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On the Solutions of First and Second Order Nonlinear Initial Value Problems

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Abstract—**In this paper, we show a simple new method of solving first and second order nonlinear differential equations in the form**

$$a_1y'(t) + a_0y(t) = f(t, y), \quad t \geq 0, \tag{1}$$

$$y(0) = \alpha_0, \tag{2}$$

and

$$ay''(t) + by'(t) + cy(t) = g(t, y, y'), \quad t \geq 0, \tag{3}$$

$$y(0) = \beta_0, y'(0) = \beta_1, \tag{4}$$

where $a_0, a_1, a, b, c, \alpha_0, \beta_0, \beta_1$ **are given constants, and** $f(t, y)$ **and** $g(t, y, y')$ **are given nonlinear functions. By substituting the given constants and functions, then using an iteration method, solutions are easily obtained. Moreover, some examples are shown exact solutions.**

Keywords: initial value problem, successive approximation, differential equation, volterra integral equation, laplace transform

1 Introduction

Finding exact solutions of nonlinear initial value problems (IVPs) is a goal for mathematicians, engineers, and scientists, and it plays an important role in real world applications. In recent years, first and second order nonlinear IVPs were considered by many authors. For instance, [1-2] used the Adomian decomposition method (ADM) to solve nonlinear differential equations such as Duffing-Vanderpole equation, [3-5] solved nonlinear IVPs by the Laplace Adomian decomposition method (LADM), [6-7] obtained approximate solutions by the method of differential transforms (DTM), and the variational iteration methods (VIM) were used by many authors [8-9].

Although ADM, LADM and DTM are effective and famous methods for solving nonlinear equations, there are limitations for using. For example, ADM, LADM and DTM require infinite series to get solutions which sometimes it is difficult to investigate closed form solution from infinite series. And we have to use some analytical methods to complete those schemes by inverse transformations of infinite series in order to obtain solutions. Furthermore, VIM needs Lagrange multiplier before using iteration formula.

Recently, Sita [10] introduced an alternative method for finding solutions of nonlinear higher order IVPs by converting IVP into Volterra integral equation. Then by the use of the successive approximation, a high accuracy solution will be obtained.

However, in the work of [10], it is introduced in a general form and requires the inverse Laplace transforms of some functions in main results of [10]. Motivated by the above-mentioned work, in this work, a new method needs no transformation or linearization or Lagrange multiplier, and it shows some formulas for solving the first order IVP (1)-(2) and the second order IVP (3)-(4). A solution is easily obtained by just having some basic knowledge of integrations and substituting given constants into the formula. Then an iterative method is needed to seek an approximate or exact solution. Finally, some examples establish that this alternative method is very simple and high performance.

2 Basic idea of Laplace transforms

In this section, we are going to review some basic idea of the Laplace transforms to use as important tools of our main results.

Definition 2.1 The Laplace transform of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $F(s)$, defined by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt.$$

Definition 2.2 Let the Laplace transform of $f(t)$ is $\mathcal{L}\{f(t)\} = F(s)$, then we say that the inverse Laplace transform of $F(s)$ is $f(t)$. Or it is defined by

$$\mathcal{L}^{-1}\{F(s)\} = f(t).$$

Property 2.1 Let ω, c_1 and c_2 be given constants.
(P1) Inverse Laplace transforms of some functions

$$\mathcal{L}^{-1}\{\frac{1}{s^2+\omega^2}\} = \frac{\sin \omega t}{\omega}, \quad \mathcal{L}^{-1}\{\frac{s}{s^2+\omega^2}\} = \cos \omega t,$$

$$\mathcal{L}^{-1}\{\frac{1}{s^2-\omega^2}\} = \frac{\sinh \omega t}{\omega}, \quad \mathcal{L}^{-1}\{\frac{s}{s^2-\omega^2}\} = \cosh \omega t.$$

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(P2) Linearity property

$$\mathcal{L}\{c_1f(t)+c_2g(t)\}=c_1\mathcal{L}\{f(t)\}+c_2\mathcal{L}\{g(t)\},$$

$$\mathcal{L}^{-1}\{c_1F(s)+c_2G(s)\}=c_1\mathcal{L}^{-1}\{F(s)\}+c_2\mathcal{L}^{-1}\{G(s)\}.$$

