On the Stokes Flow over Ellipsoidal Type Bodies

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***Abstract*—In the paper the Stokes flow over the ellipsoidal type bodies in a pipe is considered. The velocity of the flow is described by the axisymmetric Stokes system for the low Reynolds number with the appropriate boundary conditions. Effective solutions for different cases are obtained. The shear stresses and velocity are calculated. The graphics of the velocity profile and shear stresses are constructed.**

***Index Terms*—boundary value problems, Stokes flow**

I. Introduction

THE study of a viscous fluid flow in a pipe-like domains is important and has a lot of applications in engineering [1]-[10].

We consider the Stokes flow over ellipsoidal type body (or system of bodies) which moves in the pipe when the pressure fall is a constant. This process is described by the Stokes axisymmetric equation with the appropriate boundary conditions.

Our purpose is to calculate the velocity of the fluid and shear stresses.

II. Statement of the problem

Let us consider the fluid flow with a low Reynolds number over some system of ellipsoidal bodies with the same axis of symmetry in the cylindrical channel of the diameter of $2h\,(\mathrm{h}>0)$. We admit, that this system of ellipsoidal bodies move at a constant speed $\vec{V}_0(V_x^0, V_y^0, V_z^0)$. We study the axisymmetric case, when the axis of symmetry is $0x$ and consider the axisymmetric Stokes system for the components of velocity with the equation of continuity in a moving coordinate system

$$\Delta V_x + \frac{1}{r}\frac{\partial V_x}{\partial r} = C, \tag{1}$$

$$\Delta V_r + \frac{1}{r}\frac{\partial V_r}{\partial r} - \frac{V_r}{r^2} = 0, \tag{2}$$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_r}{\partial r} + \frac{V_r}{r} = 0, \tag{3}$$

where V_x, V_r are the components of the velocity, $r = \sqrt{y^2+z^2}$, $C, (C>0)$, is a definite constant.

The boundary conditions are given by

$$V_x\Big|_{r=\pm h} = -V^0, \quad V_x\Big|_S = 0, \tag{4}$$

$$V_r\Big|_{r=\pm h} = 0, \quad V_r\Big|_S = 0, \tag{5}$$

where $V^0 = V_x^0, (V_x^0 > 0)$, S is a contour of the moving bodies.

The shear stresses τ_1, τ_2 will be defined from the formulas [1]-[6]

$$\tau_1 = \rho\nu\frac{\partial V_x}{\partial r}, \quad \tau_2 = \rho\nu\left(\frac{\partial V_r}{\partial r} - \frac{V_r}{r}\right), \tag{6}$$

where ρ is a density of the fluid, ν -is a viscosity .

Our purpose is to define the bounded solutions of the system (1), (2), (3), (4), (5) and shear stresses. The contour S and the width of a channel $2h$ will be defined according to the solutions.

III. Solution of the problem

By direct verification we obtain that the solution of the system (1), (2), (3), (4), (5), are given by the pair of functions

$$V_x = q\left\{\frac{x+a}{\left(\sqrt{(x+a)^2+r^2}\right)^3} - \frac{x-a}{\left(\sqrt{(x-a)^2+r^2}\right)^3}\right\} + Cr^2 - C_0, \tag{7}$$

$$V_r = q\left\{\frac{r}{\left(\sqrt{(x+a)^2+r^2}\right)^3} - \frac{r}{\left(\sqrt{(x-a)^2+r^2}\right)^3}\right\}, \tag{8}$$

where q, a and C_0 are some positive parameters. For the different q, a and C_0 we obtain the fluid flow over ellipsoidal type bodies in the pipe of the different width.

By the formulas (6) the shear stresses corresponding to (7), (8) will be given by

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$$\tau_1 = -3r\rho\nu q\left\{\frac{(x+a)}{\left(\sqrt{(x+a)^2+r^2}\right)^5}-\frac{(x-a)}{\left(\sqrt{(x-a)^2+r^2}\right)^5}\right\} + 2r\rho\nu C,$$

$$\tau_2 = -3r^2\rho\nu q\left\{\frac{1}{\left(\sqrt{(x+a)^2+r^2}\right)^5}-\frac{1}{\left(\sqrt{(x-a)^2+r^2}\right)^5}\right\}.$$

