Fractional Partial Differential Equations Driven by Fractional Gaussian Noise

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Abstract—**Some fraction parabolic partial differential equations driven by fraction Gaussian noise are considered. Initial-value problems for these equations are studied. Some properties of the solutions are given under suitable conditions and with Hurst parameter less than half.**

Keywords:Fractional parabolic stochastic partial differential equations, fractional calculus , fraction Brownian motion

1 Introduction

In this note stochastic partial differential equations of the form:

$$dv(x,t) = dB_H(t) + f(x,t,L_2u(x,t))dt, \qquad (1.1)$$

are considered, where $0 < H < \frac{1}{2}$, $t > 0, x \in R^n$,

$$v(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - L_1 u(x,t), \qquad (1.2)$$

$$L_1 u = \sum_{|q|\le 2m} a_q(x) D^q u,\ L_2 u = \sum_{|q|\le 2m-1} b_q(x) D^q u,$$

$$D^q = D_1^{q_1} ... D_n^{q_n}, D_j = \frac{\partial}{\partial x_j}, 0 < \alpha < 1,$$

R^n is the n-dimensional Euclidean space, $q = (q_1, ..., q_n)$ is an n-dimensional multi index $|q| = q_1 + ... + q_n$, $B_H(t)$ is fractional Brownian motion with Hurst parameter $H \in [0, \frac{1}{2}]$, $B_H(0) = E[B_H(t)] = 0$, for all $t \in R = (-\infty, \infty)$ and

$$E[B_H(t)B_H(s)] = \frac{1}{2}\left\{|s|^{2H} + |t|^{2H} - |s-t|^{2H}\right\},\ s,t, \in R,$$

($E[X]$ denotes the expectation of a random variable X).

If $H = \frac{1}{2}$, then $B_H(t)$ coincides with classical Brownian motion $B(t)$. For $H \ne \frac{1}{2}$, $B_H(t)$ is not a semi martingale, so one cannot use the general theory of stochastic calculus for semi martingale on $B_H(t)$, $(see[1], [2], [3])$.

Denote by K^* the linear operator defined on the set of all step functions to a subset of the set of all square integrable function $L_2[0,T]$, such that:

$$(K_H^*\varphi)(s) = K_H(t,s)\,\varphi(s) + \int_s^T [\varphi(r) - \varphi(s)]\,\frac{\partial K_H(r,s)}{\partial\, r} d\,r,$$

where

$$K_H(t,s) = \left(\Gamma(H+\frac{1}{2})\right)^{-1} (t-s)^{H-\frac{1}{2}} F(a,b,c,z),$$

Γ denotes the gamma function, $a = H - \frac{1}{2}$, $b = \frac{1}{2} - H$, $c = H + \frac{1}{2}$ $z = 1 - \frac{t}{s}$ and F is the Gauss hyper geometric function. The process B_H has an integral representation:

$$B_H(t) = \int_0^t K_H(t,s)\ dB(s), \qquad (1.3)$$

where $B = \{B(t) : t \in [0,t]\}$ is the Brownian motion defined by

$$B(t) = B[(K_H^*)^{-1}(\chi_{[0,1]})], \qquad (1.4)$$

where $(\chi_{[0,1]})$ is the indicator function).
Let $f : R \to R$ such that $E[f^2(B_H(t))] < \infty$, then

$$f(B_H(t)) = E[f(B(T))] + \int_0^t \psi(t,\omega) d\ B_H(t), \qquad (1.5)$$

where

$$\psi(t,\omega) = \left[\frac{\partial}{\partial x} E\left\{f(x + B_H(T-t)\right\}\right]_{x=B_H(T)},$$

see [1].