(P3) Shifting property

$$\mathcal{L}\{e^{\omega t}f(t)\}=F(s-\omega),$$

$$\mathcal{L}^{-1}\{F(s-\omega)\}=e^{\omega t}f(t).$$

(P4) Laplace transform of derivatives

$$\mathcal{L}\{f'(t)\}=s\mathcal{L}\{f(t)\}-f(0),$$

$$\mathcal{L}\{f''(t)\}=s^2\mathcal{L}\{f(t)\}-sf(0)-f'(0).$$

Definition 2.3 The convolution of $f(t)$ and $g(t)$ is defined as

$$f(t)*g(t)=\int_0^t f(x)g(t-x)dx.$$

Property 2.2 Properties of the convolution
(P5) $f*g=g*f$,
(P6) $\mathcal{L}^{-1}\{F(s).G(s)\}=f*g$.
For more details about the method of Laplace transform, see [11].

3 Main Results

3.1 Lemma

Lemma 3.1 *Suppose that* $a\epsilon R^+$,*and* $b,c,m,p\epsilon R$. *Let* $\lambda=b^2-4ac$, *then the inverse Laplace transforms satisfy the followings:*

(H1) for $\lambda>0$,

$$\mathcal{L}^{-1}\{\frac{ms+p}{as^2+bs+c}\}=k_1e^{\alpha_1t}+k_2e^{\alpha_2t},$$

where

$$k_1=\frac{(\sqrt{\lambda}-b)m+2ap}{2a\sqrt{\lambda}},k_2=\frac{(\sqrt{\lambda}+b)m-2ap}{2a\sqrt{\lambda}},$$

$$\alpha_1=\frac{\sqrt{\lambda}-b}{2a},\alpha_2=\frac{-\sqrt{\lambda}-b}{2a},$$

(H2) for $\lambda<0$,

$$\mathcal{L}^{-1}\{\frac{ms+p}{as^2+bs+c}\}=e^{\alpha t}(c_1\cos\omega t+c_2\sin\omega t),$$

where

$$\alpha=\frac{-b}{2a},c_1=\frac{m}{a},c_2=\frac{2ap-bm}{a\sqrt{|\lambda|}},\omega=\frac{\sqrt{|\lambda|}}{2a}.$$

Proof Since

$$\begin{aligned}\frac{ms+p}{as^2+bs+c}&=\frac{m(s+\frac{b}{2a})+(p-\frac{bm}{2a})}{a(s+\frac{b}{2a})^2+(c-\frac{b^2}{4a})}\\&=\frac{m}{a}[\frac{(s+\frac{b}{2a})}{(s+\frac{b}{2a})^2+(\frac{4ac-b^2}{4a^2})}]\\&\quad+(\frac{2ap-bm}{2a^2})(\frac{1}{(s+\frac{b}{2a})^2+(\frac{4ac-b^2}{4a^2})}).\end{aligned}$$

By taking inverse Laplace transforms and using (P2), we obtain

$$\begin{aligned}\mathcal{L}^{-1}\{\frac{ms+p}{as^2+bs+c}\}&=\mathcal{L}^{-1}\{\frac{m}{a}[\frac{(s+\frac{b}{2a})}{(s+\frac{b}{2a})^2+(\frac{4ac-b^2}{4a^2})}]\}\\&\quad+\mathcal{L}^{-1}\{(\frac{2ap-bm}{2a^2})\frac{1}{(s+\frac{b}{2a})^2+(\frac{4ac-b^2}{4a^2})}\}.\end{aligned}$$

By using (P3), we have

$$\begin{aligned}\mathcal{L}^{-1}\{\frac{ms+p}{as^2+bs+c}\}&=e^{\frac{-bt}{2a}}[\frac{m}{a}\mathcal{L}^{-1}\{\frac{s}{s^2+(\frac{4ac-b^2}{4a^2})}\}]\\&\quad+e^{\frac{-bt}{2a}}[(\frac{2ap-bm}{2a^2})\mathcal{L}^{-1}\{\frac{1}{s^2+(\frac{4ac-b^2}{4a^2})}\}].\end{aligned}$$

Let $\lambda=b^2-4ac>0$ and by using (P1), then

$$\mathcal{L}^{-1}\{\frac{ms+p}{as^2+bs+c}\}=e^{\frac{-bt}{2a}}[\frac{m}{a}\cosh\frac{\sqrt{\lambda}}{2a}t+(\frac{2ap-bm}{a\sqrt{\lambda}})\sinh\frac{\sqrt{\lambda}}{2a}t].$$

