Proceedings of the World Congress on Engineering 2013 Vol III,
WCE 2013, July 3 - 5, 2013, London, U.K.

Efficient NLP Approaches for Direct Computation of Collapse Load Bounds of Structures under Interval Inputs

S. Tangaramvong, F. Tin-Loi, and W. Gao

Abstract—**The paper proposes a pair of novel and robust nonlinear programming (NLP) approaches to furnish within a single step a maximum collapse load bound in one case and a minimum collapse load bound in the other of a rigid perfectly plastic structure subject to interval applied loads and/or material capacities. The schemes adopt mathematical programming techniques to reformulate each of the two interval formulations as a standard NLP problem that can be efficiently solved by any available NLP code. Such techniques do not require combinatorial search procedures as typically suggested in the literature. The accuracy of the computed results is validated, to some extent, through comparisons with Monte Carlo simulations.**

Index Terms—**convex model, interval analysis, limit analysis, nonlinear programming, plasticity**

I. Introduction

It is a well-known that uncertainties exist in the specification of properties for structural design purposes. Obvious instances are those caused by material capacities, externally applied load regimes and structural geometry. Ignoring such effects can lead to either overestimated or underestimated strength estimation of the structure under consideration. There is also considerable concern regarding the applicability of existing analysis methods developed on the ubiquitous assumption that uncertain data can be approximated using their average, maximum or minimum values [1].

A better approach, in the absence of full probabilistic information, is to incorporate the influence of uncertainties through the concept of interval analysis or so-called "convex model" [2]. The uncertain parameters are presumed to be nonprobabilistic, and are deemed to vary independently within specified ranges. The method can provide engineers with simple yet fruitful preliminary information through the calculation of approximate bounds on some key quantities, before any probabilistic schemes are implemented, if possible.

The paper focuses on the use of interval analysis in computing collapse load bounds for ductile perfectly plastic structures. We propose a pair of novel nonlinear programming (NLP) approaches to efficiently compute, within a single step, a maximum collapse load limit in one case and a minimum collapse load in the other of the structure subject to interval applied loads and/or interval material capacities. The governing formulations, based on the classical deterministic plastic limit analysis relations, lead to linear programming problems with interval coefficients (LPICs) [3]. To solve the computationally challenging LPICs, we reformulate them as NLP problems that can be processed directly using any available NLP code. The main feature of this approach is the elimination of any combinatorial search procedures, as commonly suggested in the literature. One of a number of successfully solved examples is provided to illustrate applicability of the proposed methods. The accuracy of the computed results is validated, to some extent, using computationally expensive, Monte Carlo simulated runs.

A word regarding notation is in order. Vectors and matrix quantities are indicated in bold. A real vector $\mathbf{x}$ of size m is indicated by $\mathbf{x} \in \Re^m$ and a real $m \times n$ matrix $\mathbf{A}$ by $\mathbf{A} \in \Re^{m \times n}$. For brevity, a vector of functions $\mathbf{f}(\mathbf{x}) : \Re^m \to \Re^n$ is simply written as $\mathbf{f} \in \Re^n$.

II. Deterministic plastic limit analysis

Classical plastic limit analysis determines, within a single step, the maximum load capacity at collapse of structures made of rigid perfectly plastic materials, and subject to deterministic input parameters (see e.g. [4]). The approach is based on the two well-known lower (static) and upper (kinematic) bound theorems that can be mathematically cast as a pair of standard linear programming (LP) problems for a discrete model constructed within a standard finite element framework and a piecewise linearized yield surface approximation [5-7].

In terms of standard notation and description [5-7], the deterministic static limit analysis problem in variables $(\alpha, \mathbf{Q})$ can be written as follows:

$$\begin{aligned} &\max_{\alpha,\mathbf{Q}} \quad \alpha \\ &\text{subject to } \mathbf{f}\alpha - \mathbf{C}^{\mathrm{T}}\mathbf{Q} = -\mathbf{f}_d, \\ &\qquad\quad \mathbf{N}^{\mathrm{T}}\mathbf{Q} \le \mathbf{r}. \end{aligned} \tag{1}$$

Manuscript received February 28, 2013; revised April 08, 2013. This work was supported by an Australian Research Council.

S. Tangaramvong is with School of Civil and Environmental Engineering, The University of New South Wales, Sydney, NSW 2052, Australia (corresponding author—phone: +61 2 9385 6223; fax: +61 2 9385 6139; e-mail: stangaramvong@gmail.com).

F. Tin-Loi is with School of Civil and Environmental Engineering, The University of New South Wales, Sydney, NSW 2052, Australia (e-mail: f.tinloi@unsw.edu.au).

W. Gao is with School of Civil and Environmental Engineering, The University of New South Wales, Sydney, NSW 2052, Australia (e-mail: w.gao@unsw.edu.au).

