Probabilistic - Optimal Power Flow for Radial Distribution Systems with Wind Generators

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***Abstract*- Radial Distribution Systems (RDS) connect a large number of renewable generators that are inherently uncertain. From being unidirectional power flow systems, RDS now enable bi-directional power flow. Depending upon availability of power from renewables, they receive or feed power to the connected transmission system. RDS optimal power flow (OPF), is an important tool in this new era for utilities, to minimize losses and operate efficiently.**

With large scale integration of wind generators to distribution systems, they must be appropriately represented using probabilistic models capturing their intermittent nature in these OPF algorithms. This paper proposes characterizing the solution of a Probabilistic Optimal Power Flow (P-OPF) for RDS using the Cumulant Method. This method makes it possible to linearly relate the probabilistic parameters of renewables at the optimal solution point to the state of the RDS. To assess the accuracy of the proposed P-OPF Cumulant Method, wind generators and system probabilistic data are incorporated in a 33-bus and 129-bus test system. The results are compared with those of Monte Carlo simulations (MCS). It is shown that the proposed method possesses high degree of accuracy, is significantly faster and more practical than an MCS approach.

NOMENCLATURE

PD, QD	Vector of bus-wise real and reactive power loads
SD	Apparent power loads
T	Tap Setting
ST	Total Apparent Power
PT	Total Real Power
QT	Total Reactive Power
PW	Vector of real power output of WGs
PL	System Real Power Loss
TPL	System Total Real Power Loss
V	Vector of bus voltages magnitudes
NB	Number of buses in the system
SB	The difference between generation and load
QS	Reactive power injected into the ith bus
U	Decision vector in the optimization
Y	problem
	Dependent vector in the optimization
α	problem
β	Independent random input variables
	Output variable created by linear
	combination of "n" independent random
s	input variables
$\Phi(s)$	Complex frequency
$\Psi(s)$	Moment Generating Function
$K_{\beta,m}$	Cumulant Generating Function
	Vector containing the mth order cumulants
$K_{\alpha,m}$	of the system unknown variables
	Vector containing the mth order cumulants
OPF	of the system known variables
H	Optimal Power Flow
$K_{sB,n}$	Hessian of the Lagrangian
	Vector of -order Cumulants for the random
P-OPF	bus power injections
PDF	Probabilistic Optimal Power Flow
RDS	Probability Density Function
WG	Radial Distribution Systems
MCS	Wind Generator
CM	Monte Carlo Simulations
LBIPM	Cumulant Method
KKT	Logarithmic-Barrier Interior Point Method
	Karush–Kuhn–Tucker Conditions

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I. INTRODUCTION

OPF (optimal power flow) is a versatile tool used for electric transmission systems for a variety of purposes. The most common amongst them are: (a) real power OPF where real power output of generators are scheduled such that the total cost of generation is minimized [1], and, (b) reactive power OPF where generators voltages, reactive power compensation and settings of transformers taps are set to route reactive power optimally such that real power transmission losses are the least and all the voltages are within prescribed limits [2]. OPF has not been easily extended to distribution systems as their system Jacobian is ill-conditioned owing to higher R/X ratio of their lines [15]. In the recent past, numerous Jacobian based OPF methods have been researched and published [3]. Today, with a rush to integrate wind generators to electric power systems, largely to distributions systems, distribution systems OPF must account for wind generators as well.

In essence, an OPF for distribution systems must contend with the challenge that it must account for wind generators that are uncertain in their output and their near term forecasts can be best represented by a normal distribution with mean and variance values [16]. Further to understand the effect of probabilistic nature of loads and availability of wind on the OPF solution, such as the optimal values of transformer taps and capacitor settings, it is necessary to propose an efficient probabilistic OPF method that includes the load and generator probabilistic models.

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The ultimate goal is to determine the probability density function of typical variables such as voltage and power flow that form a part of OPF solution.