Since $\cosh x=\frac{e^x+e^{-x}}{2}$ and $\sinh x=\frac{e^x-e^{-x}}{2}$, we have that

$$\begin{aligned}\mathcal{L}^{-1}\{\frac{ms+p}{as^2+bs+c}\}&=(\frac{(\sqrt{\lambda}-b)m+2ap}{2a\sqrt{\lambda}})e^{\frac{\sqrt{\lambda}-b}{2a}t}\\&\quad+(\frac{(\sqrt{\lambda}+b)m-2ap}{2a\sqrt{\lambda}})e^{\frac{\sqrt{-\lambda}-b}{2a}t}.\end{aligned}$$

On the other hand, by setting $\lambda<0$, we have that

$$\begin{aligned}\mathcal{L}^{-1}\{\frac{ms+p}{as^2+bs+c}\}&=e^{\frac{-b}{2a}t}[\frac{m}{a}\cos\frac{\sqrt{|\lambda|}}{2a}t\\&\quad+(\frac{2ap-bm}{a\sqrt{|\lambda|}})\sin\frac{\sqrt{|\lambda|}}{2a}t].\end{aligned}$$

The proof is completed.

3.2 Theorems

Theorem 3.1 *Suppose that* $a_1\epsilon R^+$,*and* $a_0,\alpha_0\epsilon R$, *and* $f:[o,\infty)\times R\to R$. *A nonlinear initial value problem*

$$a_1y'(t)+a_0y(t)=f(t,y),\quad t\geq 0,$$

$$y(0)=\alpha_0,$$

is equal to the Volterra integral equation as

$$y(t)=\alpha_0e^{\frac{-a_0}{a_1}t}+\frac{1}{a_1}e^{\frac{-a_0}{a_1}t}\int_0^t e^{\frac{a_0}{a_1}x}f(x,y(x))dx.$$

Theorem 3.2 *A nonlinear initial value problem (1)- (2) has a closed form solution if*

$$y(t) = \lim_{i\to\infty} y_i(t),$$

where

$$y_{i+1}(t) = \alpha_0 e^{\frac{-a_0}{a_1}t} + \frac{1}{a_1} e^{\frac{-a_0}{a_1}t} \int_0^t e^{\frac{a_0}{a_1}x} f(x, y_i(x))dx,$$

for $i = 0, 1, 2, ...,$ *and* $y_0(t) = \alpha_0 e^{\frac{-a_0}{a_1}t}$.

Theorem 3.3 *Suppose that* $a\epsilon R^+$,*and* $b, c, \beta_0, \beta_1 \epsilon R$, *and* $g : [o, \infty) \times R^2 \to R$. *Given* $\lambda = b^2 - 4ac$, *a nonlinear initial value problem*

$$ay''(t) + by'(t) + cy(t) = g(t, y, y'), \quad t \geq 0,$$

$$y(0) = \beta_0, y'(0) = \beta_1,$$

is equal to the Volterra integral equation as followings: Let $m = a\beta_0$ *and* $p = a\beta_1 + b\beta_0$,
(A1) for $\lambda > 0$,

$$y(t) = k_1 e^{\alpha_1 t} + k_2 e^{\alpha_2 t} + \frac{1}{\sqrt{\lambda}} \int_0^t (e^{\alpha_1(t-x)} - e^{\alpha_2(t-x)}) g(x, y(x), y'(x))dx,$$

where

$$k_1 = \frac{(\sqrt{\lambda} - b)m + 2ap}{2a\sqrt{\lambda}}, k_2 = \frac{(\sqrt{\lambda} + b)m - 2ap}{2a\sqrt{\lambda}},$$

$$\alpha_1 = \frac{\sqrt{\lambda} - b}{2a}, \alpha_2 = \frac{-\sqrt{\lambda} - b}{2a},$$

(A2) for $\lambda < 0$,

$$y(t) = e^{\alpha t}(c_1 \cos\omega t + c_2 \sin\omega t) + \frac{2}{\sqrt{|\lambda|}} \int_0^t e^{\alpha(t-x)} \sin\omega(t-x) g(x, y(x), y'(x))dx,$$

where
$\alpha = \frac{-b}{2a}$, $c_1 = \frac{m}{a}$, $c_2 = \frac{2ap-bm}{a\sqrt{|\lambda|}}$ *and* $\omega = \frac{\sqrt{|\lambda|}}{2a}$.