It is assumed that the adopted structural system has been suitably discretized into n elements, d degrees of freedom, m generalized stresses/strain rates and y yield functions.

The static limit analysis formulation given in (1) is a standard LP problem. It maximizes the statically admissible load α subjected to equilibrium and plasticity constraints. More explicitly, the linear equilibrium in (1.1) between generalized stresses $\mathbf{Q} \in \Re^m$ and externally applied forces $\mathbf{f}\alpha + \mathbf{f}_d \in \Re^d$ is described through a constant compatibility matrix $\mathbf{C} \in \Re^{m \times d}$, where α denotes a load (scalar) multiplier, $\mathbf{f}$ and $\mathbf{f}_d$ are basic variable and fixed applied forces, respectively. The plasticity conditions are expressed in (1.2) through a normality matrix $\mathbf{N} \in \Re^{m \times y}$ and yield capacities $\mathbf{r} \in \Re^y$, where the matrix $\mathbf{N}$ collects all unit normal directions to the yield surfaces [5-7].

The dual or kinematic problem in variables ($\dot{\mathbf{u}}, \dot{\boldsymbol{\lambda}}$) is as follows (this, in fact, can be constructed without knowledge of the mechanical problem using the well-known duality property of LP theory):

$$\begin{aligned} &\min_{\dot{\mathbf{u}},\dot{\boldsymbol{\lambda}}} && -\mathbf{f}_d^{\mathrm{T}}\dot{\mathbf{u}} + \mathbf{r}^{\mathrm{T}}\dot{\boldsymbol{\lambda}} \\ &\text{subject to} && \mathbf{f}^{\mathrm{T}}\dot{\mathbf{u}} = 1, \\ &&& -\mathbf{C}\dot{\mathbf{u}} + \mathbf{N}\dot{\boldsymbol{\lambda}} = \mathbf{0}, \\ &&& \dot{\boldsymbol{\lambda}} \geq \mathbf{0}. \end{aligned} \tag{2}$$

For this pair of LP problems (1) and (2) the "duality" property that defines an identical value of the two respective optimal objective functions (if they exist), namely

$$\alpha = -\mathbf{f}_d^{\mathrm{T}}\dot{\mathbf{u}} + \mathbf{r}^{\mathrm{T}}\dot{\boldsymbol{\lambda}}. \tag{3}$$

In our specific problem, the optimal solutions of the two LP problems (2) and (3) represent the collapse loads.

Mechanically, the LP problem (2) describes the kinematic limit analysis counterpart which minimizes the plasticity dissipation subject to constraints expressing positivity of the external work, and linear compatibility between nodal displacement rates $\dot{\mathbf{u}} \in \Re^d$ and plastic multiplier rates $\dot{\boldsymbol{\lambda}} \in \Re^y$. The optimal kinematic variables $\dot{\mathbf{u}}$ and $\dot{\boldsymbol{\lambda}}$ can be extracted as by-products from Lagrange multipliers associated with the solutions of the primal LP problem (1).

III. Interval Plastic Limit Analysis

The main focus of this section, in fact of the paper, is to incorporate the effects of uncertain inputs into the classical plastic limit analysis framework. More explicitly, we consider the uncertainties caused by externally applied forces $\mathbf{f}$ and $\mathbf{f}_d$ and/or plastic capacities $\mathbf{r}$ that are still deterministic but their values can independently vary within specified interval ranges [2]. In essence, we define $\underline{\mathbf{r}} \leq \mathbf{r} \leq \overline{\mathbf{r}}$, $\underline{\mathbf{f}} \leq \mathbf{f} \leq \overline{\mathbf{f}}$ and $\underline{\mathbf{f}}_d \leq \mathbf{f} \leq \overline{\mathbf{f}}_d$ as their respective (componentwise) convex bounds, namely $\underline{r}^l \leq r^l \leq \overline{r}^l$ for $l = 1,\ldots,y$; $\underline{f}^k \leq f^k \leq \overline{f}^k$ and $\underline{f}_d^k \leq f_d^k \leq \overline{f}_d^k$ for $k = 1,\ldots,d$; where overbar and underscore symbols denote upper and lower bound values associated with the interval parameters.

The proposed approaches, if successfully processed, provide estimates of two extreme, namely maximum $\overline{\alpha}$ and minimum $\underline{\alpha}$, collapse load limit solutions and associated parameters for the interval quantities.