Also the pair of functions

$$V_{xm} = \frac{\partial^m}{\partial x^m}\left(V_x\right)+C_1r^2-C_0 = q\sum_{k=0}^{n}\left(\frac{a_k r^{2k}(x+a)^{\alpha}}{\sqrt{(x+a)^2+r^2}^{\,2k+1+2n+2\alpha}}-\frac{a_k r^{2k}(x-a)^{\alpha}}{\sqrt{(x-a)^2+r^2}^{\,2k+1+2n+2\alpha}}\right) + Cr^2 - C_0, \tag{9}$$

$$V_{rm} = \frac{\partial^m}{\partial x^m}\left(V_r\right) = q\sum_{k=1}^{n}\left(\frac{b_k r^{2k-1}(x+a)^{\beta}}{\sqrt{(x+a)^2+r^2}^{\,2k+1+2n}}-\frac{b_k r^{2k-1}(x-a)^{\beta}}{\sqrt{(x-a)^2+r^2}^{\,2k+1+2n}}\right), \tag{10}$$

will be the solution of the system (1), (2), (3), (4), (5) . The corresponding shear stresses will be calculated according to the formula (6).

In the formulas (9), (10), $a_k, b_k, \alpha, \beta, n$ are the definite constants ($\alpha = 0,\ \beta = 1$ for odd m, $\alpha = 1, \beta = 0$ for even m). Also the linear combinations of (7) and (9), (8) and (10) will be the solutions of the system (1), (2), (3), (4), (5).

The contour of the bodies will be given by the formula:

1. For (7), (8)

$$\left\{\frac{x+a}{\left(\sqrt{(x+a)^2+r^2}\right)^3}-\frac{x-a}{\left(\sqrt{(x-a)^2+r^2}\right)^3}\right\}+C_1r^2-A=0,$$

$$\left\{\frac{r}{\left(\sqrt{(x+a)^2+r^2}\right)^3}-\frac{r}{\left(\sqrt{(x-a)^2+r^2}\right)^3}\right\}\approx 0.$$

2. For (9), (10)

$$\sum_{k=0}^{n}\left(\frac{a_k r^{2k}(x+a)^{\alpha}}{\sqrt{(x+a)^2+r^2}^{\,2k+1+2n+2\alpha}}-\frac{a_k r^{2k}(x-a)^{\alpha}}{\sqrt{(x-a)^2+r^2}^{\,2k+1+2n+2\alpha}}\right) + Cr^2 - C_0 = 0,$$

$$\sum_{k=1}^{n}\left(\frac{b_k r^{2k-1}(x+a)^{\beta}}{\sqrt{(x+a)^2+r^2}^{\,2k+1+2n}}-\frac{b_k r^{2k-1}(x-a)^{\beta}}{\sqrt{(x-a)^2+r^2}^{\,2k+1+2n}}\right)\approx 0.$$

Below some examples are given by using Maple. In the Fig. 1 and Fig. 5 the lateral cross-section of the pipe with ellipsoidal bodies inside is shown for the different parameter a in the formulas. The corresponding graphics of velocity modulus and shear stresses are constructed (Fig. 2, Fig. 3, Fig. 4, Fig. 6, Fig. 7, Fig. 8).

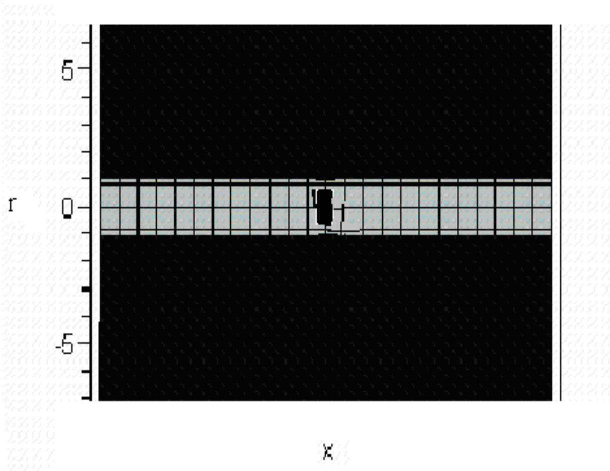

Fig. 1. The lateral cross-section of the pipe of width $2h = 2$, in case of $a = 1/5; C = C_0 = 1; q = 1/10$.