Uncertainties of the power systems components have been addressed with many researchers by adapting probabilistic techniques in the Power Flow solution in transmission systems since 1960s [4]. Later, the probabilistic methods were applied to the optimal dispatch [5] and for the first time the term P-OPF was used in [6]. However, in contrast to the transmission system case, distribution systems have not been studied to the same extent. In [7] the authors proposed a probabilistic optimal capacitor planning method using Cumulant technique to find the probabilistic information of the size of newly installed capacitor banks in the distribution systems with high penetration of wind generations.

This paper uses the idea given in [7] to propose and construct a distribution system OPF using a set of 3N equations such that the Jacobian is robust [8]. The objective of the OPF is to minimize losses in the distribution system by optimally scheduling all the reactive power sources and ensuring that voltages are within the prescribed limits. Then, it proposes to use Cumulant Method (CM) to directly relate probabilistic values of the loads and output power of wind generators to the optimal settings of the distribution system [9]. This approach has been reported for uncertainty without specific application to wind generators and in transmission system by A. Schellenberg et. al. [10].

This paper is outlined as follows. In sections 2 and 3, the model of the radial distribution system for OPF solution is presented and the Cumulant method is described respectively. Section 4 presents the numerical results of the method as tested on the 33-bus IEEE test system with three wind generators and a 129-bus test system with nine wind generators. Section 5 concludes the paper.

II. System Model and Probabilistic OPF

A. Problem Formulation

This subsection uses the radial distribution system (RDS) model from [8]. Fig. 1 shows a single-line representation of a tree-like distribution systems structure.

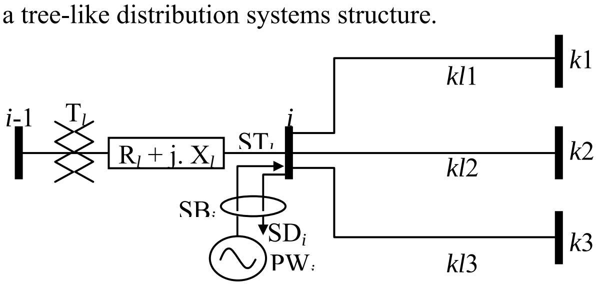

Fig. 1 A tree-like distribution system with wind generator

Consider the ith bus in Fig. 1. It has a wind turbine connected to it that injects only real power equal to PW_i. Its bus load is represented by $SD_i = PD_i + j.QD_i$. The total power injected into this bus is $SB_i = SD_i - PW_i$. It is the difference between generation and load at that bus. Consider the l^{th} line/transformer between buses i-1 and i. The tap setting of this transformer/line is represented by T_l and it has an impedance of $Zl = Rl + j{\cdot}Xl$. The total apparent power reaching the downstream end of this line equals ST_l . The real power loss on this line equals:

$$PL_l = R_l \cdot |ST_l|^2 \cdot V_i^{-2} \qquad (1)$$

The total real power loss in all feeders of the system equals:

$$TPL = \sum_{l=1}^{nl} R_l \cdot |ST_l|^2 \cdot V_i^{-2} \qquad (2)$$

where Vi is the bus voltage magnitude, *nb* is the number of buses in the system and nl is the number of lines/transformers.

In Fig. 1, the complex power balance at the i^{th} can be expressed as:

$$ST_i = SD_i + \left[\sum_{(l,k)=(kl1,k1)}^{(kl3,k3)} Z_l . |ST_l|^2 . V_k^{-2} + ST_k \right] - PW_i - j \cdot QS_i \quad (3)$$

Where QS_i is the reactive power injected into the i^{th} bus. Equation (3) is a complex equation and yields a set of 2(NB-1) equations. Writing the voltage drop equation across line *l* gives:

$$V_i^4 + 2.V_i^2 . \left[PT_l . R_l + QT_l . X_l - \frac{1}{2} . \frac{V_{i-1}^2}{T_l^2} \right] - |Z_l|^2 . |ST_l|^2 = 0 \quad (4)$$

Equations (3) and (4) provide 3(NB-1) equations that completely model a RDS.