The formulation adopted to capture the maximum collapse load $\overline{\alpha}$ can be straightforwardly formed from the deterministic static limit analysis counterpart given in (1) as follows:

$$\begin{aligned} &\max_{\alpha,\mathbf{Q},\mathbf{f},\mathbf{f}_d} && \alpha \\ &\text{subject to} && \underline{\mathbf{r}} \leq \mathbf{r} \leq \overline{\mathbf{r}}, \\ &&& \underline{\mathbf{f}} \leq \mathbf{f} \leq \overline{\mathbf{f}}, \\ &&& \underline{\mathbf{f}}_d \leq \mathbf{f}_d \leq \overline{\mathbf{f}}_d, \\ &&& \max_{\alpha,\mathbf{Q}} \quad \alpha \\ &&& \text{subject to} \quad \mathbf{f}\alpha - \mathbf{C}^{\mathrm{T}}\mathbf{Q} = -\mathbf{f}_d \\ &&& \qquad\qquad \mathbf{N}^{\mathrm{T}}\mathbf{Q} \leq \mathbf{r}. \end{aligned} \tag{4}$$

Problem (4) is a bi-level optimization problem consisting of an inner-level maximization and an outer-level maximization. In particular, an inner-level problem maximizes α subjected to the constraints describing static limit analysis problem (1), whilst an outer-level problem maximizes the load among all legal collapse load solutions influenced by the interval forces and plastic capacities. Successfully processing (4) yields the "most favorable" solution of the interval static maximization [3].

An alternative formulation adopted to find $\overline{\alpha}$ and founded on the deterministic kinematic counterpart in (2) reads:

$$\begin{aligned} &\max_{\dot{\mathbf{u}},\dot{\boldsymbol{\lambda}},\mathbf{f},\mathbf{f}_d} && -\mathbf{f}_d^{\mathrm{T}}\dot{\mathbf{u}} + \mathbf{r}^{\mathrm{T}}\dot{\boldsymbol{\lambda}} \\ &\text{subject to} && \underline{\mathbf{r}} \leq \mathbf{r} \leq \overline{\mathbf{r}}, \\ &&& \underline{\mathbf{f}} \leq \mathbf{f} \leq \overline{\mathbf{f}}, \\ &&& \underline{\mathbf{f}}_d \leq \mathbf{f}_d \leq \overline{\mathbf{f}}_d, \\ &&& \min_{\dot{\mathbf{u}},\dot{\boldsymbol{\lambda}}} \quad -\mathbf{f}_d^{\mathrm{T}}\dot{\mathbf{u}} + \mathbf{r}^{\mathrm{T}}\dot{\boldsymbol{\lambda}} \\ &&& \text{subject to} \quad \mathbf{f}^{\mathrm{T}}\dot{\mathbf{u}} = 1 \\ &&& \qquad\qquad -\mathbf{C}\dot{\mathbf{u}} + \mathbf{N}\dot{\boldsymbol{\lambda}} = \mathbf{0} \\ &&& \qquad\qquad \dot{\boldsymbol{\lambda}} \geq \mathbf{0}. \end{aligned} \tag{5}$$

The bi-level problem (5) minimizes plastic dissipation subject to the constraints expressing the kinematic limit analysis problem (2) in an inner-level optimization, and at the same time maximizes the dissipation among all possible plastic dissipation solutions under the interval parameters in an outer-level optimization. Successfully solving (5) captures the "least favorable" solution of an interval kinematic minimization [3] and hence the maximum collapse load limit $\overline{\alpha}$.

ISBN: 978-988-19252-9-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

An important property underpinning the two interval limit analysis problems (4) and (5) is that the optimal maximum collapse load bound $\overline{\alpha}$ obtained from the most favorable solution of an interval static maximization problem is identical to that from the least favorable solution of the interval kinematic minimization problem.

The well-known duality property of LP theory implies that for any specified values of applied forces and plastic capacities an identical collapse load limit can be obtained from the optimal solutions of the two deterministic static and kinematic problems (1) and (2). With a similar range of interval inputs, the two interval counterparts (4) and (5) contain a similar feasible set of all possible collapse load solutions generated by their inner-level optimization, such that the outer-level optimization selects a maximum. Therefore, the above statement holds.

To compute the minimum collapse load $\underline{\alpha}$, the interval formulation formed from the deterministic static LP problem (1) takes the form of the following bi-level optimization problem:

$$\begin{aligned}
&\min_{\alpha,\mathbf{Q},\mathbf{f},\mathbf{f}_d} \quad \alpha \\
&\text{subject to} \quad \underline{\mathbf{r}} \le \mathbf{r} \le \overline{\mathbf{r}}, \\
&\qquad \underline{\mathbf{f}} \le \mathbf{f} \le \overline{\mathbf{f}}, \\
&\qquad \underline{\mathbf{f}}_d \le \mathbf{f}_d \le \overline{\mathbf{f}}_d, \\
&\qquad \max_{\alpha,\mathbf{Q}} \quad \alpha \\
&\qquad \text{subject to} \quad \mathbf{f}\alpha - \mathbf{C}^{\mathrm{T}}\mathbf{Q} = -\mathbf{f}_d \\
&\qquad\qquad \mathbf{N}^{\mathrm{T}}\mathbf{Q} \le \mathbf{r}.
\end{aligned} \tag{6}$$

Clearly, (6) calculates in an outer-level optimization a minimum load among all possible collapse load solutions generated by an inner-level maximization problem. The optimal result of (6) represents the least favorable solution of the interval static maximization [3].