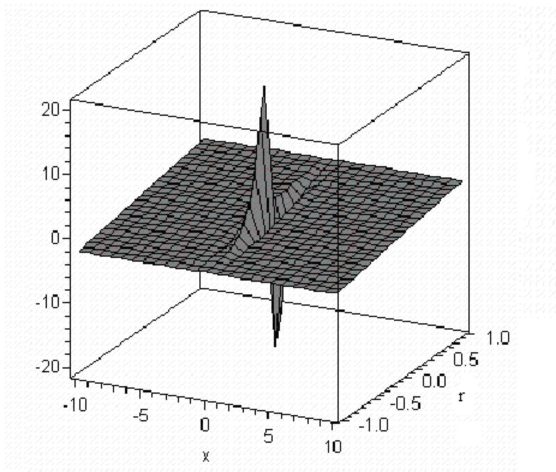

Fig. 2. The shear stress τ_1 corresponding to the case $a = 1/5$; $C = C_0 = 1$; $q = 1/10$.

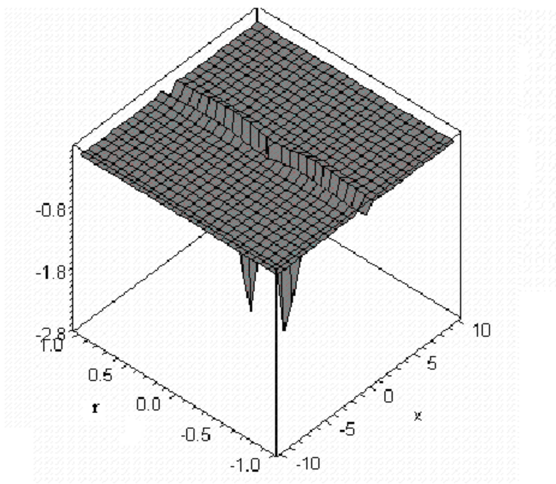

Fig. 3. The shear stress τ_2 corresponding to the case $a = 1/5$; $C = C_0 = 1; q = 1/10$.

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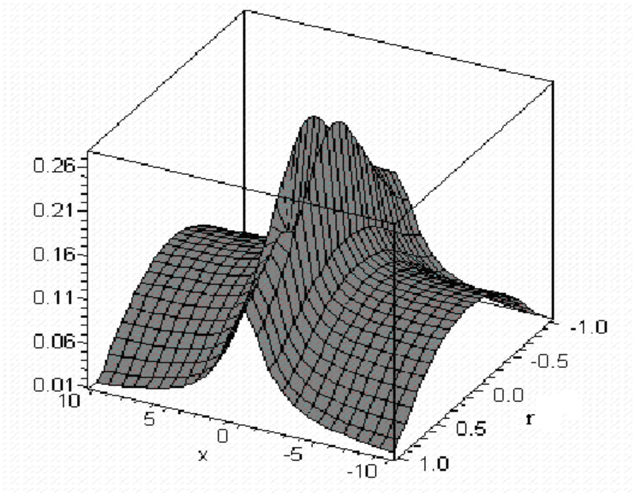

Fig. 4. The profile of the velocity corresponding to the case $a = 1/5$; $C = C_0 = 1; q = 1/10$.

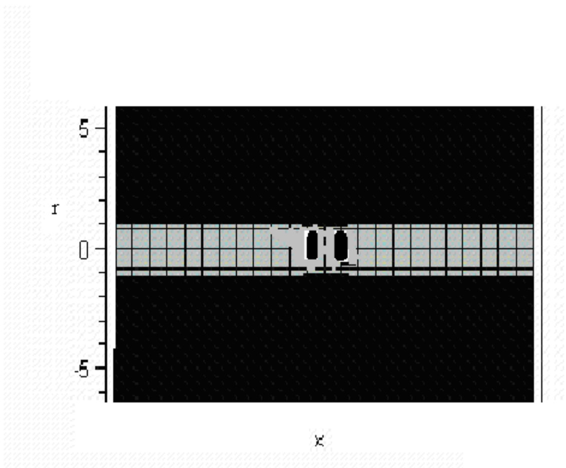

Fig. 5. The lateral cross-section of the pipe of width $2h = 2$, in case of $a = 1; C = C_0 = 1; q = 1/10$.