B. OPF for Radial Distribution System

The objective of Radial Distribution System OPF is to minimize the total real power loss. By referring to the set of equations (2) – (4), one may construct an optimal power flow formulation for a radial distribution system as below:

Objective Function:

$$\text{Minimize: } TPL = \sum_{l=1}^{nl} R_l \cdot |ST_l|^2 \cdot V_i^{-2} \qquad (5)$$

Constraints:

$$ST_i = SD_i + \left[\sum_{(l,k)=(kl1,k1)}^{(kl3,k3)} Z_l . |ST_l|^2 . V_k^{-2} + ST_k \right] - PW_i - j \cdot QS_i \quad (6)$$

$$V_i^4 + 2.V_i^2 . \left[PT_l . R_l + QT_l . X_l - \frac{1}{2} . \frac{V_{i-1}^2}{T_l^2} \right] - |Z_l|^2 . |ST_l|^2 = 0 \quad (7)$$

$$U_{MIN} < U < U_{MAX} \qquad (8)$$

$$V_{MIN} < V < V_{MAX} \qquad (9)$$

where the decision vector is U = [QS , T] and dependent vector is Y = [V, P, Q]. Equations (6)-(7) are equality constraints which correspond to the complex power balance equation and the voltage drop equation across line *l*, respectively. Equations (8) and (9) limit the control and dependent vectors. The optimization problem described by (5)-(9) is solved by using the Logarithmic-Barrier Interior Point Method (LBIPM) [11].

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C. Optimal Solution

The formulation (5)-(9) is solved using the Lower Bound Interior Point Method [11]. This yields the optimal solution of decision and dependent vectors U and Y. In addition, the Hessian of the Lagrangian formed from the optimization formulation (5)-(9) is evaluated H (U, Y, λ). It provides a linear relation between incremental changes of dependent vector in terms of the decision vector.

III. Cumulant Technique and P-OPF

In probability theory, Cumulants and moments are two sets of quantities of a random variable which are mathematically equivalent. However, in some cases preference is to use Cumulants due to their simplicity over using moments [12]. In this section some properties of Cumulants used to adapt the Cumulant Technique to the radial distribution system P-OPF are presented.

Consider a linear combination of 'n' independent random input variables α used to create a new random output variable β as follows [10]:

$$\beta = c_1 . \alpha_1 + c_2 . \alpha_2 + c_3 . \alpha_3 + \ldots + c_n . \alpha_n \qquad (10)$$

where ci is the i^{th} coefficient in the linear combination. The above expansion can be written in terms of the moment generation function of random variable β, i.e., $\Phi_\beta(s)$, as

$$\Phi_\beta(s) \;\; = E\left[e^{s\beta}\right] = E\left[e^{s(c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n)}\right] \qquad (11)$$

where s is the Laplace operator. Assuming that α_1, α_2, α_3, …, α_n are independent, the above relationship can be written as

$$\Phi_\beta(s) \;\; = E[e^{s\beta}] = E\left[e^{sc_1\alpha_1}\right] \cdot E\left[e^{sc_2\alpha_2}\right] \cdots E\left[e^{sc_n\alpha_n}\right] =$$
$$\Phi_{\alpha 1}(sc_1) . \Phi_{\alpha 2}(sc_2) \ldots \Phi_{\alpha n}(sc_n) \qquad (12)$$

The Cumulant generating function $\Psi_x(s)$ can be written in terms of the moment generating function Φ_x as [12]

$$\Psi_\beta(s) = \ln(\Phi_\beta(s)). \qquad (13)$$

By taking the natural logarithm at both sides of the (12) and using (13), (12) is written in the terms of cumulant generating function as below:

$$\Psi_\beta(s) = \Psi_{\alpha 1}(sc_1) + \Psi_{\alpha 2}(sc_2) + \ldots + \Psi_{\alpha n}(sc_n). \qquad (14)$$