The interval kinematic formulation founded on the direct deterministic counterpart (2) to capture $\underline{\alpha}$ is written as:

$$\begin{aligned}
&\min_{\dot{\mathbf{u}},\dot{\boldsymbol{\lambda}},\mathbf{f},\mathbf{f}_d} \quad -\mathbf{f}_d^{\mathrm{T}}\dot{\mathbf{u}} + \mathbf{r}^{\mathrm{T}}\dot{\boldsymbol{\lambda}} \\
&\text{subject to} \quad \underline{\mathbf{r}} \le \mathbf{r} \le \overline{\mathbf{r}}, \\
&\qquad \underline{\mathbf{f}} \le \mathbf{f} \le \overline{\mathbf{f}}, \\
&\qquad \underline{\mathbf{f}}_d \le \mathbf{f}_d \le \overline{\mathbf{f}}_d, \\
&\qquad \min_{\dot{\mathbf{u}},\dot{\boldsymbol{\lambda}}} \quad -\mathbf{f}_d^{\mathrm{T}}\dot{\mathbf{u}} + \mathbf{r}^{\mathrm{T}}\dot{\boldsymbol{\lambda}} \\
&\qquad \text{subject to} \quad \mathbf{f}^{\mathrm{T}}\dot{\mathbf{u}} = 1 \\
&\qquad\qquad -\mathbf{C}\dot{\mathbf{u}} + \mathbf{N}\dot{\boldsymbol{\lambda}} = \mathbf{0} \\
&\qquad\qquad \dot{\boldsymbol{\lambda}} \ge \mathbf{0}.
\end{aligned} \tag{7}$$

The optimal result of (7) represents the most favorable solution of the interval kinematic minimization [3].

Similarly, the duality property of LP theory stated for the maximum collapse load case in both (4) and (5) is valid for the minimum collapse load case in (6) and (7). Therefore, the optimal minimum collapse limit computed from the least favorable solution of the interval static maximization problem is identical to that from the most favorable solution of the interval kinematic minimization problem.

For a compact representation of the interval problems (4) to (7), we substitute in the two LP problems (1) and (2) the direct deterministic counterparts **r**, **f**, and $\mathbf{f}_d$ with the new respective interval symbols, i.e. $[\underline{\mathbf{r}},\overline{\mathbf{r}}]$, $[\underline{\mathbf{f}},\overline{\mathbf{f}}]$ and $[\underline{\mathbf{f}}_d,\overline{\mathbf{f}}_d]$, satisfying their specified interval ranges, i.e. $\underline{\mathbf{r}} \le [\underline{\mathbf{r}},\overline{\mathbf{r}}] \le \overline{\mathbf{r}}$, $\underline{\mathbf{f}} \le [\underline{\mathbf{f}},\overline{\mathbf{f}}] \le \overline{\mathbf{f}}$ and $\underline{\mathbf{f}}_d \le [\underline{\mathbf{f}}_d,\overline{\mathbf{f}}_d] \le \overline{\mathbf{f}}_d$. This leads to two generic interval limit analysis formulations, namely an interval static limit analysis problem in variables (α,**Q**):

$$\begin{aligned}
&\max_{\alpha,\mathbf{Q}} \quad \alpha \\
&\text{subject to} \quad [\underline{\mathbf{f}},\overline{\mathbf{f}}]\alpha - \mathbf{C}^{\mathrm{T}}\mathbf{Q} = -[\underline{\mathbf{f}}_d,\overline{\mathbf{f}}_d], \\
&\qquad \mathbf{N}^{\mathrm{T}}\mathbf{Q} \le [\underline{\mathbf{r}},\overline{\mathbf{r}}],
\end{aligned} \tag{8}$$

and an interval kinematic limit analysis problem in variables ($\dot{\mathbf{u}}$, $\dot{\boldsymbol{\lambda}}$):

$$\begin{aligned}
&\min_{\dot{\mathbf{u}},\dot{\boldsymbol{\lambda}}} \quad -[\underline{\mathbf{f}}_d,\overline{\mathbf{f}}_d]^{\mathrm{T}}\dot{\mathbf{u}} + [\underline{\mathbf{r}},\overline{\mathbf{r}}]^{\mathrm{T}}\dot{\boldsymbol{\lambda}} \\
&\text{subject to} \quad [\underline{\mathbf{f}},\overline{\mathbf{f}}]^{\mathrm{T}}\dot{\mathbf{u}} = 1, \\
&\qquad -\mathbf{C}\dot{\mathbf{u}} + \mathbf{N}\dot{\boldsymbol{\lambda}} = \mathbf{0}, \\
&\qquad \dot{\boldsymbol{\lambda}} \ge \mathbf{0}.
\end{aligned} \tag{9}$$

Problems (8) and (9) are known in the mathematical programming literature as LPICs [3], where the general term "interval coefficients" describes the interval data in the right-hand-side and/or coefficients of variables.