It is supposed that:
(1) All the coefficients a_q, b_q satisfy a uniform Hölder condition on R^n,
(2)All the coefficients a_q, b_q are bounded on R^n,
(3)The operator $\frac{\partial}{\partial t} - \sum_{|q|=2m} a_q(x) D^q$ is uniformly parabolic on R^n.
This means that

$$(-1)^{m-1} \sum_{|q|=2m} a_q(x) y^q \ge c|y|^{2m}, c > 0,$$

for all $x, y \in R^n, y \ne (0, ..., 0)$, where $y^q = y_1^{q_1} ... y_n^{q_n}$, $|y|^2 = y_1^2 + ... + y_n^2$ and c is a positive constant,

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(4) The function f is continuous on $R^n \times [0,T] \times R$.
It is assumed that

$$u(x,0) = u_0(x), \frac{\partial u(x,0)}{\partial t} = u_1(x), \qquad (1.5)$$

where u_0, u_1 are given sufficiently smooth bounded functions on R^n.
Without loss of generality, we can assume that $u_0(x) = u_1(x) = 0$
In sections 2,3 the solution of the stochastic Cauchy problem (1.1),(1.5) is studied.
The fractional Brownian motion has many different impotent applications with amazing range. This amazing range makes fractional Brownian motion a very interesting object to study, (see [4-7]).

2 Formal Solutions

The solution of equation (1.2) is formally given by:

$$v(x,t) = B_H(t) + \int_0^t f(x,\theta, L_2 u(x,\theta) d\theta, \qquad (2.1)$$

where

$$u(x,t) = \alpha \int_0^t \int_0^\infty \int_{R^n} \zeta_\alpha(\theta) G^*(x,\xi,t,s,\theta) d\xi \; d\theta \; ds, \qquad (2.2)$$

where

$$G^* = \theta(t-s)^{\alpha-1} v(\xi,s) G(x,\xi,(t-s)^\alpha \theta)$$

and G is the fundamental solution of the parabolic equation:

$$\frac{\partial u(x,t)}{\partial t} = \sum_{|q|\le 2m} a_q(x) D^q u(x,t).$$

The function G satisfies the following inequality:

$$|D^q G(x,\xi,t)| \le \gamma t^{c_1} exp[-c_2 \rho], \qquad (2.3)$$

where

$$\rho = |x-\xi|^{m_1} t^{m_2}, m_1 = \frac{2m}{2m-1},$$

$$m_2 = -\frac{1}{2m-1}, c_1 = -\frac{n+|q|}{2m},$$

γ and c_2 are positive constants, [8-10]. The definition of the function $\zeta_\alpha(\theta)$can be found in [8].

3 Fractional Integral Representation

Let I^α_{a+} be the fractional integral operator defined by

$$I^\alpha_{a+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \; \alpha > 0.$$

Denote by $I^\alpha_{a+}(L_2[a,b])$ the image of $L_2[a,b]$ by the operator I^α_{a+}. The operator K_H on $L_2(0,T)$ associated with kernel $K_H(t,s)$ is an isomorphism from

$$L_2[0,T] \text{ onto } I^{H+\frac{1}{2}}_{0+}(L_2[0,T])$$

and it can be expressed in terms of fractional integrals by

$$(K_H \; g)(s) = I^{2H}_{0+} s^{\frac{1}{2}-H} I^{\frac{1}{2}-H}_{0+} s^{H-\frac{1}{2}} g,$$

$$(K_H \; g)(s) = \int_0^t K(t,s) f(s) ds.$$

The inverse operator K_H^{-1} is given by

$$K_H^{-1} g = s^{\frac{1}{2}-H} D^{\frac{1}{2}-H}_{0+} s^{H-\frac{1}{2}} D^{2H}_{0+} \; g,$$

for all $g \in I^{H+\frac{1}{2}}_{0+}(L_2[0,T])$. If g is absolutely continuous, it can be proved that

$$K_H^{-1} g = s^{H-\frac{1}{2}} I^{\frac{1}{2}-H}_{0+} s^{\frac{1}{2}-H} \; g', g' = \frac{dg}{ds}, \qquad (3.1)$$