To obtain the different orders of cumulants, we can set $s = 0$ to compute the different order derivatives of the cumulant generating function. A general equation for the m^{th} order cumulant of Ψ is

$$\Psi_\beta^{(m)}(0) = c_1^m . \Psi_{\alpha 1}^{(m)}(0) + c_2^m \Psi_{\alpha 2}^{(m)}(0) + \ldots + c_n^m \Psi_{\alpha n}^{(m)}(0) \qquad (15)$$

$$K_{\beta,m} = c_1^m . K_{\alpha 1,m} + c_2^m . K_{\alpha 2,m}(0) + \ldots + c_n^m K_{\alpha n,m}(0). \qquad (16)$$

where $K_{\beta,m}$ is a vector containing the mth order cumulants of the system unknown variables and $K_{\alpha,m}$ is a vector containing the m^{th} order cumulants of the random bus generation and loading.

A. Adaptation to P-OPF

By applying Newton method to the Lagrangian function L (U, Y, λ) for (5)-(9), the following system is obtained:

$$\nabla L(U,Y,\lambda) + H(U,Y,\lambda) \times \begin{bmatrix} \Delta U \\ \Delta Y \\ \Delta\lambda \end{bmatrix} = 0 \qquad (17)$$

where $\nabla L(U,Y,\lambda)$ and H(U,Y, λ) are the gradient and the Hessian of the Lagrangian respectively. Rearranging (16) and replacing $\nabla L(U,Y,\lambda)$ with a vector of change in the bus power injections for uncertain wind power, ΔSB, the vector of changes can be linearly mapped with ΔSB by using the inverse of the Hessian:

$$\begin{bmatrix} \Delta U \\ \Delta Y \\ \Delta\lambda \end{bmatrix} = -H(U,Y,\lambda)^{-1} \times \begin{bmatrix} \nabla_{U,Y} L(U,Y,\lambda) \\ \Delta SB \end{bmatrix} \qquad (18)$$

By replacing ΔSB with a vector containing the n^{th} order cumulants of loads and generation, the Cumulants of system variables, ΔSB can obtain using the inverse of the Hessian as follow:

$$K_{(U,Y,\lambda),n} = \left(-H^{-1}\right)^{(n)} \times \begin{bmatrix} 0 \\ K_{SB,n} \end{bmatrix} \qquad (19)$$

where $K_{(U,Y,\lambda),n}$ is a vector of n^{th} -order cumulants for the optimal settings of the distribution system and $K_{SB,n}$ is a vector of n^{th} -order Cumulants for the random bus power injections. Consequently, the Hessian contains the constant multipliers. Once the cumulants of the random variables of the OPF solution are computed from the input random variables, PDFs are recreated by using Gram-Charlier/Edgeworth Expansion theory [13].

IV. Numerical Results

This section provides the results based on applying the Cumulant method to the 33-bus and 129-bus test systems. In both systems, the loads and the power output of wind generators are considered Gaussian random variables with the mean values set to the nominal bus loading and mean capacity of wind generators respectively. The standard deviation is such that the 99% confidence interval is equal to ±%15 of the nominal loading value. In order to show the efficiency and accuracy of the Cumulant method, the results have been compared with MCS with 5000 samples.

A. 33-Bus Test System Case Study

The method is firstly applied to the 33-bus, 32-branch IEEE test system described in [14]. The system, however, is modified to accommodate the probabilistic data of the loads and wind generators. The modified system and loads data can be found in the Appendix. Three wind turbines are connected to buses 3, 17 and 32 with a mean capacity equal to 500 kW each. The results of both mean and standard deviation values obtained by comparing with those of 5000 sample points MCS, are discussed as follows.