The study of LPICs and their solution algorithms have increasingly attracted a number of recent research work (see e.g. [2,3,8,9]). An LPIC embodies the formulation of numerous applications in operation research and engineering mechanics that involve nonprobabilistic uncertain information (e.g. see [2,8]). Finding exact bounds to the solution of general LPICs is very difficult in view of their combinatorial nature. Most of the available techniques (e.g. [3,9]) suffer from the need to perform computationally expensive exhaustive searches. Unfortunately, simply resorting to such schemes as deterministic Monte Carlo simulations often fails to capture the exact (optimal) bound solutions.

IV. Direct NLP approaches

This section proposes a pair of novel NLP algorithms that can efficiently compute the two extreme bound solutions $\overline{\alpha}$ and $\underline{\alpha}$ without the need to perform any combinatorial search procedures. The approaches take advantage of the important duality property underpinning interval limit analysis formulations described, in that it enables the maximum $\overline{\alpha}$ and minimum $\underline{\alpha}$ collapse load solutions to be computed from the counterparts of the interval static maximization problem (8) and the interval kinematic

ISBN: 978-988-19252-9-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

minimization problem (9), respectively.

By adopting a maximum value range concept [9], the maximum collapse limit $\overline{\alpha}$ can be obtained by replacing $[\underline{\mathbf{r}},\overline{\mathbf{r}}]$ in the interval inequalities (8.2) with their upper bound values $\overline{\mathbf{r}}$. To automate a choice of interval applied forces $[\underline{\mathbf{f}},\overline{\mathbf{f}}]$ and $[\underline{\mathbf{f}}_d,\overline{\mathbf{f}}_d]$, two respective additional variables $\boldsymbol{\mu} = [\mu^1,\ldots,\mu^d] \in \Re^d$ and $\boldsymbol{\beta} = [\beta^1,\ldots,\beta^d] \in \Re^d$ are introduced. In essence, for each component k the two interval parameters $[\underline{f}^k,\overline{f}^k]$ and $[\underline{f}_d^k,\overline{f}_d^k]$ are replaced by the associated deterministic reformulations:

$$\underline{f}^k + \mu^k \Delta f^k,\ 0 \le \mu^k \le 1, \tag{10}$$

$$\underline{f}_d^k + \beta^k \Delta f_d^k,\ 0 \le \beta^k \le 1, \tag{11}$$

where $\Delta f^k = \overline{f}^k - \underline{f}^k$ and $\Delta f_d^k = \overline{f}_d^k - \underline{f}_d^k$. Such deterministic substitution of interval applied loads accepts not only the interval values that can arbitrarily lie within the intervals, namely $0 < \mu^k < 1$ and $0 < \beta^k < 1$, but also those explicitly on the extreme bounds, i.e. $\mu^k = 1$ (or $\beta^k = 1$) at an upper bound $\overline{f}^k$ (or $\overline{f}_d^k$); and $\mu^k = 0$ (or $\beta^k = 0$) at an lower bound $\underline{f}^k$ (or $\underline{f}_d^k$).

Therefore, the above implementation enables computation for the maximum collapse limit $\overline{\alpha}$ to be obtained by processing the following standard NLP problem formed from the static LPIC (8) in variables (α,**Q**,**μ**,**β**):

$$\begin{aligned}
&\max_{\alpha,\mathbf{Q},\boldsymbol{\mu},\boldsymbol{\beta}} \quad \alpha \\
&\text{subject to } \left(\underline{f}^k + \mu^k \Delta f^k\right)\alpha - \sum_{j=1}^{m} C^{jk} Q^j = \\
&\qquad -\left(\underline{f}_d^k + \beta^k \Delta f_d^k\right),\ \text{ for } k = 1,\ldots,d, \\
&\qquad \sum_{j=1}^{m} N^{jl} Q^j \le \overline{r}^l,\ \text{ for } l = 1,\ldots,y, \\
&\qquad 0 \le \mu^k \le 1,\ \text{ for } k = 1,\ldots,d, \\
&\qquad 0 \le \beta^k \le 1,\ \text{ for } k = 1,\ldots,d.
\end{aligned} \tag{12}$$

Successfully processing the optimal solution of the NLP problem (12) is guaranteed to provide the global maximum collapse load limit $\overline{\alpha}$ to the original LPIC (8).