Theorem 3.4 *A nonlinear initial value problem (3)- (4) has a closed form solution if*

$$y(t) = \lim_{i\to\infty} y_i(t),$$

where $i = 0, 1, 2, ...,$
(B1) for $\lambda > 0$, *and* $y_0(t) = k_1 e^{\alpha_1 t} + k_2 e^{\alpha_2 t}$,

$$y_{i+1}(t) = k_1 e^{\alpha_1 t} + k_2 e^{\alpha_2 t} + \frac{1}{\sqrt{\lambda}} \int_0^t (e^{\alpha_1(t-x)} - e^{\alpha_2(t-x)}) g(x, y_i(x), y_i'(x))dx,$$

(B2) for $\lambda < 0$, *and* $y_0(t) = e^{\alpha t}(c_1 \cos\omega t + c_2 \sin\omega t)$,

$$y_{i+1}(t) = e^{\alpha t}(c_1 \cos\omega t + c_2 \sin\omega t) + \frac{2}{\sqrt{|\lambda|}} \int_0^t e^{\alpha(t-x)} \sin\omega(t-x) g(x, y_i(x), y_i'(x))dx.$$

3.3 Proofs of Theorems

A proof of Theorem 3.1
Consider the differential equation (1), and by taking Laplace transforms and using (P2) and (P4), we have

$$a_1\mathcal{L}\{y'(t)\} + a_0\mathcal{L}\{y(t)\} = \mathcal{L}\{f(t, y)\},$$

$$a_1(s\mathcal{L}\{y\} - y(0)) + a_0\mathcal{L}\{y\} = \mathcal{L}\{f(t, y)\}.$$

From the initial condition (2), we get that

$$(a_1 s + a_0)\mathcal{L}\{y\} = a_1\alpha_0 + \mathcal{L}\{f(t, y)\},$$

i.e.,

$$\mathcal{L}\{y\} = \frac{a_1\alpha_0}{a_1 s + a_0} + \frac{\mathcal{L}\{f\}}{a_1 s + a_0}.$$

By taking inverse Laplace transforms and using (P2) and (P6), we have

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{\frac{a_1\alpha_0}{a_1 s + a_0}\} + \mathcal{L}^{-1}\{\frac{\mathcal{L}\{f\}}{a_1 s + a_0}\} \\ &= \alpha_0\mathcal{L}^{-1}\{\frac{1}{s + \frac{a_0}{a_1}}\} + f * \mathcal{L}^{-1}\{\frac{1}{a_1 s + a_0}\} \\ &= \alpha_0 e^{\frac{-a_0}{a_1}t} + f * \frac{1}{a_1} e^{\frac{-a_0}{a_1}t}. \end{aligned}$$

This completes the proof by the definition of convolution and (P5).

A proof of Theorem 3.3
Consider the differential equation (3), and by taking Laplace transforms and using (P2) and (P4), we have that

$$a\mathcal{L}\{y''(t)\} + b\mathcal{L}\{y'(t)\} + c\mathcal{L}\{y(t)\} = \mathcal{L}\{g(t, y, y')\},$$

$$a(s^2\mathcal{L}\{y\} - sy(0) - y'(0)) + b(s\mathcal{L}\{y\} - y(0)) + c\mathcal{L}\{y\} = \mathcal{L}\{g(t, y, y')\}.$$

From the initial condition (4), we get that

$$\mathcal{L}\{y\} = \frac{ms + p}{as^2 + bs + c} + \frac{\mathcal{L}\{g(t, y, y')\}}{as^2 + bs + c},$$

where $m = a\beta_0$ and $p = a\beta_1 + b\beta_0$.
By taking the inverse Laplace transforms and using (P2), we have

$$y(t) = \mathcal{L}^{-1}\{\frac{ms + p}{as^2 + bs + c}\} + \mathcal{L}^{-1}\{\frac{\mathcal{L}\{g\}}{as^2 + bs + c}\}. \quad (5)$$

Consider the second term of Eq.(5), and using (P5), (P6) and (H1), and by setting $m = 0$, $p = 1$ and $\lambda > 0$, we have

$$\begin{aligned} \mathcal{L}^{-1}\{\frac{L\{g\}}{as^2 + bs + c}\} &= g * \mathcal{L}^{-1}\{\frac{1}{as^2 + bs + c}\} \\ &= g * \frac{1}{\sqrt{\lambda}}(e^{\alpha_1 t} - e^{\alpha_2 t}). \end{aligned}$$

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Moreover, by using (H2) with $m = 0$ and $p = 1$, for $\lambda < 0$, we have

$$\begin{aligned}\mathcal{L}^{-1}\{\frac{L\{g\}}{as^2+bs+c}\} &= g * \mathcal{L}^{-1}\{\frac{1}{as^2+bs+c}\} \\ &= g * \frac{2}{\sqrt{|\lambda|}} e^{\alpha t}\sin\omega t.\end{aligned}$$

Finally, we complete the proof by Definition 2.3 and Lemma 3.1.