1) *Mean Values:* From Table I it can be seen that the percentage errors of the voltage mean values are very small with the maximum value equal to 0.0077% which occurs at bus 18. The maximum percentage error of the real power and reactive power mean values is equal to 3.44% and 7.97% at bus 29 and 5 respectively. The corresponding maximum error for capacitor value

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occurs at bus 18 and is as low as 0.1901%. These results show a small difference between two methods in systems variables mean values which can be seen more clearly in Fig. 2.

2) *Variance Value:* The maximum percentage errors for the system variables variance values are presented in Table II. These values for the voltage, active power, reactive power and capacitor value variance are equal to 1.55%, 1.91%, 2.13% and 1.85% which occur at buses 24, 9, 11 and 30 respectively. These small error values for the variance of the systems variables are shown in Fig. 3.

In summary, the percentage errors of the mean and variance values for the system variables are well below 8% which implies a close match between two methods. Also, it is worth noting that largest errors (8%) occur for reactive power and voltage magnitude variables due to inherent nonlinearity. This nonlinearity for voltage and reactive power usually happens in the buses with capacitor banks connected to them.

TABLE I
MAXIMUM ERROR OF MEAN VALUES OF THE SYSTEM VARIABLES

	Bus No.	CM (per unit)	MCS (per unit)	Error (%)
Voltage	18	0.98	0.98	0.0077
Active Power	29	0.03	0.03	3.44
Reactive Power	5	-0.0045	-0.0049	7.97
Capacitor Value	18	0.15	0.15	0.19

TABLE II
MAXIMUM ERROR OF VARIANCE VALUES OF THE SYSTEM VARIABLES

	Bus No.	CM	MCS	Error (%)
Voltage	24	0.0013	0.0012	1.55
Active Power	9	0.079	0.0811	1.91
Reactive Power	11	0.0062	0.0061	2.13
Capacitor Value	30	0.091	0.093	1.85

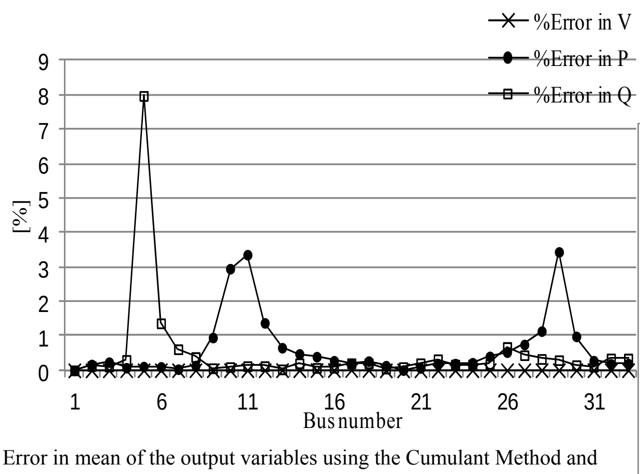

Fig. 2 Error in mean of the output variables using the Cumulant Method and MCS technique - 33-Bus System

The analysis using Cumulant method is captured in graphs of Fig. 4 and 5 wherein mean capacitor and bus voltage magnitude values are shown with potential spread using corresponding 3σ values. This analysis and graphing can be rapidly completed using the proposed method.

Table III presents the time comparison between Cumulant method and Monte Carlo Simulation technique. It is evident that to identify this spread in values of capacitor settings and voltages at buses, though being important information, is difficult to obtain using the conventional Monte Carlo Simulation technique due to long solution time. However, using the proposed cumulant method, using a few additional steps such as computation of the Hessian of the Lagrangian, yields the variance values of the optimized control (capacitor) and dependent (voltage) variables.