Likewise, the algorithm for finding the minimum collapse load bound $\underline{\alpha}$ employs a minimum value range concept [9] such that the interval plastic capacities $[\underline{\mathbf{r}},\overline{\mathbf{r}}]$ can be a priori set to their lower bound values $\underline{\mathbf{r}}$. The two interval applied forces $[\underline{\mathbf{f}},\overline{\mathbf{f}}]$ and $[\underline{\mathbf{f}}_d,\overline{\mathbf{f}}_d]$ are replaced by their respective deterministic reformulations in (10) and (11). Thus, to compute the minimum collapse load limit $\underline{\alpha}$ the following standard NLP problem in variables ($\dot{\mathbf{u}}$, $\dot{\boldsymbol{\lambda}}$,**μ**,**β**) is solved:

$$\begin{aligned}
&\min_{\dot{\mathbf{u}},\dot{\boldsymbol{\lambda}},\boldsymbol{\mu},\boldsymbol{\beta}} \quad -\sum_{k=1}^{d}\left(\underline{f}_d^k + \beta^k \Delta f_d^k\right)\dot{u}^k + \sum_{l=1}^{y} \underline{r}^l \dot{\lambda}^l \\
&\text{subject to } \sum_{k=1}^{d}\left(\underline{f}^k + \mu^k \Delta f^k\right)\dot{u}^k = 1, \\
&\qquad -\sum_{k=1}^{d} C^{jk}\dot{u}^k + \sum_{l=1}^{y} N^{jl}\dot{\lambda}^l = 0,\ \text{ for } j = 1,\ldots,m, \\
&\qquad 0 \le \mu^k \le 1,\ \text{ for } k = 1,\ldots,d, \\
&\qquad 0 \le \beta^k \le 1,\ \text{ for } k = 1,\ldots,d, \\
&\qquad \dot{\lambda}^l \ge 0,\ \text{ for } l = 1,\ldots,y.
\end{aligned} \tag{13}$$

Successfully processing the optimal solution of the NLP problem (13) is guaranteed to provide the global minimum collapse load limit $\underline{\alpha}$ to the original LPIC (9).

Some useful remarks associated with the proposed pairs of NLP approaches in (12) and (13) are mentioned in the following:

1) Whilst for both the NLP problems (12) and (13) a priori determined values of plastic capacities either to $\overline{\mathbf{r}}$ in (12) or to $\underline{\mathbf{r}}$ in (13) are not strictly required, they often reduce the computational effort involved.
2) The interval reformulation takes the form of a standard NLP problem that can be robustly and efficiently solved using any available general purpose NLP code.
3) Such NLP problems can be processed within a similar mathematical programming framework to classical deterministic limit analysis problems. Optimal interval values and collapse mechanisms corresponding to each of the collapse load bounds $\overline{\alpha}$ and $\underline{\alpha}$ can be extracted as by-products.
4) The proposed NLP schemes offer greater flexibility, and hence are suitable for possible extension to more sophisticated yet important interval analysis problems, such as those incorporating other plasticity models.

V. Illustrate Example

The application and efficiency of the proposed NLP approaches in computing the optimal extreme collapse load limits $\overline{\alpha}$ and $\underline{\alpha}$ of structures subjected to interval applied loads and plastic capacities are illustrated in this section.

The NLP algorithms were programmed as MATLAB codes that were linked with GAMS, an acronym for the General Algebraic Modeling System [10], mathematical modeling framework through use of a MATLAB-GAMS interfacing software [11]. The specific NLP solver used is GAMS/CONOPT [12].

Some key advantages of GAMS [10], it is worthwhile mentioning, are the availability of a number of industry standard mathematical programming solvers, ability to warm-start successive solves, simplicity of modeling, a facility for large-size data structure manipulation and an automatic differentiation capability. GAMS and all of its solvers are freely accessible through the NEOS server over the internet [13].

The computing times were not reported since each NLP solve only took some seconds to furnish the result.

We consider a nine story portal frame in Fig 1, previously employed to investigate the influences of material softening behaviors and geometry nonlinearities [14,15].

We incorporate the uncertainties caused by interval external forces and material capacities, albeit still assuming small deformations and rigid perfectly plastic materials. The interval loads are applied only at the nodal points within a global axis system, where each nodal loading magnitude varies independently within its specified interval range.

More specifically, the frame was subjected simultaneously to the interval vertical variable forces of $[-6.6\alpha,-5.4\alpha]$ kN and various lateral variable loads, namely $[0.9\alpha,1.1\alpha]$, $[1.8\alpha,2.2\alpha]$ and $[2.7\alpha,3.3\alpha]$ kN, as shown in Fig 1.

