 2 and 3, it is easily seen that the boundary value methods (BVM3 and BVM4) are comparable to the results obtained in Hojjati and Parand [9] and Horedt [10].

Problem 5.2 We also consider the second order differential equation:

$$y'' + \frac{2}{x}y'(x) + 4\left(2e^y + e^{\frac{y}{2}}\right) = 0 \quad (27)$$

Problem (27) has an analytical solution: $y(x) = -2\ln\left(1 + x^2\right)$. To solve (27) numerically, we again recast it to a system of first order ordinary differential equation of the form,

$$\begin{array}{ll} y' = z & y(0) = 0 \\ z' = -\frac{2}{x}z - 4\left(2e^y + e^{\frac{y}{2}}\right) & z(0) = 0 \end{array} \quad (28)$$

The problem is then solved using the derived SDBDF methods by BVM. The Numerical results obtained are presented in Tables 4, 5, 6 and 7 for solution using $h = 0.05$ and $h = 0.01$. **Remark:** See from Tables 4, 5, 6 and 7, that methods of BVM4 performs better than the methods of BVM3, which justifies that the higher the order of convergence of the BVM the higher the accuracy to be expected viz a viz the step length.

Table 4: Numerical Results for Problem 5.2 for BVM3 using $h = 0.05$

x	Exact	BVM3	Error
0.25	-0.121249243633	-0.121239888642	9.35E-06
0.50	-0.446287102628	-0.446275599224	1.15E-05
0.75	-0.892574205257	-0.892570430056	3.77E-06
1.00	-1.386294361199	-1.386294361120	2.80E-06

Table 5: Numerical Results for Problem 5.2 for BVM3 using $h = 0.01$

x	Exact	BVM3	Error
0.25	-0.121249243633	-0.121249230231	9.35E-08
0.50	-0.446287102628	-0.446287082373	2.02E-08
0.75	-0.892574205257	-0.892574195535	9.72E-09
1.00	-1.386294361199	-1.386294365367	4.24E-09

Table 6: Numerical Results for Problem 5.2 for BVM4 using $h = 0.05$

x	Exact	BVM4	Error
0.25	-0.121249243633	-0.121249559058	3.15E-07
0.50	-0.446287102628	-0.446286113311	9.89E-07
0.75	-0.892574205257	-0.892573230121	9.75E-07
1.00	-1.386294361199	-1.386294072225	2.88E-07

Problem 5.3 The final example in this paper is also given by:

$$y'' + \frac{2}{x}y'(x) - 2\left(2x^2 + 3\right)y = 0 \quad (29)$$

Equation (29) is of Lane-Emden problem type with an analytical solution given by: $y(x) = e^{x^2}$. Equation (29) transforms to the system of first order ordinary differential equation given as,

$$\begin{array}{ll} y' = z & y(0) = 0 \\ z' = -\frac{2}{x}z + 2\left(2x^2 + 3\right)y & z(0) = 0 \end{array} \quad (30)$$

The Tables 8, 9, 10 and 11 shows the numerical results obtained using the BVM3 and BVM4 methods.

Remark: Again numerical results of Problem 5.3 as presented in Tables 8, 9, 10 and 11 shows that methods of BVM4 performs better than the methods of BVM3 which affirms that the higher the order of convergence of the BVM, the higher the accuracy to be expected viz a viz the step length.

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Table 7: Numerical Results for Problem 5.2 for BVM4 using $h = 0.01$

x	Exact	BVM4	Error
0.25	-0.121249243633	-0.121249243938	3.04E-10
0.50	-0.446287102628	-0.446287102319	3.09E-10
0.75	-0.892574205257	-0.892574204718	5.39E-10
1.00	-1.386294361199	-1.386294361120	3.34E-10

Table 8: Numerical Results for Problem 5.3 for BVM3 using $h = 0.05$

x	Exact	BVM3	Error
0.25	1.0644944589179	1.0644898251043	4.63E-06
0.50	1.2840254166877	1.2839987348882	2.67E-05
0.75	1.7550546569603	1.7549853685547	6.92E-05
1.00	2.7182818284590	2.7180041217896	2.78E-04

Table 9: Numerical Results for Problem 5.3 for BVM3 using $h = 0.01$

x	Exact	BVM3	Error
0.25	1.0644944589179	1.0644944535753	5.34E-09
0.50	1.2840254166877	1.2840253903544	2.63E-08
0.75	1.7550546569603	1.7550545627475	9.42E-08
1.00	2.7182818284590	2.7182814883158	3.40E-07

Table 10: Numerical Results for Problem 5.3 for BVM4 using $h = 0.05$

x	Exact	BVM4	Error
0.25	1.0644944589179	1.0644960766097	1.62E-06
0.50	1.2840254166877	1.2840299847767	4.57E-06
0.75	1.7550546569603	1.7550684409046	1.37E-05
1.00	2.7182818284590	2.7183249275467	4.31E-04

6 Conclusion

We have been able to derive some mixed boundary value methods via the multistep collocation technique. The methods obtained have been represented as a boundary value methods using the representation of block schemes (21). Properties such as order of convergence and region of absolute stability were highlighted using tables and figures respectively. These BVMs methods were implemented on second order nonlinear ordinary differential equations of Lane-Emden's Type and their results were found to be sufficiently accurate for various values of step length.

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Table 11: Numerical Results for Problem 5.3 for BVM4 using $h = 0.01$

x	Exact	BVM4	Error
0.25	1.0644944589179	1.0644944591979	2.80E-10
0.50	1.2840254166877	1.2840254176494	9.61E-10
0.75	1.7550546569603	1.7550546601576	3.20E-09
1.00	2.7182818284590	2.7182818393239	1.09E-08

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