where D^α_{a+} is the fractional derivative defined by

$$D^\alpha_{a+} \; g(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{g(s)}{(t-s)^\alpha} \; ds,$$

see [3],[6]. A weak solution of equation (2.1) is defined by a couple of adapted processes (B_H, v), for every fixed x on a filtered probability space $(\Omega, F, P, \{F_t : t \in [0,T]\})$, such that
(a) B_H is an F_t - fractional Brownian motion,
(b) v and B_H satisfy (2.1).
Suppose that equation (2.5) has a weak solution. Then using the definitions of the operators K_H, K_H^{-1} and the representation(1.1), one can write equation (2.1) in the form

$$v(x,t) = \int_0^t K_H(t,s) d\tilde{B}(x,s), \qquad (3.2)$$

$$\tilde{B}(x,t) = B(t) + \int_0^t \eta(x,s) ds,$$

$$\eta(x,s) = K_H^{-1} g(x,.)(s),$$

$$g(x,\theta) = \int_0^\theta f(x,s,L_2 u(x,s)) ds.$$

Theorem 3.1. Let $H < \frac{1}{2}$ and v be a weak solution of equation (2.5). If f is a Borel function on $R^n \times [0,T] \times R$ and satisfies the linear growth condition

$$|f(x,t,u)| \le C(1+|u|), \qquad (3.3)$$

for all $u \in R, x \in R^n, t \in [0,T]$, (where C is a positive constant), then $g(x,.) \in I^{H+\frac{1}{2}}_{0+}(L_2[0,T])$.
proof. From (2.1), (2.2), (2.3) and (3.3) it can be deduced that

$$V(t) \le |B_H(t)| + Ct + C_1 \int_0^t V(s) ds,$$

where $C_1 > 0$ is a constant and $V(t) = Sup_x |v(x,t)|$. The last inequality leads to

$$V(t) \le |B_H(t)| + C_1 \int_0^t e^{C_1(t-\theta)} |B_H(\theta)| d\theta + C_2(e^{C_1 t} - 1). \qquad (3.4)$$

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Thus from (3.4) we get

$$\int_0^t V^2(s)ds \le C_3 \int_0^t B_H^2(s)ds + C_4, \qquad (3.5)$$

where $C_2 > 0, C_3 > 0$ are constants. From (3.3) and (3.5), we get

$$\int_0^T g^2(x,\theta)d\theta \le C_4 T + C_5 \int_0^T B_H^2(s)ds \ + C_6, \qquad (3.6)$$

where C_4, C_5 and C_6 are positive constants.
It is easy to see that

$$\begin{aligned} |I_0^{H+\frac{1}{2}}| &= \frac{1}{\Gamma(\alpha)} | \int_0^t (t-s)^{H-\frac{1}{2}} g(x,s)ds| \\ &\le \frac{1}{\Gamma(\alpha)} (\int_0^t (t-s)^{2H-1} ds)^{\frac{1}{2}} (\int_0^t g^2(x,s)ds)^{\frac{1}{2}}.) \end{aligned}$$

The required result follows from (3.5) and (3.6).
It is clear that $K_H^{-1} g(x,.) \in L_2[0,T]$ a.s. if and only if $g(x,.) \in I_{0+}^{H+\frac{1}{2}}(L_2[0,T])$ a.s.
Let $\zeta(x,T) = exp[-\int_0^T \eta(x,s)dB(s) - \frac{1}{2}\int_0^T \eta^2(x,s)ds]$.
If f is bounded, then $\zeta(x,T)$ defines for every $x \in R^n$ a random variable such that the measure $\tilde{P}$ given by $d\tilde{P} = \zeta(x,T)dP$ is a probability measure equivalent to P. If $E\tilde{P}$ denotes the expectation with respect to $\tilde{P}$, then