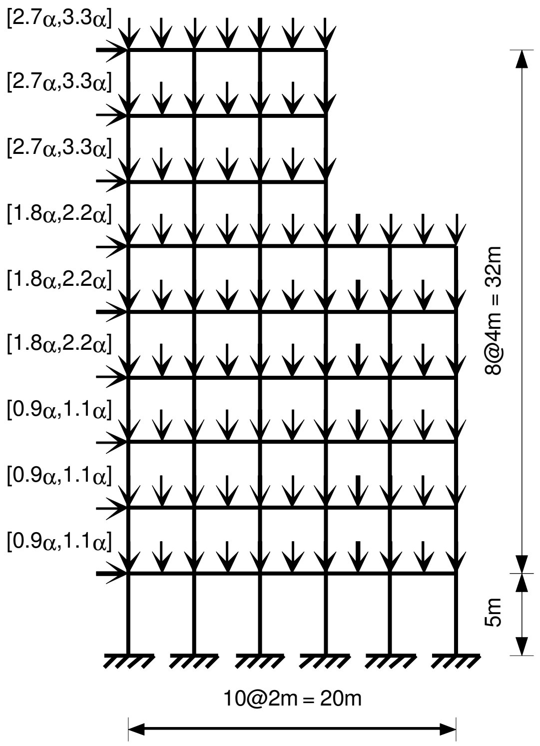

Fig 1. Nine story portal frame.

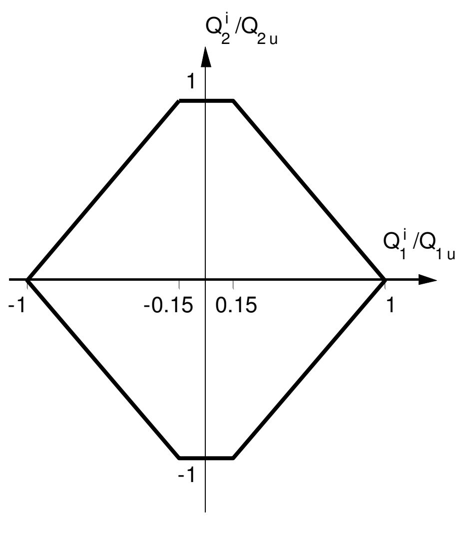

Fig 2. Piecewise linear hexagonal yield model.

Material properties employed were rigid perfectly plastic steel sections: 400WC328 for all columns, Q_{2u} = 1988 kNm, Q_{1u} = 11704 kN; and 460UB82.1 for all beams, Q_{2u} = 552 kNm, Q_{1u} = 3150 kN, where Q_{2u} and Q_{1u} are flexural and axial plastic nominal capacities of each member end, respectively. Moreover, for each element the flexural capacities of two plastic hinges were assumed to be identical. For these hinges, a widely accepted (for a standard I-section) hexagonal piecewise linear yield locus [14] in Fig 2 was adopted throughout.

The interval plastic capacities represent an interval size variation of [0.9,1.1] that still maintains a homothethic hexagonal yield shape of Fig 2. Obviously, a size variation of 1 denotes a nominal value.

The frame structure was discretized into 126 elements, 93 nodes, 261 degrees of freedom, 378 generalized stresses/strains and 1512 yield functions.

Firstly, the NLP problem (12) was successfully solved within a single step to obtain the optimal maximum collapse limit of $\overline{\alpha} = 114.55$. The corresponding collapse mechanism is displayed in Fig 3a.

Then, the optimal minimum collapse limit of $\underline{\alpha} = 76.68$ was computed directly from the NLP problem (13). The collapse mechanism associated with this $\underline{\alpha}$ is depicted in Fig 3b. Incidentally, hinge dispositions of the two optimal solutions $\overline{\alpha}$ and $\underline{\alpha}$ are identical.

The above results $\overline{\alpha}$ and $\underline{\alpha}$ are compared with those (approximately) found from 100,000 Monte Carlo simulations. The Monte Carlo runs reported $\overline{\alpha} = 99.06$ and $\overline{\alpha} = 85.26$. These values, to some extent, validate the accuracy of the proposed NLP approaches. Clearly, such simulation schemes provided a larger value of $\overline{\alpha}$ and at the same time a smaller value of $\overline{\alpha}$.

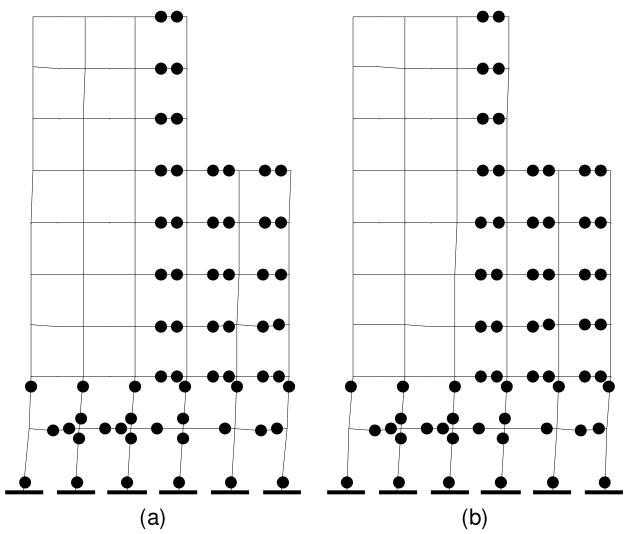

Fig 3. Collapse mechanisms (a) maximum collapse limit and (b) minimum collapse limit; • denotes plastic hinge.