Proofs of Theorems 3.2 and 3.4
The proofs are completed by the successive approximate theorem in [10].

4 Examples

Example 1

$$y' - y = -2ty; y(0) = 1, \tag{6}$$

exact solution : $y(t) = e^{t-t^2}$.

By Theorem 3.1, we convert Eq.(6) to the integral equation as

$$y = e^t - 2e^t \int_0^t e^{-x} x y(x) dx.$$

To find a solution, we use Theorem 3.2. Thus

$$y_{i+1} = e^t - 2e^t \int_0^t e^{-x} x y_i(x) dx,$$

where $i = 0, 1, 2, ...$ and $y_0(t) = e^t$.
So we get

$$y_1(t) = e^t - 2e^t \int_0^t e^{-x} x e^x dx = e^t(1 - t^2),$$

$$y_2(t) = e^t - 2e^t \int_0^t e^{-x} x e^x (1 - x^2) dx = e^t(1 - t^2 + \frac{t^4}{2}),$$

$$y_3(t) = e^t(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!}),$$

$$y_4(t) = e^t(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \frac{t^8}{4!}),$$

$$\vdots$$

$$y_i(t) = e^t(1 - t^2 + \frac{t^4}{2!} + ... + (-1)^i \frac{t^{2i}}{i!}),$$

$$\lim_{i\to\infty} y_i(t) = y(t) = e^t e^{-t^2} = e^{t-t^2}.$$

This is an exact solution.

Example 2

$$y'' - 2y' = y^2 - e^{4t}; y(0) = 1, y'(0) = 2, \tag{7}$$

exact solution : $y(t) = e^{2t}$.

We use Theorem 3.3 to convert Eq.(7) to the integral equation as

$$y = e^{2t} + \frac{1}{2}\int_0^t (e^{2(t-x)} - 1)(y^2(x) - e^{4x}) dx.$$

And we use Theorem 3.4 to propose an iteration formula. So we get

$$y_{i+1}(t) = e^{2t} + \frac{1}{2}\int_0^t (e^{2(t-x)} - 1)(y_i^2(x) - e^{4x}) dx,$$

where $i = 0, 1, 2, ...$ and $y_0(t) = e^{2t}$.
Hence, we see that

$$y_1(t) = e^{2t} + \frac{1}{2}\int_0^t (e^{2(t-x)} - 1)(e^{4x} - e^{4x}) dx = e^{2t},$$

$$y_2(t) = e^{2t} + \frac{1}{2}\int_0^t (e^{2(t-x)} - 1)(e^{4x} - e^{4x}) dx = e^{2t},$$

$$\vdots$$

$$y_i(t) = e^{2t},$$

$$\lim_{i\to\infty} y_i(t) = y(t) = e^{2t}.$$

Example 3 Vanderpole Oscillator Problem

$$y'' + y' + y + y^2 y' = 2\cos t - \cos^3 t; y(0) = 0, y'(0) = 1, \tag{8}$$

exact solution : $y(t) = \sin t$.

From Eq.(8), we have

$$y'' + y = 2\cos t - \cos^3 t - (1 + y^2) y'. \tag{9}$$

By Theorem 3.3, we convert Eq.(9) into the integral equation and use Theorem 3.4. Then we have

$$\begin{aligned} y_{i+1} &= \sin t \\ &+ \int_0^t \sin(t-x)(2\cos x - \cos^3 x - (1 + y_i^2(x)) y_i'(x)) dx,\end{aligned}$$

where $i = 0, 1, 2, ...$ and $y_0(t) = \sin t$.
Thus we have

$$y_i(t) = \sin t,$$

and

$$\lim_{i\to\infty} y_i(t) = y(t) = \sin t.$$

5 Conclusions

A new simple way for solving a nonlinear IVP was proposed. It provided a formula of a solution by just using a basic knowledge of integration. Some examples were given to show the effectiveness of a new method.

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