$$E^{\tilde{P}}[\zeta(x,T) = 1. \qquad (3.7)$$

From (3.1), (3.7), theorem (3.1) and Girsanov theorem, we see that v is an F_t - fractional Brownian motion with Hurst parameter H under the probability $\tilde{P}$, (see [7]).
Lemma 3.1. If f is bounded ,then

$$E^P[\zeta^\alpha(x,T)] \le exp[C|(2\alpha-1)(\alpha-1)|T],$$

where C is a positive constant.
Proof. We can deduce from the results in [7] that

$$E^{\tilde{P}} exp(-2\alpha \int_0^T \eta(x,s)dB(s) - 2\alpha^2 \int_0^T \eta^2(x,s)ds) = 1,$$

for all $\alpha \in R$

Using (3.1),[comp7]
we get

$$\begin{aligned} |\eta(x,s)| &= s^{H-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H} s^{\frac{1}{2}-H} f(x,s,L_2 u(x,s)) \\ &\le \frac{M_1}{\Gamma(\frac{1}{2}-H)} s^{H-\frac{1}{2}} \int_0^s (s-\theta)^{-\frac{1}{2}-H} \theta^{\frac{1}{2}-H} d\theta, \end{aligned}$$

where M_1 is a positive constant,($|f| \le M_1$). Thus

$$E^{\tilde{P}} exp[2|\alpha^2 + \frac{\alpha}{2}| \int_0^T \eta^2(x,s)ds \le exp[2|\alpha^2 + \frac{\alpha}{2}|M_2 T]],$$

where M_2 is a positive constant.
Using the fact that

$$E^P[\zeta^\alpha(x,T)] = E^{\tilde{P}}[\zeta^{\alpha-1}(x,T)],$$

we get the required result.
We can deduce from (3.1) that the operator K_H^{-1} preserves the adaptability property. In other words the process $\eta(x,s)$ is adapted.
Let b be a positive Borel function defined on $[0,T] \times R$ such that the following integral.

$$\| b \|_{q,\gamma} = [\int_R b^q(t,v)dv)^{\frac{\gamma}{q}} dt]^{\frac{1}{\gamma}}$$

exists, where $q > 1, \gamma > \frac{q}{q-H}$.
In this case we say that b belongs to $L_{q,\gamma}$, then by using lemma (3.1) the results of Naulart and Ouknine in [7] can be directly generalized to obtain the following estimations

$$E \int_0^T b(t, v(x,t)dt \le C \| b \|_{q,\gamma},$$

$$E\ exp[\int_0^T b(t,v(x,t)dt \le Q(\| b \|_{q,\gamma})],$$

where C is a positive constant and Q is a real analytic function, [11].
Theorem 3.2. If f is continuous on $R^n \times [0,T] \times R$ and satisfies the Lipschitz condition;

$$|f(x,t,u) - f(x,t,v)| \le C|u-v|$$

for all $x \in R^n, t \in [0,T], u,v \in R$, where C is a positive constant, then there is weak solution v of equation (2.5). Moreover

$$E[v^2(x,t)] < \infty.$$

Proof. We shall use the method of successive approximations.
Set

$$v_{k+1}(x,t) = B_H(t) + \int_0^t f(x,\theta,L_2 u_k(x,\theta))d\theta,$$

$$u_k(x,t) = \alpha \int_0^t \int_0^\infty \int_{R^n} \zeta_\alpha(\theta) G_k^*(x,\xi,t,s,\theta)d\xi\ d\theta\ ds,$$

$$v_0(x,t) = 0,$$

where

$$G_k^* = \theta(t-s)^{\alpha-1} v_k(\xi,s) G(x,\xi,(t-s)^\alpha \theta)$$

Thus

$$|v_{k+1}(x,t) - v_k(x,t)| \le \frac{C^k}{(k-1)!} \int_0^t (t-\theta)^{k-1} |B_H(\theta)| d\theta.$$

it follows that the sequence $\{v_k\}$ uniformly converges with respect to x to a stochastic process v.

This complete the proof of the theorem (see [10-15]).

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