Simply assuming nominal or average values of the interval applied loads and plastic capacities enabled the classical deterministic limit analysis problem (1) to be solved. The computations reported a collapse load solution of $\alpha = 93.72$ that is some 18% less and 22% greater than the

ISBN: 978-988-19252-9-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

two optimal $\overline{\alpha}$ and $\underline{\alpha}$, respectively.

VI. Concluding Remarks

Robust and efficient NLP approaches have been proposed to by-pass traditional, often computationally expensive, combinatorial search procedures in handling the influences of uncertain, nonprobabilistic but interval, inputs. In essence, the classical limit analysis approach for rigid perfectly plastic structures has been extended to accommodate interval applied loads and/or plastic capacities.

In the presence of interval quantities, the paper has developed a pair of novel NLP techniques that can provide the optimal maximum collapse load limit in one case and the optimal minimum collapse load limit in the other.

A number of numerical examples motivated by practical engineering structures, one of which has been provided in this paper, have been successfully processed using the proposed approaches. The computing efforts in obtaining each of the optimal bound solutions are as small as those required when processing a traditional deterministic limit analysis counterpart, thereby illustrating the efficiency of the developed schemes. The accuracy of the reported results has partially been validated by Monte Carlo simulated runs.

The present methods can lead to useful extensions in incorporating other sources of uncertainties (which are difficult to handle) mainly related to nonlinear interval functions, such as geometric imperfections and nonlinear yield models.

References

[1] R.E. Melchers, *Structural Reliability: Analysis and Prediction*. New York: Wiley, 1987.

[2] I. Elishakoff, *Probabilistic Methods in the Theory of Structures*. New York: Wiley, 1983.

[3] J.W. Chinneck, and K. Ramadan, "Linear programming with interval coefficients", *The Journal of the Operational Research Society*, vol. 51, pp. 209-220, 2000.

[4] J.A. Kamenjarzh, *Limit Analysis of Solids and Structures*. Boca Raton: CRC Press, 1996.

[5] G. Maier, "A matrix structural theory of piecewise linear elastoplasticity with interacting yield planes", *Meccanica*, vol. 5, pp. 54-66, 1970.

[6] G. Cocchetti, and G. Maier, "Elastic-plastic and limit-state analyses of frames with softening plastic-hinge models by mathematical programming", *International Journal of Solids and Structures*, vol. 40, pp. 7219-7244, 2003.

[7] S. Tangaramvong, F. Tin-Loi, and T. Senjuntichai, "An MPEC approach for the critical post-collapse behavior of rigid-plastic structures", *International Journal of Solids and Structures*, vol. 48, pp. 2732-2742, 2011.

[8] Y. Kanno and I. Takewaki, "Worst case plastic limit analysis of trusses under uncertain loads via mixed 0-1 programming", *Journal of Mechanics of Materials and Structures*, vol. 2, pp. 245-273, 2007.

[9] T. Shaocheng, "Interval number and fuzzy number linear programmings", *Fuzzy Sets and Systems*, vol. 66, pp. 301-306, 1994.

[10] A. Brooke, D. Kendrick, A. Meeraus, and R. Raman, *GAMS: A User's Guide*, Washington, DC: GAMS Development Corporation, 1998.

[11] M.C. Ferris, "MATLAB and GAMS: interfacing optimization and visualization software", Computer Sciences Department, University of Wisconsin, Madison, Wisconsin. Technical Report TR98-19, 1998.

[12] A.S. Drud, "CONOPT-a large-scale GRG code", *ORSA Journal on Computing*, vol. 6, pp. 207-216, 1994.

[13] E.D. Dolan, R. Fourer, J.J. Moré, and T.S. Munson, "Optimization on the NEOS server", *SIAM News*, vol. 35 pp. 1-5, 2002.

[14] S. Tangaramvong and F. Tin-Loi, "Limit analysis of elastoplastic frames considering 2nd-order geometric nonlinearity and displacement constraints", *International Journal of Mechanical Sciences*, vol. 51, pp. 179-191, 2009.

[15] S. Tangaramvong and F. Tin-Loi, "The influence of geometric effects on the behavior of strain softening frames", *Computational Mechanics*, vol. 46, pp. 661-678, 2010.