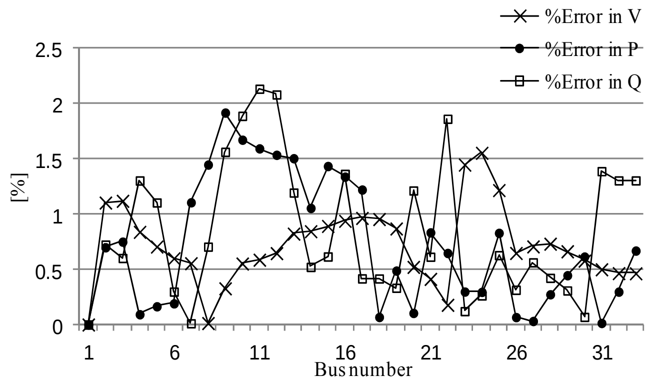

Fig. 3 Error in Standard Deviation of the output variables using the Cumulant Method and MCS technique– 33-Bus System

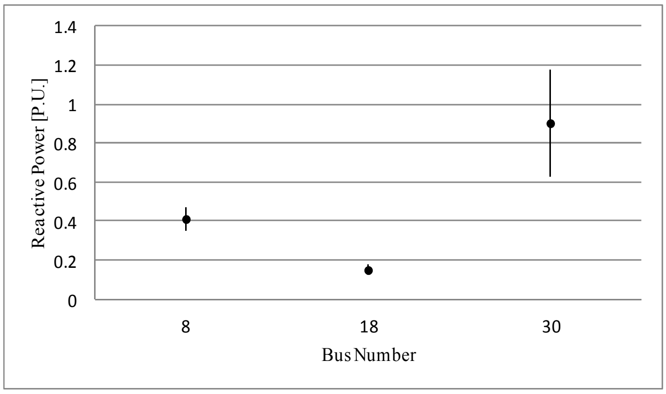

Fig. 4 Mean capacitor values are shown with potential spread using corresponding 3σ values.

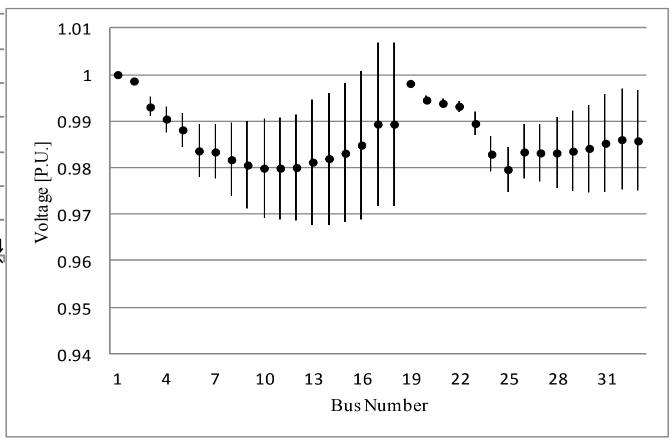

Fig. 5 Mean bus voltage magnitude values are shown with potential spread using corresponding 3σ values.

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This is an important benefit for the RDS operator as s/he would like to how far the capacitor values and voltage solutions might travel from the mean forecasted and anticipate/plan to avoid violations.

TABLE III
TIME COMPARISON BETWEEN CUMULANT METHOD AND MONTE CARLO SIMULATION TECHNIQUE

Execution Time (seconds)	CM	MCS
System #1	4.44 seconds	2888.05 seconds.

B. The 129-Bus Test System Case Study

To show the accuracy and efficiency of the Cumulant method, it also has been tested on a large system of 129-bus with nine wind generators each having a mean capacity equal to 500 kW. The results have been compared with 5000 sample points of MCS.

Table IV shows the mean absolute percentage error of mean and standard deviation of the problem variables.

TABLE IV
MEAN ABSOLUTE PERCENTAGE ERROR OF MEAN AND STANDARD DEVIATION OF THE PROBLEM VARIABLES

Variable	Mean	Standard Deviation
Voltage	0.0025	1.4285
Real Power	0.2206	0.8180
Reactive Power	0.4580	1.2150
Capacitor Size	0.0589	1.2537

TABLE V
TIME COMPARISON BETWEEN CUMULANT METHOD AND MONTE CARLO SIMULATION TECHNIQUE

Execution Time (seconds)	CM	MCS
System #2	12.79 seconds	6504.87 seconds.