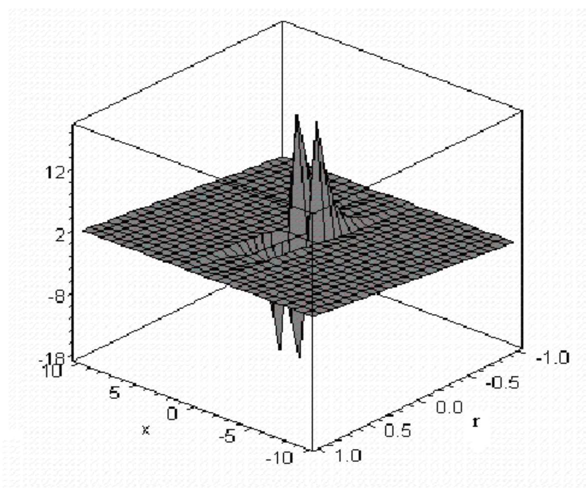

Fig. 6. The shear stress τ_1 corresponding to the case $a = 1$; $C = C_0 = 1; q = 1/10$.

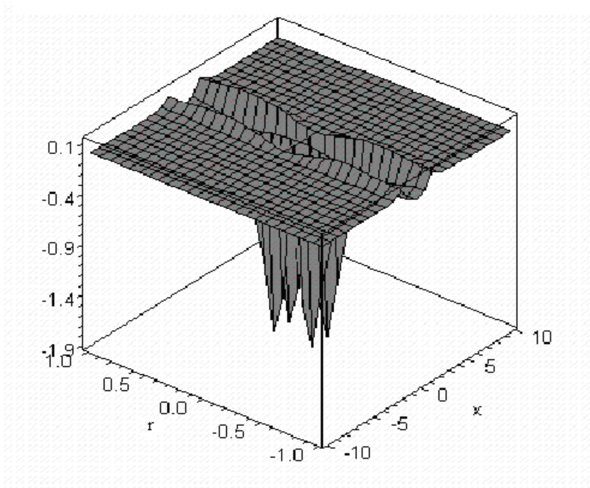

Fig. 7. The shear stress τ_2 corresponding to the case $a = 1$; $C = C_0 = 1; q = 1/10$.

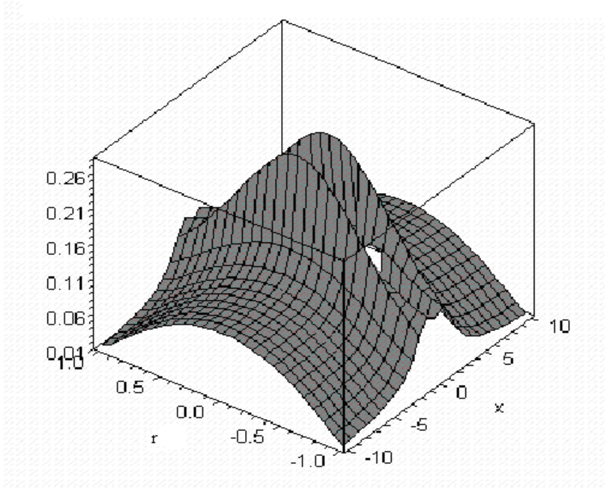

Fig. 8. The profile of the velocity corresponding to the case $a = 1$; $C = C_0 = 1; q = 1/10$.

REFERENCES

[1] G. G. Stokes, "On the steady motion of incompressible fluids". *Transactions of the Cambridge Philosophical Society* **7**: 439–453, Mathematical and Physical Papers, Cambridge University Press, 1880.

[2] H. Lamb, *Hydrodynamics* (6th ed.). Cambridge University Press, 1994.

[3] L. M. Milne-Thompson, Theoretical Hydrodynamics. (5-th ed) Macmillan, 1968.

[4] G. K. Bachelor, An Introduction to Fluid Dynamics, Cambridge Univ. Press, 1967.

[5] L. D. Landay, E. M. Lifshitz, Fluid Mechanics, Course of Theoretical Physics, 6, Pergamon Press, 1987.

[6] R. Temam, Navier-Stokes Equations, Theory and numerical Analysis, AMS Chelsea, 2001.

[7] B. J. Kirby, *Micro- and Nanoscale Fluid Mechanics: Transport in Microfluidic Devices*. Cambridge University Press, 2010.

[8] H. Ockendon, & J. R. Ockendon, *Viscous Flow*, Cambridge University Press, 1995.

[9] A. Chwang and T. Wu, "Hydromechanics of low-Reynolds-number flow. Part 2. Singularity method for Stokes flows". *J. Fluid Mech. 62*(6), part 4, 1974.

[10] S. Kim, S. J. Karrila, *Microhydrodynamics: Principles and Selected Applications*, Dover, 2005.