The method works well for small and reasonably sized systems.

V. CONCLUSION

This paper describes the Cumulant method for probabilistic optimization of radial distribution systems. Random variables used in distribution system are the output of wind generators and bus loads. Both are modeled using Gaussian distribution function. While a Logarithmic Barrier Interior Point Method (LBIPM) based solver is used to solve the probabilistic nonlinear optimization problem, the cumulants of the system variables are easily computed using the inverse of the Hessian of the Lagrangian function.

The method was implemented and tested on a 33-bus and 129-bus IEEE test system. In order to illustrate efficiency and accuracy of the cumulant method, the resultant data using cumulant method are benchmarked with those obtained from Monte Carlo Simulation Technique with 5000 samples. The errors were found well below 8%. An execution time comparison demonstrates superiority of the Cumulant method. Less computational burden and complexity makes the proposed method very practical and advantageous.

This method can be advantageously used by RDS operators to know possible swings in optimal capacitor settings and voltage solution giving them additional insight into operation of the system and anticipate potential operational challenges.

APPENDIX

Table VI presents the mean values used for the load demands, and Table VII presents the feeder data.

TABLE VI
33 BUS RDS - MEAN VALUE OF THE LOADS

Bus Number	Mean	
	Real Power (kW)	Reactive Power (kW)
1	0	0.0
2	0.1	0.06
3	-0.5	0.0
4	0.12	0.08
5	0.06	0.03
6	0.2	0.1
7	0.2	0.1
8	0.2	0.1
9	0.06	0.02
10	0.06	0.02
11	0.045	0.03
12	0.06	0.035
13	0.06	0.035
14	0.06	0.04
15	0.06	0.01
16	0.09	0.04
17	-0.5	0.0
18	0.09	0.04
19	0.09	0.04
20	0.09	0.04
21	0.09	0.04
22	0.09	0.04
23	0.09	0.05
24	0.42	0.2
25	0.42	0.2
26	0.06	0.025
27	0.06	0.025
28	0.06	0.02
29	0.12	0.07
30	0.2	0.6
31	0.15	0.07
32	-0.5	0.1
33	0.06	0.04

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TABLE VII
33 BUS RDS - LINE DATA

From Bus	To Bus	R (Ω)	X (Ω)	Rating (MVA)	System Voltage (kV)
1	2	0.0922	0.047	100	12.66
2	3	0.493	0.2511	100	12.66
3	4	0.3662	0.1864	100	12.66
4	5	0.3811	0.1941	100	12.66
5	6	0.819	0.707	100	12.66
6	7	0.1872	0.6188	100	12.66
7	8	1.7114	1.2351	100	12.66
8	9	1.03	0.74	100	12.66
9	10	1.044	0.74	100	12.66
10	11	0.1966	0.065	100	12.66
11	12	0.3744	0.1238	100	12.66
12	13	1.468	1.155	100	12.66
13	14	0.5416	0.7129	100	12.66
14	15	0.591	0.526	100	12.66
15	16	0.7463	0.545	100	12.66
16	17	1.289	1.721	100	12.66
17	18	0.732	0.574	100	12.66
18	19	0.164	0.1565	100	12.66
19	20	1.5042	1.3554	100	12.66
20	21	0.4095	0.4784	100	12.66
21	22	0.7089	0.9373	100	12.66
22	23	0.4512	0.3083	100	12.66
23	24	0.898	0.7091	100	12.66
24	25	0.896	0.7011	100	12.66
25	26	0.203	0.1034	100	12.66
26	27	0.2842	0.1447	100	12.66
27	28	1.059	0.9337	100	12.66
28	29	0.8042	0.7006	100	12.66
29	30	0.5075	0.2585	100	12.66
30	31	0.9744	0.963	100	12.66
31	32	0.3105	0.3619	100	12.66
32	33	0.341	0.5302	100	12.66

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